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Research Article

Gronwall-OuIang-Type Integral Inequalities on Time Scales

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We present several Gronwall-OuIang-type integral inequalities on time scales. Firstly, an OuIang inequality on time scales is discussed. Then we extend the Gronwall-type inequalities to multiple integrals. Some special cases of our results contain continuous Gronwall-type inequalities and their discrete analogues. Several examples are included to illustrate our results at the end.

1. Introduction

OuIang inequalities and their generalizations have proved to be useful tools in oscillation theory, boundedness theory, stability theory, and other applications of differential and difference equations. A nice introduction to continuous and discrete OuIang inequalities can be found in [1, 2], and studies in [3–5] give some of their generalizations to multiple integrals and higher-dimensional spaces. Like Gronwall's inequality, OuIang's inequality is also used to obtain a priori bounds on unknown functions. Therefore, integral inequalities of this type are usually known as Gronwall-OuIang-type inequalities [6].

The calculus on time scales has been introduced by Hilger [7] in order to unify discrete and continuous analysis. For the general basic ideas and background, we refer to [8, 9]. In this paper, we are concerned with Gronwall-OuIang-type integral inequalities on time scales, which unify and extend the corresponding continuous inequalities and their discrete analogues. We also provide a more useful and explicit bound than that in [10–12].

2. OuIang Inequality

We first give Gronwall's inequality on time scales which could be found in [8, Corollary 6.7]. Throughout this section, we fix $t_0 \in \mathbb{T}$ and let $\mathbb{T}_{t_0}^{+} = \{t \in \mathbb{T} : t \geq t_0\}$.

Lemma 2.1. *Let $y \in C_{\mathrm{rd}}$, $p \in \mathcal{R}^+$, $p(t) \geq 0$, for all $t \in \mathbb{T}_{t_0}^+$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau \quad \forall t \in \mathbb{T}_{t_0}^+ \tag{2.1}$$

implies that

$$y(t) \leq \alpha e_p(t, t_0) \quad \forall t \in \mathbb{T}_{t_0}^+. \tag{2.2}$$

Above, $\mathcal{R}$ is defined as the set of all regressive and rd-continuous functions, $\mathcal{R}^+$ is the positive regressive part of $\mathcal{R}$, the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ is defined by

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{with } p, q \in \mathcal{R}, \tag{2.3}$$

and $e_p(t, t_0)$ is the exponential function on time scales; for more details on time scales, see [8, 9].

Now we will give the OuIang inequality on time scales.

Theorem 2.2. *Let u and v be real-valued nonnegative rd-continuous functions defined on $\mathbb{T}_{t_0}^+$. If*

$$u^2(t) \leq c + \int_{t_0}^{t} u(\tau)v(\tau)\Delta\tau \quad \forall t \in \mathbb{T}_{t_0}^+, \tag{2.4}$$

where c is a positive constant, then

$$u(t) \leq \sqrt{c} + \frac{1}{2}\int_{t_0}^{t} v(\tau)\Delta\tau \quad \forall t \in \mathbb{T}_{t_0}^+. \tag{2.5}$$

Proof. Let

$$w(t) = \int_{t_0}^{t} u(\tau)v(\tau)\Delta\tau. \tag{2.6}$$

From (2.4), we have

$$u^2(t)v^2(t) \leq v^2(t)(c + w(t)). \tag{2.7}$$

The definition of w gives

$$w^{\Delta}(t) \leq v(t)\sqrt{c + w(t)}, \tag{2.8}$$

Dividing both sides of (2.8) by $\sqrt{c+w(t)}$ and integrating from t_0 to $t \in \mathbb{T}_{t_0}^+$, we have

$$\int_{t_0}^{t} \frac{w^{\Delta}(\tau)}{\sqrt{c+w(\tau)}}\Delta\tau \le \int_{t_0}^{t} v(\tau)\Delta\tau. \tag{2.9}$$

According to the chain rule [8, Theorem 1.93] and since w is increasing,

$$\begin{aligned} 2\int_{t_0}^{t} \left(\sqrt{c+w}\right)^{\Delta}(\tau)\Delta\tau &= \int_{t_0}^{t} \frac{2w^{\Delta}(\tau)}{\sqrt{c+w(\tau)}+\sqrt{c+w(\sigma(\tau))}}\Delta\tau \\ &\le \int_{t_0}^{t} \frac{w^{\Delta}(\tau)}{\sqrt{c+w(\tau)}}\Delta\tau \\ &\le \int_{t_0}^{t} v(\tau)\Delta\tau, \end{aligned} \tag{2.10}$$

so

$$\sqrt{c+w(t)} - \sqrt{c} \le \frac{1}{2}\int_{t_0}^{t} v(\tau)\Delta\tau. \tag{2.11}$$

Combining (2.4) and (2.11) yields (2.5) and completes the proof. □

In 1979, Dafermos [13] published a so-called Gronwall-type inequality (see also [3]). In the same way as Theorem 2.2, we now extend this result to general time scales.

Theorem 2.3. *Let y and g be nonnegative rd-continuous functions on $\mathbb{T}_{t_0}^+$. Let α, M, N be nonnegative constants and $-\alpha \in \mathcal{R}^+$. If*

$$y^2(t) \le M^2 y^2(t_0) + 2\int_{t_0}^{t}\left[\alpha y^2(\tau) + N g(\tau) y(\tau)\right]\Delta\tau \quad \forall t \in \mathbb{T}_{t_0}^+, \tag{2.12}$$

then

$$y(t) \le M y(t_0) e_{\ominus(-\alpha)}(t,t_0) + \int_{t_0}^{t} N g(\tau) e_{\ominus(-\alpha)}(t,\tau)\Delta\tau \quad \forall t \in \mathbb{T}_{t_0}^+. \tag{2.13}$$

Proof. Let

$$z(t) = M^2 y^2(t_0) + 2\int_{t_0}^{t}\left[\alpha y^2(\tau) + N g(\tau) y(\tau)\right]\Delta\tau. \tag{2.14}$$

Then,

$$\begin{aligned} z^{\Delta}(t) &= 2\alpha y^2(t) + 2Ng(t)y(t) \le 2\alpha z(t) + 2Ng(t)\sqrt{z(t)} \\ &\le \alpha\sqrt{z(\sigma(t))}\left(\sqrt{z(t)} + \sqrt{z(\sigma(t))}\right) + Ng(t)\left(\sqrt{z(t)} + \sqrt{z(\sigma(t))}\right). \end{aligned} \tag{2.15}$$

Hence,

$$\frac{z^{\Delta}(t)}{\sqrt{z(t)} + \sqrt{z(\sigma(t))}} - \alpha\sqrt{z(\sigma(t))} \le Ng(t). \tag{2.16}$$

Multiplying both sides of (2.16) by $e_{-\alpha}(t, t_0)$, we have

$$\left(\sqrt{z}e_{-\alpha}(\cdot, t_0)\right)^{\Delta}(t) \le Ng(t)e_{-\alpha}(t, t_0). \tag{2.17}$$

Integrating (2.17) from t_0 to t, we obtain that

$$\sqrt{z(t)}e_{-\alpha}(t, t_0) \le My(t_0) + \int_{t_0}^{t} Ng(\tau)e_{-\alpha}(\tau, t_0)\Delta\tau. \tag{2.18}$$

Combining (2.12) and (2.18), and using [8, Theorems 2.36 and 2.48] yields (2.13) and completes the proof. □

Remark 2.4. If $\alpha = 0$ and $N = 1/2$, then Theorem 2.3 reduces to Theorem 2.2.

Remark 2.5. If we multiply inequality (2.16) by another exponential function on time scales, for example, $e_{\ominus(2\alpha)}(t, t_0)$, we could get another kind of inequality, which is a special case of Theorem 3.4.

3. Gronwall-OuIang-Type Inequality

Pachpatte discussed several integral inequalities arising in the theory of differential equations and difference equations [3, 4]. Now, we extend some of these results to time scales. First, we give some notations and definitions which are used in our subsequent discussion.

To simplify the expression, we let $0 \in \mathbb{T}$, choose rd-continuous functions r_i $(1 \le i \le n)$ such that

$$r_i(t) > 0, \quad 1 \le i \le n-1, \qquad r_n(t) = 1 \quad \forall t \in \mathbb{T}_0^+, \tag{3.1}$$

and define the differential operators L_i, $0 \le i \le n$, by

$$L_0 x = x, \qquad L_i x = \frac{1}{r_i}(L_{i-1}x)^{\Delta}, \quad 1 \le i \le n. \tag{3.2}$$

For $t \in \mathbb{T}_0^+$ and a nonnegative function r defined on $\mathbb{T}_0^+$, we set

$$A[t, r_1, \ldots, r_{n-1}, r] = \int_0^t r_1(t_1) \cdots \int_0^{t_{n-2}} r_{n-1}(t_{n-1}) \int_0^{t_{n-1}} r(t_n) \Delta t_n \Delta t_{n-1} \cdots \Delta t_1. \tag{3.3}$$

Theorem 3.1. *Let F and r be real-valued nonnegative rd-continuous functions on $\mathbb{T}_0^+$, and let $q > 1$ be a constant. If*

$$F^q(t) \leq c + A[t, r_1, \ldots, r_{n-1}, rF] \quad \forall t \in \mathbb{T}_0^+, \tag{3.4}$$

where $c > 0$ is a constant, then

$$F(t) \leq \left\{ c^{(q-1)/q} + \frac{q-1}{q} A[t, r_1, \ldots, r_{n-1}, r] \right\}^{1/(q-1)} \quad \forall t \in \mathbb{T}_0^+. \tag{3.5}$$

Proof. Let

$$z(t) = c + A[t, r_1, \ldots, r_{n-1}, rF]. \tag{3.6}$$

From (3.6), it is easy to observe that

$$L_n z = rF \leq r z^{1/q}. \tag{3.7}$$

From (3.7) and using the facts that z and z^Δ are nonnegative, and

$$\left(z^{1/q}\right)^\Delta = \frac{1}{q} z^\Delta \int_0^1 \left(z + \mu z^\Delta h\right)^{1/q-1} \mathrm{d}h \geq 0, \tag{3.8}$$

we have

$$\frac{L_n z}{\left(z^{1/q}\right)^\sigma} \leq \frac{L_n z}{z^{1/q}} \leq r \leq r + \frac{L_{n-1} z \left(z^{1/q}\right)^\Delta}{z^{1/q} \left(z^{1/q}\right)^\sigma}, \tag{3.9}$$

that is,

$$\left(\frac{L_{n-1} z}{z^{1/q}} \right)^\Delta \leq r. \tag{3.10}$$

Integrating (3.10) with respect to t_n from 0 to t and using the fact that $L_{n-1} z(0) = 0$, we obtain that

$$\frac{L_{n-1} z(t)}{z^{1/q}(t)} \leq \int_0^t r(t_n) \Delta t_n, \tag{3.11}$$

which implies that

$$\frac{(L_{n-2}z)^{\Delta}(t)}{r_{n-1}(t)z^{1/q}(t)} \leq \int_0^t r(t_n)\Delta t_n. \tag{3.12}$$

Again as above, from (3.12), we observe that

$$\frac{(L_{n-2}z)^{\Delta}(t)}{\left(z^{1/q}\right)^{\sigma}(t)} \leq \frac{(L_{n-2}z)^{\Delta}(t)}{z^{1/q}(t)} \leq r_{n-1}(t)\int_0^t r(t_n)\Delta t_n \leq r_{n-1}(t)\int_0^t r(t_n)\Delta t_n + \frac{L_{n-2}z(t)\left(z^{1/q}\right)^{\Delta}(t)}{z^{1/q}(t)\left(z^{1/q}\right)^{\sigma}(t)}, \tag{3.13}$$

that is,

$$\left(\frac{L_{n-2}z}{z^{1/q}}\right)^{\Delta}(t) \leq r_{n-1}(t)\int_0^t r(t_n)\Delta t_n. \tag{3.14}$$

By setting $t = t_{n-1}$ in (3.14) and integrating with respect to t_{n-1} from 0 to t and using the fact that $L_{n-2}z(0) = 0$, we get

$$\frac{L_{n-2}z(t)}{z^{1/q}(t)} \leq \int_0^t r_{n-1}(t_{n-1})\int_0^{t_{n-1}} r(t_n)\Delta t_n \Delta t_{n-1}. \tag{3.15}$$

Continuing this way, we obtain that

$$\frac{L_1 z(t)}{z^{1/q}(t)} \leq \int_0^t r_2(t_2)\cdots\int_0^{t_{n-2}} r_{n-1}(t_{n-1})\int_0^{t_{n-1}} r(t_n)\Delta t_n \Delta t_{n-1}\cdots\Delta t_2, \tag{3.16}$$

that is,

$$\frac{z^{\Delta}(t)}{z^{1/q}(t)} \leq r_1(t)\int_0^t r_2(t_2)\cdots\int_0^{t_{n-2}} r_{n-1}(t_{n-1})\int_0^{t_{n-1}} r(t_n)\Delta t_n \Delta t_{n-1}\cdots\Delta t_2. \tag{3.17}$$

For $z^{\Delta}(t) \geq 0$, from the chain rule in [8, Theorem 1.90],

$$\begin{aligned}\left(\frac{1}{1-1/q}z^{-1/q+1}\right)^{\Delta} &= z^{\Delta}\int_0^1 \left(z + h\mu z^{\Delta}\right)^{-1/q} \mathrm{d}h \\ &= z^{-1/q}z^{\Delta}\int_0^1 \left(1 + h\mu\frac{z^{\Delta}}{z}\right)^{-1/q} \mathrm{d}h \\ &\leq z^{-1/q}z^{\Delta}.\end{aligned} \tag{3.18}$$

Letting $t = t_1$ in (3.17) and integrating with respect to t_1 from 0 to t, we have

$$\begin{aligned}
&\frac{q}{q-1}\left((z(t))^{(q-1)/q} - (z(0))^{(q-1)/q}\right) \\
&\quad \le \int_0^t \frac{z^{\Delta}(t_1)}{z^{1/q}(t_1)}\Delta t_1 \\
&\quad \le \int_0^t r_1(t_1)\int_0^{t_1} r_2(t_2)\cdots\int_0^{t_{n-2}} r_{n-1}(t_{n-1})\int_0^{t_{n-1}} r(t_n)\Delta t_n \Delta t_{n-1}\cdots\Delta t_2\Delta t_1,
\end{aligned} \tag{3.19}$$

which means that

$$F(t) \le z^{1/q}(t) \le \left\{c^{(q-1)/q} + \frac{q-1}{q}A[t, r_1, r_2, \ldots, r_{n-1}, r]\right\}^{1/(q-1)}. \tag{3.20}$$

This completes the proof. □

Remark 3.2. Theorem 3.1 also holds for $c = 0$. To show this, assume (3.4) holds for $c = 0$, that is,

$$F^q(t) \le A[t, r_1, \ldots, r_{n-1}, rF] \quad \forall t \in \mathbb{T}_0^+. \tag{3.21}$$

Now, let $d > 0$ be arbitrary. Then

$$F^q(t) \le d + A[t, r_1, \ldots, r_{n-1}, rF] \quad \forall t \in \mathbb{T}_0^+, \tag{3.22}$$

that is, (3.4) holds for $c = d$. By Theorem 3.1, (3.5) also holds for $c = d$, that is,

$$F(t) \le \left\{d^{(q-1)/q} + \frac{q-1}{q}A[t, r_1, \ldots, r_{n-1}, r]\right\}^{1/(q-1)} \quad \forall t \in \mathbb{T}_0^+. \tag{3.23}$$

Since (3.23) holds for arbitrary $d > 0$, we may let $d \to 0^+$ in (3.23) to arrive at

$$F(t) \le \left\{\frac{q-1}{q}A[t, r_1, \ldots, r_{n-1}, r]\right\}^{1/(q-1)} \quad \forall t \in \mathbb{T}_0^+, \tag{3.24}$$

that is, (3.5) holds for $c = 0$.

Theorem 3.3. *Let u, v, and h_j for $j = 1, 2, 3, 4$ be real-valued nonnegative rd-continuous functions on $t \in \mathbb{T}_0^+$ and let $q > 1$ be a constant. If c_1, c_2, and α are nonnegative constants such that*

$$u^q(t) \le c_1 + A[t, r_1, \ldots, r_{n-1}, h_1 u] + A[t, r_1, \ldots, r_{n-1}, h_2\overline{v}] \quad \forall t \in \mathbb{T}_0^+, \tag{3.25}$$

$$v^q(t) \le c_2 + A[t, r_1, \ldots, r_{n-1}, h_3\overline{u}] + A[t, r_1, \ldots, r_{n-1}, h_4 v] \quad \forall t \in \mathbb{T}_0^+, \tag{3.26}$$

where $\overline{u} = e^q_{\ominus\alpha}(\cdot,0)u$ *and* $\overline{v} = e^q_{\alpha}(\cdot,0)v$, *then for all* $t \in \mathbb{T}_0^+$,

$$\begin{aligned} u(t) &\le e_\alpha(t,0)\left\{[2^{q-1}(c_1+c_2)]^{(q-1)/q} + \frac{q-1}{q}A[t,r_1,\ldots,r_{n-1},2^{q-1}h]\right\}^{1/(q-1)}, \\ v(t) &\le \left\{[2^{q-1}(c_1+c_2)]^{(q-1)/q} + \frac{q-1}{q}A[t,r_1,\ldots,r_{n-1},2^{q-1}h]\right\}^{1/(q-1)}, \end{aligned} \tag{3.27}$$

where $h(t) = \max\{h_1(t)+h_3(t), h_2(t)+h_4(t)\}$.

Proof. Multiplying (3.25) by $e^q_{\ominus\alpha}(t,0)$ yields

$$\begin{aligned} e^q_{\ominus\alpha}(t,0)u^q(t) &\le c_1 e^q_{\ominus\alpha}(t,0) + A[t,r_1,\ldots,r_{n-1},h_1u]e^q_{\ominus\alpha}(t,0) + A[t,r_1,\ldots,r_{n-1},h_2\overline{v}]e^q_{\ominus\alpha}(t,0) \\ &\le c_1 + A[t,r_1,\ldots,r_{n-1},h_1\overline{u}] + A[t,r_1,\ldots,r_{n-1},h_2v]. \end{aligned} \tag{3.28}$$

Define

$$F(t) = e_{\ominus\alpha}(t,0)u(t) + v(t). \tag{3.29}$$

By taking the qth power on both sides of (3.29) and using the elementary inequality $(d_1 + d_2)^q \le 2^{q-1}(d_1^q + d_2^q)$, where d_1, d_2 are nonnegative reals, and also noticing (3.26) and $e_{\ominus\alpha}(t,0) \le 1$, we get

$$\begin{aligned} F^q(t) &\le 2^{q-1}\left[e^q_{\ominus\alpha}(t,0)u^q(t) + v^q(t)\right] \\ &\le 2^{q-1}\{c_1 + A[t,r_1,\ldots,r_{n-1},h_1\overline{u}] + A[t,r_1,\ldots,r_{n-1},h_2v] \\ &\qquad + c_2 + A[t,r_1,\ldots,r_{n-1},h_3\overline{u}] + A[t,r_1,\ldots,r_{n-1},h_4v]\} \\ &= 2^{q-1}\{(c_1+c_2) + A[t,r_1,\ldots,r_{n-1},(h_1+h_3)\overline{u}] + A[t,r_1,\ldots,r_{n-1},(h_2+h_4)v]\} \\ &\le 2^{q-1}(c_1+c_2) + A\left[t,r_1,\ldots,r_{n-1},2^{q-1}hF\right]. \end{aligned} \tag{3.30}$$

Now, Theorem 3.1 yields

$$F(t) \le \left\{[2^{q-1}(c_1+c_2)]^{(q-1)/q} + \frac{q-1}{q}A[t,r_1,\ldots,r_{n-1},2^{q-1}h]\right\}^{1/(q-1)}. \tag{3.31}$$

Noticing that (3.29) implies $v \le F$ and $u \le e_\alpha(\cdot,0)F$, the bounds in (3.27) follow, which concludes the proof. □

Theorem 3.4. *Let* $q > 1$ *and* B *be the set of all nonnegative real-valued rd-continuous functions defined on* $[0,t] \cap \mathbb{T}$. *Let* K *and* L *be monotone increasing linear operators on* B. *If there exists a* *positive constant c such that, for $y \in B$,*

$$y^q(t) \le c + \int_0^t \{qL[y^q](\tau) + K[y](\tau)\}\Delta\tau \quad \forall t \in \mathbb{T}_0^+, \tag{3.32}$$

then, for all $t \in \mathbb{T}_0^+$,

$$y(t) \le e_{q\overline{L}}^{1/q}(t,0)\left\{c^{(q-1)/q} + \frac{q-1}{q}\int_0^t (1+\mu(\tau)(\ominus(q\overline{L}))(\tau))\overline{K}(\tau)e_{q\overline{L}}^{1/q-1}(\tau,0)\Delta\tau\right\}^{1/(q-1)}, \tag{3.33}$$

where $\overline{L} = L[id]$, $\overline{K} = K[id]$ with $id(s) \equiv 1$ for all $s \in \mathbb{T}$.

Proof. Let

$$z(t) = c + \int_0^t \{qL[y^q](\tau) + K[y](\tau)\}\Delta\tau. \tag{3.34}$$

Hence, $z(s) \le z(t)$ for all $0 \le s \le t$, so that $z \le z(t)id$ on $[0,t]$, and thus

$$L[z] \le L[z(t)id] = z(t)L[id] = z(t)\overline{L}. \tag{3.35}$$

Hence $L[z](t) \le z(t)\overline{L}(t)$, and therefore $L[z] \le z\overline{L}$. Similarly, $K[z^{1/q}] \le z^{1/q}\overline{K}$. Using this and (3.32), we obtain that

$$z^\Delta = qL[y^q] + K[y] \le qL[z] + K\left[z^{1/q}\right] \le q\overline{L}z + \overline{K}z^{1/q}. \tag{3.36}$$

By the product rule [8, Theorem 1.20], we have

$$\begin{aligned}
\left(e_{\ominus(q\overline{L})}(\cdot,0)z\right)^\Delta &= \ominus\left(q\overline{L}\right)e_{\ominus(q\overline{L})}(\cdot,0)z + e^\sigma_{\ominus(q\overline{L})}(\cdot,0)z^\Delta \\
&= \ominus\left(q\overline{L}\right)e_{\ominus(q\overline{L})}(\cdot,0)z + \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)e_{\ominus(q\overline{L})}(\cdot,0)z^\Delta \\
&= e_{\ominus(q\overline{L})}(\cdot,0)\left(\ominus\left(q\overline{L}\right)z + \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)z^\Delta\right) \\
&\le e_{\ominus(q\overline{L})}(\cdot,0)\left(\ominus\left(q\overline{L}\right)z + \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)\left(q\overline{L}z + \overline{K}z^{1/q}\right)\right) \\
&= e_{\ominus(q\overline{L})}(\cdot,0)\left(\frac{-q\overline{L}}{1+\mu q\overline{L}}z + \left(1+\mu\frac{-q\overline{L}}{1+\mu q\overline{L}}\right)\left(q\overline{L}z + \overline{K}z^{1/q}\right)\right) \\
&= e_{\ominus(q\overline{L})}(\cdot,0)\left(\frac{-q\overline{L}}{1+\mu q\overline{L}}z + \frac{q\overline{L}}{1+\mu q\overline{L}}z + \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)\overline{K}z^{1/q}\right) \\
&= e_{\ominus(q\overline{L})}(\cdot,0)\left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)\overline{K}z^{1/q}.
\end{aligned} \tag{3.37}$$

In summary,

$$w^{\Delta} \leq \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)\overline{K}w^{1/q}e_{q\overline{L}}^{1/q-1}(\cdot,0), \quad \text{where } w=\frac{z}{e_{q\overline{L}}(\cdot,0)}. \tag{3.38}$$

Obviously

$$ww^{\sigma}>0, \quad \text{which implies } \frac{w^{\Delta}}{w}\in\mathcal{R}^{+}, \tag{3.39}$$

so that the chain rule [9, Theorem 2.37] yields

$$\left(\frac{1}{1-1/q}w^{-1/q+1}\right)^{\Delta}=w^{-1/q}w^{\Delta}\int_{0}^{1}\left(1+h\mu\frac{w^{\Delta}}{w}\right)^{-1/q}\mathrm{d}h \leq w^{-1/q}w^{\Delta}. \tag{3.40}$$

Dividing both sides of (3.38) by $w^{1/q}$ provides that

$$w^{-1/q}w^{\Delta} \leq \left(1+\mu\left(\ominus\left(q\overline{L}\right)\right)\right)\overline{K}e_{q\overline{L}}^{1/q-1}(\cdot,0). \tag{3.41}$$

Integrating both sides of (3.41) from 0 to t and noticing (3.40), we find that

$$\frac{q}{q-1}\left(w^{1-1/q}(t)-w^{1-1/q}(0)\right) \leq \int_{0}^{t}\left(1+\mu(\tau)\left(\ominus\left(q\overline{L}\right)\right)(\tau)\right)\overline{K}e_{q\overline{L}}^{1/q-1}(\tau,0)\Delta\tau. \tag{3.42}$$

Substitute the expression of $w(t)$, we have

$$\frac{z(t)}{e_{q\overline{L}}(t,0)} \leq \left\{c^{(q-1)/q}+\frac{q-1}{q}\int_{0}^{t}(1+\mu(\tau)(\ominus(q\overline{L}))(\tau))\overline{K}e_{q\overline{L}}^{1/q-1}(\tau,0)\Delta\tau\right\}^{q/(q-1)}, \tag{3.43}$$

which gives the desired inequality (3.32). This concludes the proof. □

Remark 3.5. As in the discussion in Remark 3.2, Theorem 3.4 also holds true for $c=0$.

4. Some Applications

In this section, we indicate some applications of our results to obtain the estimates of the solutions of certain integral equations for which inequalities obtained in the literature thus far do not apply directly. As an application of Theorem 2.2, we consider the second-order dynamic equation

$$y^{\Delta\Delta}+p^{\sigma}(t)\left(y+y^{\sigma}\right)=0. \tag{4.1}$$

Theorem 4.1. *Assume that p is a differentiable positive function such that p^{Δ} is rd-continuous. If there exist $t_0 \in \mathbb{T}$ and $M > 0$ such that*

$$\frac{1}{\sqrt{p(t)}} e_{|p^{\Delta}|/2p}(t, t_0) \leq M \quad \forall t \in \mathbb{T}_{t_0}^{+}, \tag{4.2}$$

then all nonoscillatory solutions of (4.1) *are bounded.*

Proof. Let y be a nonoscillatory solution of (4.1). Without loss of generality, we assume there exists $t_0 \in \mathbb{T}$ such that

$$y(t) > 0 \quad \forall t \in \mathbb{T}_{t_0}^{+}. \tag{4.3}$$

Then

$$y^{\Delta\Delta}(t) = -p^{\sigma}(t)\big(y(t) + y^{\sigma}(t)\big) < 0 \quad \forall t \in \mathbb{T}_{t_0}^{+}. \tag{4.4}$$

Hence, y^{Δ} is strictly decreasing on $\mathbb{T}_{t_0}^{+}$. Thus, either

$$y^{\Delta}(t) > 0 \quad \forall t \in \mathbb{T}_{t_0}^{+} \tag{4.5}$$

or there exists $t_1 \in \mathbb{T}_{t_0}^{+}$ such that

$$y^{\Delta}(t) < 0 \quad \forall t \in \mathbb{T}_{t_1}^{+}. \tag{4.6}$$

We now claim that (4.6) is impossible to hold. To show this, let us assume that (4.6) is true. Then y is strictly decreasing on $\mathbb{T}_{t_1}^{+}$ and

$$y(t) = y(t_1) + \int_{t_1}^{t} y^{\Delta}(\tau)\Delta\tau \leq y(t_1) + y^{\Delta}(t_1)(t - t_1) \quad \forall t \in \mathbb{T}_{t_1}^{+}. \tag{4.7}$$

Hence, there exists $t_2 \in \mathbb{T}_{t_1}^{+}$ such that

$$y(t) < 0 \quad \forall t \in \mathbb{T}_{t_2}^{+}, \tag{4.8}$$

contradicting $y(t) > 0$ for all $t \in \mathbb{T}_{t_0}^{+}$. Similarly, we can prove that if $y(t) < 0$, then $y^{\Delta\Delta}(t) > 0$ and $y^{\Delta}(t) \leq 0$ for $t \in \mathbb{T}_{t_1}^{+}$.

Multiplying (4.1) on both sides by y^{Δ} and taking integral from t_1 to t, we have

$$\int_{t_1}^{t} y^{\Delta}(\tau) y^{\Delta\Delta}(\tau)\Delta\tau + \int_{t_1}^{t} p^{\sigma}(\tau)\big(y(\tau) + y^{\sigma}(\tau)\big) y^{\Delta}(\tau)\Delta\tau = 0. \tag{4.9}$$

From the integration by parts in [8, Theorem 1.77],

$$\left(y^{\Delta}(t)\right)^2-\left(y^{\Delta}(t_1)\right)^2-\int_{t_1}^{t}y^{\Delta\Delta}(\tau)y^{\Delta\sigma}(\tau)\Delta\tau+p(t)y^2(t)-p(t_1)y^2(t_1)-\int_{t_1}^{t}p^{\Delta}(\tau)y^2(\tau)\Delta\tau=0, \tag{4.10}$$

Thus, with $c_1=p(t_1)y^2(t_1)+(y^{\Delta}(t_1))^2>0$, we have

$$\left(\sqrt{p(t)}y(t)\right)^2\le c_1+\int_{t_1}^{t}\frac{|p^{\Delta}(\tau)||y(\tau)|}{\sqrt{p(\tau)}}\sqrt{p(\tau)}|y(\tau)|\Delta\tau\quad\forall t\in\mathbb{T}_{t_1}^{+}. \tag{4.11}$$

Theorem 2.2 gives that

$$\left|\sqrt{p(t)}y(t)\right|\le\sqrt{c_1}+\frac{1}{2}\int_{t_1}^{t}\frac{|p^{\Delta}(\tau)||y(\tau)|}{\sqrt{p(\tau)}}\Delta\tau=\sqrt{c_1}+\int_{t_1}^{t}\frac{|p^{\Delta}(\tau)|}{2p(\tau)}\left|\sqrt{p(\tau)}y(\tau)\right|\Delta\tau\quad\forall t\in\mathbb{T}_{t_1}^{+}. \tag{4.12}$$

Applying Gronwall's inequality from Lemma 2.1 yields

$$\left|\sqrt{p(t)}y(t)\right|\le\sqrt{c_1}e_{|p^{\Delta}|/2p}(t,t_1)\quad\forall t\in\mathbb{T}_{t_1}^{+}. \tag{4.13}$$

Hence,

$$|y(t)|\le\sqrt{c_1}\frac{1}{\sqrt{p(t)}}e_{|p^{\Delta}|/2p}(t,t_1)\le\sqrt{c_1}M\quad\forall t\in\mathbb{T}_{t_1}^{+}, \tag{4.14}$$

which completes the proof. □

The proof in Theorem 4.1 corrects an inaccuracy in the proof of [1, Theorem 1]. We can also obtain the following results.

Corollary 4.2. *Let $\mathbb{T}=\mathbb{R}$. If p is a continuously differentiable positive function such that p' is nonnegative, then all nonoscillatory solutions of* (4.1) *are bounded.*

Proof. For $\mathbb{T} = \mathbb{R}$, we have

$$\begin{aligned} \frac{1}{\sqrt{p(t)}} e_{|p^{\Delta}|/2p}(t,0) &= \frac{1}{\sqrt{p(t)}} e^{\int_0^t (p'(\tau)/2p(\tau))\mathrm{d}\tau} \\ &= \frac{1}{\sqrt{p(t)}} e^{(1/2)\ln(p(t)/p(0))} \\ &= \frac{1}{\sqrt{p(t)}} \left(\frac{p(t)}{p(0)}\right)^{1/2} \\ &= \frac{1}{\sqrt{p(0)}}, \end{aligned} \tag{4.15}$$

and hence the statement follows from Theorem 4.1. □

Example 4.3. Consider the nonlinear one-dimensional integral equation of the form

$$u^q(t) = f(t) + \int_0^t k(t,s) g(s,u(s))\Delta s, \tag{4.16}$$

where $f : \mathbb{T}_0^+ \to \mathbb{R}$, $k : \mathbb{T}_0^+ \times \mathbb{T}_0^+ \to \mathbb{R}$, $g : \mathbb{T}_0^+ \times \mathbb{R} \to \mathbb{R}$ are rd-continuous functions, and $q > 1$ is a constant. When $\mathbb{T} = \mathbb{R}$, its physical meaning is to model the water percolation phenomena, and Okrasiński has studied the existence and uniqueness of solutions [14].

Here, we assume that every solution u of (4.16) exists on the interval $\mathbb{T}_0^+$. We suppose that the functions f, k, g in (4.16) satisfy the conditions

$$|f(t)| \le c_1, \qquad |k(t,s)| \le c_2, \qquad |g(t,u)| \le r(t)|u|, \tag{4.17}$$

where c_1, c_2 are nonnegative constants and $r : [0,\infty) \cap \mathbb{T} \to \mathbb{R}_+$ is an rd-continuous function. From (4.16) and using (4.17), it is easy to observe that

$$|u(t)|^q \le c_1 + \int_0^t c_2 r(s)|u(s)|\Delta s. \tag{4.18}$$

Now an application of Theorem 3.1 with $n = 1$ gives

$$|u(t)| \le \left\{ c_1^{(q-1)/q} + \frac{q-1}{q} \int_0^t c_2 r(s)\Delta s \right\}^{1/(q-1)}, \tag{4.19}$$

which gives the bound on u.

Now, we consider (4.16) under the conditions

$$|f(t)| \le c_1 e^q_{\ominus\alpha}(t,0), \qquad |k(t,s)| \le h(s) e^q_{\ominus\alpha}(t,0), \qquad |g(t,u)| \le r(t)|u|, \tag{4.20}$$

where c_1 and r are as above, $\alpha > 0$ is a constant, $h : \mathbb{T}_0^+ \to \mathbb{R}_+$ is an rd-continuous function, and

$$\int_0^\infty h(s)r(s)e_{\ominus\alpha}(s,0)\Delta s < \infty. \tag{4.21}$$

From (4.16) and (4.20), it is easy to observe that

$$|e_\alpha(t,0)u(t)|^q \le c_1 + \int_0^t h(s)r(s)e_{\ominus\alpha}(s,0)|e_\alpha(s,0)u(s)|\Delta s. \tag{4.22}$$

Applying Theorem 3.1 with $n = 1$ yields

$$e_\alpha(t,0)|u(t)| \le \left\{c_1^{(q-1)/q} + \frac{q-1}{q}\int_0^t h(s)r(s)e_{\ominus\alpha}(s,0)\Delta s\right\}^{1/(q-1)}. \tag{4.23}$$

So,

$$|u(t)| \le c^* e_{\ominus\alpha}(t,0), \quad \text{where } c^* = c_1^{(q-1)/q} + \frac{q-1}{q}\int_0^\infty h(s)r(s)e_{\ominus\alpha}(s,0)\Delta s > 0. \tag{4.24}$$

From (4.24), we see that the solution $u(t)$ of (4.16) approaches zero as $t \to \infty$.

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