

Research Article

On Hadamard-Type Inequalities Involving Several Kinds of Convexity

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Received 14 May 2010; Accepted 23 August 2010

Academic Editor: Sin E. I. Takahasi

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We do not only give the extensions of the results given by Gill et al. (1997) for log-convex functions but also obtain some new Hadamard-type inequalities for log-convex m -convex, and (α, m) -convex functions.

1. Introduction

The following inequality is well known in the literature as Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1.1)$$

where $f : I \rightarrow R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, note worthy extension, generalizations, and numerous applications on this inequality; see ([1–6]) where further references are given.

Let I be an interval in \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$ is said to be convex if, for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.2)$$

(see [5], Page 1). Geometrically, this means that if K, L , and M are three distinct points on the graph of f with L between K and M , then L is on or below chord KM .

Recall that a function $f : I \rightarrow (0, \infty)$ is said to be log-convex function if, for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality (see [5], Page 3)

$$f(tx + (1 - t)y) \leq [f(x)]^t [f(y)]^{(1-t)}. \quad (1.3)$$

It is said to be log-concave if the inequality in (1.3) is reversed.

In [7], Toader defined m -convexity as follows.

Definition 1.1. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be m -convex, where $m \in [0, 1]$, if one has

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) \quad (1.4)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m -concave if $-f$ is m -convex.

Denote by $K_m(b)$ the class of all m -convex functions on $[0, b]$ such that $f(0) \leq 0$ (if $m < 1$). Obviously, if we choose $m = 1$, Definition 1.1 recaptures the concept of standard convex functions on $[0, b]$.

In [8], Miheşan defined (α, m) -convexity as in the following:

Definition 1.2. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if one has

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y) \quad (1.5)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha, m) = (1, m)$, (α, m) -convexity reduces to m -convexity and for $(\alpha, m) = (1, 1)$, (α, m) -convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$.

For recent results and generalizations concerning m -convex and (α, m) -convex functions, see ([9–12]).

In the literature, the logarithmic mean of the positive real numbers p, q is defined as the following:

$$L(p, q) = \frac{p - q}{\ln p - \ln q} \quad (p \neq q) \quad (1.6)$$

(for $p = q$, we put $L(p, p) = p$).

In [13], Gill et al. established the following results.

Theorem 1.3. *Let f be a positive, log-convex function on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(t) dt \leq L(f(a), f(b)), \quad (1.7)$$

where $L(\cdot, \cdot)$ is a logarithmic mean of the positive real numbers as in (1.6).

For f a positive log-concave function, the inequality is reversed.

Corollary 1.4. *Let f be positive log-convex functions on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(t) dt \leq \min_{x \in [a, b]} \frac{(x-a)L(f(a), f(x)) + (b-x)L(f(x), f(b))}{b-a}. \quad (1.8)$$

If f is a positive log-concave function, then

$$\frac{1}{b-a} \int_a^b f(x) dx \geq \max_{x \in [a, b]} \frac{(x-a)L(f(a), f(x)) + (b-x)L(f(x), f(b))}{b-a}. \quad (1.9)$$

For some recent results related to the Hadamard's inequalities involving two log-convex functions, see [14] and the references cited therein. The main purpose of this paper is to establish the general version of inequalities (1.7) and new Hadamard-type inequalities involving two log-convex functions, two m -convex functions, or two (α, m) -convex functions using elementary analysis.

2. Main Results

We start with the following theorem.

Theorem 2.1. *Let $f_i : I \subset \mathbb{R} \rightarrow (0, \infty)$ ($i = 1, 2, \dots, n$) be log-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a} \int_a^b \prod_{i=1}^n f_i(x) dx \leq L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(b)\right), \quad (2.1)$$

where L is a logarithmic mean of positive real numbers.

For f a positive log-concave function, the inequality is reversed.

Proof. Since $f_i (i = 1, 2, \dots, n)$ are log-convex functions on I , we have

$$f_i(ta + (1-t)b) \leq [f_i(a)]^t [f_i(b)]^{(1-t)} \quad (2.2)$$

for all $a, b \in I$ and $t \in [0, 1]$. Writing (2.2) for $i = 1, 2, \dots, n$ and multiplying the resulting inequalities, it is easy to observe that

$$\begin{aligned} \prod_{i=1}^n f_i(ta + (1-t)b) &\leq \left[\prod_{i=1}^n f_i(a) \right]^t \left[\prod_{i=1}^n f_i(b) \right]^{(1-t)} \\ &= \prod_{i=1}^n f_i(b) \left[\prod_{i=1}^n \frac{f_i(a)}{f_i(b)} \right]^t \end{aligned} \quad (2.3)$$

for all $a, b \in I$ and $t \in [0, 1]$.

Integrating inequality (2.3) on $[0, 1]$ over t , we get

$$\int_0^1 \prod_{i=1}^n f_i(ta + (1-t)b) dt \leq \prod_{i=1}^n f_i(b) \int_0^1 \left[\prod_{i=1}^n \frac{f_i(a)}{f_i(b)} \right]^t dt. \quad (2.4)$$

As

$$\int_0^1 \prod_{i=1}^n f_i(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b \prod_{i=1}^n f_i(x) dx, \quad (2.5)$$

$$\int_0^1 \left[\prod_{i=1}^n \frac{f_i(a)}{f_i(b)} \right]^t dt = \frac{1}{\prod_{i=1}^n f_i(b)} L \left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(b) \right), \quad (2.6)$$

the theorem is proved. \square

Remark 2.2. By taking $i = 1$ and $f_1 = f$ in Theorem 2.1, we obtain (1.7).

Corollary 2.3. Let $f_i : I \subset \mathbb{R} \rightarrow (0, \infty)$ ($i = 1, 2, \dots, n$) be log-convex functions on I and $a, b \in I$ with $a < b$. Then

$$\begin{aligned} &\frac{1}{b-a} \int_a^b \prod_{i=1}^n f_i(x) dx \\ &\leq \min_{x \in [a,b]} \frac{(x-a)L(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)) + (b-x)L(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b))}{b-a}. \end{aligned} \quad (2.7)$$

If $f_i (i = 1, 2, \dots, n)$ are positive log-concave functions, then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \prod_{i=1}^n f_i(x) dx \\ & \geq \max_{x \in [a,b]} \frac{(x-a)L(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)) + (b-x)L(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b))}{b-a}. \end{aligned} \quad (2.8)$$

Proof. Let $f_i (i = 1, 2, \dots, n)$ be positive log-convex functions. Then by Theorem 2.1 we have that

$$\begin{aligned} \int_a^b \prod_{i=1}^n f_i(t) dt &= \int_a^x \prod_{i=1}^n f_i(t) dt + \int_x^b \prod_{i=1}^n f_i(t) dt \\ &\leq (x-a)L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)\right) + (b-x)L\left(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b)\right), \end{aligned} \quad (2.9)$$

for all $x \in [a, b]$, whence (2.7). Similarly we can prove (2.8). \square

Remark 2.4. By taking $i = 1$ and $f_1 = f$ in (2.7) and (2.8), we obtain the inequalities of Corollary 1.4.

We will now point out some new results of the Hadamard type for log-convex, m -convex, and (α, m) -convex functions, respectively.

Theorem 2.5. Let $f, g : I \rightarrow (0, \infty)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left\{ \frac{1}{b-a} \int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)] dx \right\} \\ &\leq \frac{f(a)f(b) + g(a)g(b)}{2}. \end{aligned} \quad (2.10)$$

Proof. We can write

$$\frac{a+b}{2} = \frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}. \quad (2.11)$$

Using the elementary inequality $cd \leq 1/2[c^2 + d^2]$ ($c, d \geq 0$ reals) and equality (2.11), we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{2} \left[f^2\left(\frac{a+b}{2}\right) + g^2\left(\frac{a+b}{2}\right) \right] \\
& = \frac{1}{2} \left[f^2\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) + g^2\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \right] \\
& \leq \frac{1}{2} \left\{ \left[(f(ta+(1-t)b))^{1/2} \right]^2 \left[(f((1-t)a+tb))^{1/2} \right]^2 \right. \\
& \quad \left. + \left[(g(ta+(1-t)b))^{1/2} \right]^2 \left[(g((1-t)a+tb))^{1/2} \right]^2 \right\} \\
& = \frac{1}{2} [f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)].
\end{aligned} \tag{2.12}$$

Since f, g are log-convex functions, we obtain

$$\begin{aligned}
& \frac{1}{2} [f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)] \\
& \leq \left\{ \frac{1}{2} [f(a)]^t [f(b)]^{(1-t)} [f(a)]^{(1-t)} [f(b)]^t + [g(a)]^t [g(b)]^{(1-t)} [g(a)]^{(1-t)} [g(b)]^t \right\} \\
& = \frac{f(a)f(b) + g(a)g(b)}{2}
\end{aligned} \tag{2.13}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Rewriting (2.12) and (2.13), we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2} [f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)], \tag{2.14}$$

$$\frac{1}{2} [f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)] \leq \frac{f(a)f(b) + g(a)g(b)}{2}. \tag{2.15}$$

Integrating both sides of (2.14) and (2.15) on $[0, 1]$ over t , respectively, we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)] dx \right], \\
& \frac{1}{2} \left[\frac{1}{b-a} \int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)] dx \right] \leq \frac{f(a)f(b) + g(a)g(b)}{2}.
\end{aligned} \tag{2.16}$$

Combining (2.16), we get the desired inequalities (2.10). The proof is complete. \square

Theorem 2.6. Let $f, g : I \rightarrow (0, \infty)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b [f^2(x) + g^2(x)] dx \\ &\leq \frac{f(a) + f(b)}{2} L(f(a), f(b)) + \frac{g(a) + g(b)}{2} L(g(a), g(b)), \end{aligned} \quad (2.17)$$

where $L(\cdot, \cdot)$ is a logarithmic mean of positive real numbers.

Proof. From inequality (2.14), we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq \frac{1}{2} [f(ta + (1-t)b)f((1-t)a + tb) + g(ta + (1-t)b)g((1-t)a + tb)]. \end{aligned} \quad (2.18)$$

for all $a, b \in I$ and $t \in [0, 1]$.

Using the elementary inequality $cd \leq 1/2[c^2 + d^2]$ ($c, d \geq 0$ reals) on the right side of the above inequality, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq \frac{1}{4} [f^2(ta + (1-t)b) + f^2((1-t)a + tb) + g^2(ta + (1-t)b) + g^2((1-t)a + tb)]. \end{aligned} \quad (2.19)$$

Since f, g are log-convex functions, then we get

$$\begin{aligned} &[f^2(ta + (1-t)b) + f^2((1-t)a + tb) + g^2(ta + (1-t)b) + g^2((1-t)a + tb)] \\ &\leq \{ [f(a)]^{2t} [f(b)]^{(2-2t)} + [f(a)]^{(2-2t)} [f(b)]^{2t} + [g(a)]^{2t} [g(b)]^{(2-2t)} + [g(a)]^{(2-2t)} [g(b)]^{2t} \} \\ &= \left[f^2(b) \left[\frac{f(a)}{f(b)} \right]^{2t} + f^2(a) \left[\frac{f(b)}{f(a)} \right]^{2t} + g^2(b) \left[\frac{g(a)}{g(b)} \right]^{2t} + g^2(a) \left[\frac{g(b)}{g(a)} \right]^{2t} \right]. \end{aligned} \quad (2.20)$$

Integrating both sides of (2.19) and (2.20) on $[0, 1]$ over t , respectively, we obtain

$$\begin{aligned}
& 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b [f^2(x) + g^2(x)] dx, \\
& \frac{1}{b-a} \int_a^b [f^2(x) + g^2(x)] dx \\
& \leq \frac{1}{2} \left(f^2(b) \int_0^1 \left[\frac{f(a)}{f(b)} \right]^{2t} dt + f^2(a) \int_0^1 \left[\frac{f(b)}{f(a)} \right]^{2t} dt \right. \\
& \quad \left. + g^2(b) \int_0^1 \left[\frac{g(a)}{g(b)} \right]^{2t} dt + g^2(a) \int_0^1 \left[\frac{g(b)}{g(a)} \right]^{2t} dt \right) \\
& = \frac{1}{2} \left(f^2(b) \left[\frac{[f(a)/f(b)]^{2t}}{2 \log f(a)/f(b)} \right]_0^1 + f^2(a) \left[\frac{[f(b)/f(a)]^{2t}}{2 \log f(b)/f(a)} \right]_0^1 \right. \\
& \quad \left. + g^2(b) \left[\frac{[g(a)/g(b)]^{2t}}{2 \log g(a)/g(b)} \right]_0^1 + g^2(a) \left[\frac{[g(b)/g(a)]^{2t}}{2 \log g(b)/g(a)} \right]_0^1 \right) \tag{2.21} \\
& = \frac{1}{2} \left(\frac{f^2(a) - f^2(b)}{2(\log f(a) - \log f(b))} + \frac{f^2(b) - f^2(a)}{2(\log f(b) - \log f(a))} \right. \\
& \quad \left. + \frac{g^2(a) - g^2(b)}{2(\log g(a) - \log g(b))} + \frac{g^2(b) - g^2(a)}{2(\log g(b) - \log g(a))} \right) \\
& = \frac{1}{2} \left(\frac{f(a) + f(b)}{2} L(f(a), f(b)) + \frac{f(a) + f(b)}{2} L(f(b), f(a)) \right. \\
& \quad \left. + \frac{g(a) + g(b)}{2} L(g(a), g(b)) + \frac{g(a) + g(b)}{2} L(g(b), g(a)) \right) \\
& = \left\{ \frac{f(a) + f(b)}{2} L(f(a), f(b)) + \frac{g(a) + g(b)}{2} L(g(a), g(b)) \right\}.
\end{aligned}$$

Combining (2.21), we get the required inequalities (2.17). The proof is complete. \square

Theorem 2.7. Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is nonincreasing m_1 -convex function and g is nonincreasing m_2 -convex function on $[a, b]$ for some fixed $m_1, m_2 \in (0, 1]$, then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \min\{S_1, S_2\}, \tag{2.22}$$

where

$$S_1 = \frac{1}{6} \left[(f^2(a) + g^2(a)) + m_1 f(a) f\left(\frac{b}{m_1}\right) + m_2 g(a) g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right) + m_2^2 g^2\left(\frac{b}{m_2}\right) \right], \quad (2.23)$$

$$S_2 = \frac{1}{6} \left[(f^2(b) + g^2(b)) + m_1 f(b) f\left(\frac{a}{m_1}\right) + m_2 g(b) g\left(\frac{a}{m_2}\right) + m_1^2 f^2\left(\frac{a}{m_1}\right) + m_2^2 g^2\left(\frac{a}{m_2}\right) \right] \quad (2.24)$$

Proof. Since f is m_1 -convex function and g is m_2 -convex function, we have

$$\begin{aligned} f(ta + (1-t)b) &\leq tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right), \\ g(ta + (1-t)b) &\leq tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right) \end{aligned} \quad (2.25)$$

for all $t \in [0, 1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a) \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt. \quad (2.26)$$

Using the elementary inequality $cd \leq 1/2(c^2 + d^2)$ ($c, d \geq 0$ reals), (2.25) on the right side of (2.26) and making the change of variable and since f, g is nonincreasing, we have

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\leq \frac{1}{2}(b-a) \int_0^1 \left[\{f(ta + (1-t)b)\}^2 + \{g(ta + (1-t)b)\}^2 \right] dt \\ &\leq \frac{1}{2}(b-a) \int_0^1 \left[\left(tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right) \right)^2 + \left(tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right) \right)^2 \right] dt \\ &= \frac{1}{2}(b-a) \left[\frac{1}{3}f^2(a) + \frac{1}{3}m_1^2 f^2\left(\frac{b}{m_1}\right) + \frac{1}{3}m_1 f(a) f\left(\frac{b}{m_1}\right) + \frac{1}{3}g^2(a) + \frac{1}{3}m_2^2 g^2\left(\frac{b}{m_2}\right) \right. \\ &\quad \left. + \frac{1}{3}m_2 g(a) g\left(\frac{b}{m_2}\right) \right] \quad (2.27) \\ &= \frac{(b-a)}{6} \left[(f^2(a) + g^2(a)) + m_1 f(a) f\left(\frac{b}{m_1}\right) + m_2 g(a) g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right) \right. \\ &\quad \left. + m_2^2 g^2\left(\frac{b}{m_2}\right) \right]. \end{aligned}$$

Analogously we obtain

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \leq \frac{(b-a)}{6} \left[(f^2(b)+g^2(b)) + m_1 f(b)f\left(\frac{a}{m_1}\right) + m_2 g(b)g\left(\frac{a}{m_2}\right) + m_1^2 f^2\left(\frac{a}{m_1}\right) + m_2^2 g^2\left(\frac{a}{m_2}\right) \right]. \end{aligned} \quad (2.28)$$

Rewriting (2.27) and (2.28), we get the required inequality in (2.22). The proof is complete. \square

Theorem 2.8. Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is nonincreasing (α_1, m_1) -convex function and g is nonincreasing (α_2, m_2) -convex function on $[a, b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$. Then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \min\{E_1, E_2\}, \quad (2.29)$$

where

$$\begin{aligned} E_1 = \frac{1}{2} & \left[\frac{1}{2\alpha_1 + 1} f^2(a) + \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2\left(\frac{b}{m_1}\right) \right. \\ & + \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(a)f\left(\frac{b}{m_1}\right) + \frac{1}{2\alpha_2 + 1} g^2(a) \\ & \left. + \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2\left(\frac{b}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(a)g\left(\frac{b}{m_2}\right) \right], \end{aligned} \quad (2.30)$$

$$\begin{aligned} E_2 = \frac{1}{2} & \left[\frac{1}{2\alpha_1 + 1} f^2(b) + \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2\left(\frac{a}{m_1}\right) \right. \\ & + \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(b)f\left(\frac{a}{m_1}\right) + \frac{1}{2\alpha_2 + 1} g^2(b) \\ & \left. + \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2\left(\frac{a}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(b)g\left(\frac{a}{m_2}\right) \right]. \end{aligned} \quad (2.31)$$

Proof. Since f is (α_1, m_1) -convex function and g is (α_2, m_2) -convex function, then we have

$$\begin{aligned} f(ta + (1-t)b) & \leq t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right), \\ g(ta + (1-t)b) & \leq t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right) \end{aligned} \quad (2.32)$$

for all $t \in [0, 1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a) \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt. \quad (2.33)$$

Using the elementary inequality $cd \leq 1/2(c^2 + d^2)$ ($c, d \geq 0$ reals), (2.32) on the right side of (2.33) and making the change of variable and since f, g is nonincreasing, we have

$$\begin{aligned} \int_a^b f(x)g(x)dx &\leq \frac{1}{2}(b-a) \int_0^1 \left[\{f(ta + (1-t)b)\}^2 + \{g(ta + (1-t)b)\}^2 \right] dt \\ &\leq \frac{1}{2}(b-a) \int_0^1 \left[\left(t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1}) f\left(\frac{b}{m_1}\right) \right)^2 \right. \\ &\quad \left. + \left(t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2}) g\left(\frac{b}{m_2}\right) \right)^2 \right] dt \\ &= \frac{1}{2}(b-a) \left[\frac{1}{2\alpha_1+1} f^2(a) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} m_1^2 f^2\left(\frac{b}{m_1}\right) \right. \\ &\quad + \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)} m_1 f(a) f\left(\frac{b}{m_1}\right) + \frac{1}{2\alpha_2+1} g^2(a) \\ &\quad \left. + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)} m_2^2 g^2\left(\frac{b}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)} m_2 g(a) g\left(\frac{b}{m_2}\right) \right] \end{aligned} \quad (2.34)$$

Analogously we obtain

$$\begin{aligned} \int_a^b f(x)g(x)dx &\leq \frac{1}{2}(b-a) \left[\frac{1}{2\alpha_1+1} f^2(b) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} m_1^2 f^2\left(\frac{a}{m_1}\right) \right. \\ &\quad + \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)} m_1 f(b) f\left(\frac{a}{m_1}\right) + \frac{1}{2\alpha_2+1} g^2(b) \\ &\quad \left. + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)} m_2^2 g^2\left(\frac{a}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)} m_2 g(b) g\left(\frac{a}{m_2}\right) \right]. \end{aligned} \quad (2.35)$$

Rewriting (2.34) and (2.35), we get the required inequality in (2.29). The proof is complete. \square

Remark 2.9. In Theorem 2.8, if we choose $\alpha_1 = \alpha_2 = 1$, we obtain the inequality of Theorem 2.7.

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