Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 264347, 23 pages
doi:10.1155/2010/264347

Research Article

On an Inequality of H. G. Hardy

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Received 18 June 2010; Accepted 16 October 2010

Academic Editor: Q. Lan

We state, prove, and discuss new general inequality for convex and increasing functions. As a special case of that general result, we obtain new fractional inequalities involving fractional integrals and derivatives of Riemann-Liouville type. Consequently, we get the inequality of H. G. Hardy from 1918. We also obtain new results involving fractional derivatives of Canavati and Caputo types as well as fractional integrals of a function with respect to another function. Finally, we apply our main result to multidimensional settings to obtain new results involving mixed Riemann-Liouville fractional integrals.

1. Introduction

First, let us recall some facts about fractional derivatives needed in the sequel, for more details see, for example, [1, 2].

Let $0 < a < b \le \infty$. By $C^m([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$, we denote the space of all functions $g \in C^m([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \le \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \ge \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the *Riemann-Liouville fractional integrals*, see [3]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a_+}^\alpha f$ and $I_{b_-}^\alpha f$ of order $\alpha > 0$ are defined by

$$\left(I_{a_+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1.1}$$

$$\left(I_{b_-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b), \tag{1.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We denote some properties of the operators $I_{a_+}^{\alpha} f$ and $I_{b_-}^{\alpha} f$ of order $\alpha > 0$, see also [4]. The first result yields that the fractional integral operators $I_{a_+}^{\alpha} f$ and $I_{b_-}^{\alpha} f$ are bounded in $L_p(a,b)$, $1 \le p \le \infty$, that is

$$\|I_{a_+}^{\alpha} f\|_p \le K \|f\|_p, \qquad \|I_{b_-}^{\alpha} f\|_p \le K \|f\|_p, \tag{1.3}$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{1.4}$$

Inequality (1.3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [5]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable, and expressions of the form $0 \cdot \infty$, ∞/∞, and $0/0$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$, we mean a nonnegative measurable function on the actual interval or more general set.

The paper is organized in the following way. After this Introduction, in Section 2 we state, prove, and discuss new general inequality for convex and increasing functions. As a special case of that general result, we obtain new fractional inequalities involving fractional integrals and derivatives of Riemann-Liouville type. Consequently, we get the inequality of H. G. Hardy since 1918. We also obtain new results involving fractional derivatives of Canavati and Caputo types as well as fractional integrals of a function with respect to another function. We conclude this paper with new results involving mixed Riemann-Liouville fractional integrals.

2. The Main Results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative function, and

$$K(x) = \int_{\Omega_2} k(x,y) d\mu_2(y), \quad x \in \Omega_1. \tag{2.1}$$

Throughout this paper, we suppose that $K(x) > 0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let $U(k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{2.2}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

Our first result is given in the following theorem.

Theorem 2.1. *Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (2.1). Assume that the function $x \mapsto u(x)(k(x,y)/K(x))$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x) < \infty. \tag{2.3}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\phi\left(\left|\frac{g(x)}{K(x)}\right|\right)d\mu_1(x) \leq \int_{\Omega_2} v(y)\phi(|f(y)|)d\mu_2(y) \tag{2.4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ and for all functions $g \in U(k)$.

Proof. By using Jensen's inequality and the Fubini theorem, since ϕ is increasing function, we find that

$$\begin{aligned}
\int_{\Omega_1} u(x)\phi\left(\left|\frac{g(x)}{K(x)}\right|\right)d\mu_1(x) &= \int_{\Omega_1} u(x)\phi\left(\left|\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right|\right)d\mu_1(x) \\
&\leq \int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\phi(|f(y)|)d\mu_2(y)\right)d\mu_1(x) \\
&= \int_{\Omega_2} \phi(|f(y)|)\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)\right)d\mu_2(y) \\
&= \int_{\Omega_2} v(y)\phi(|f(y)|)d\mu_2(y),
\end{aligned} \tag{2.5}$$

and the proof is complete. □

As a special case of Theorem 2.1, we get the following result.

Corollary 2.2. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a_+}f$ denotes the Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \alpha\int_y^b u(x)\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}}dx < \infty. \tag{2.6}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left|I^{\alpha}_{a_+}f(x)\right|\right)dx \leq \int_a^b v(y)\phi(|f(y)|)dy \tag{2.7}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a \le y \le x, \\ 0, & x < y \le b, \end{cases} \tag{2.8}$$

we get that $K(x) = (x-a)^{\alpha}/\Gamma(\alpha+1)$ and $g(x) = I^{\alpha}_{a_+} f(x)$, so (2.7) follows. □

Remark 2.3. In particular for the weight function $u(x) = (x-a)^{\alpha}$, $x \in (a,b)$ in Corollary 2.2, we obtain the inequality

$$\int_a^b (x-a)^{\alpha} \phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} \left|I^{\alpha}_{a_+} f(x)\right|\right) dx \le \int_a^b (b-y)^{\alpha} \phi(|f(y)|) dy. \tag{2.9}$$

Although (2.4) holds for all convex and increasing functions, some choices of ϕ are of particular interest. Namely, we will consider power function. Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (2.9) reduces to

$$\int_a^b (x-a)^{\alpha} \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} \left|I^{\alpha}_{a_+} f(x)\right|\right)^q dx \le \int_a^b (b-y)^{\alpha} |f(y)|^q dy. \tag{2.10}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then we obtain that the left hand side of (2.10) is

$$\int_a^b (x-a)^{\alpha} \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} |I^{\alpha}_{a_+} f(x)|\right)^q dx \ge (b-a)^{\alpha(1-q)} (\Gamma(\alpha+1))^q \int_a^b \left|I^{\alpha}_{a_+} f(x)\right|^q dx \tag{2.11}$$

and the right-hand side of (2.10) is

$$\int_a^b (b-y)^{\alpha} |f(y)|^q dy \le (b-a)^{\alpha} \int_a^b |f(y)|^q dy. \tag{2.12}$$

Combining (2.11) and (2.12), we get

$$\int_a^b |I^{\alpha}_{a_+} f(x)|^q dx \le \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b |f(y)|^q dy. \tag{2.13}$$

Taking power $1/q$ on both sides, we obtain (1.3).

Corollary 2.4. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{b_-} f$ denotes the Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \alpha \int_a^y u(x) \frac{(y-x)^{\alpha-1}}{(b-x)^{\alpha}} dx < \infty. \tag{2.14}$$

If $\phi : (0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left|I_{b_-}^\alpha f(x)\right|\right)dx \leq \int_a^b v(y)\phi(|f(y)|)dy \tag{2.15}$$

holds.

Proof. Similar to the proof of Corollary 2.2. □

Remark 2.5. In particular for the weight function $u(x) = (b-x)^\alpha,\ x \in (a,b)$ in Corollary 2.4, we obtain the inequality

$$\int_a^b (b-x)^\alpha\phi\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left|I_{b_-}^\alpha f(x)\right|\right)dx \leq \int_a^b (y-a)^\alpha\phi(|f(y)|)dy. \tag{2.16}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (2.16) reduces to

$$\int_a^b (b-x)^\alpha\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left|I_{b_-}^\alpha f(x)\right|\right)^q dx \leq \int_a^b (y-a)^\alpha|f(y)|^q dy. \tag{2.17}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then we obtain that the left hand side of (2.17) is

$$\int_a^b (b-x)^\alpha\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left|I_{b_-}^\alpha f(x)\right|\right)^q dx \geq (b-a)^{\alpha(1-q)}(\Gamma(\alpha+1))^q\int_a^b \left|I_{b_-}^\alpha f(x)\right|^q dx \tag{2.18}$$

and the right-hand side of (2.17) is

$$\int_a^b (y-a)^\alpha|f(y)|^q dy \leq (b-a)^\alpha\int_a^b |f(y)|^q dy. \tag{2.19}$$

Combining (2.18) and (2.19), we get

$$\int_a^b \left|I_{b_-}^\alpha f(x)\right|^q dx \leq \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)^q\int_a^b |f(y)|^q dy. \tag{2.20}$$

Taking power $1/q$ on both sides, we obtain (1.3).

Theorem 2.6. *Let $p, q > 1$, $1/p + 1/q = 1$, $\alpha > 1/q$, $I_{a_+}^\alpha f$ and $I_{b_-}^\alpha f$ denote the Riemann-Liouville fractional integral of f, then the following inequalities*

$$\int_a^b |I_{a_+}^\alpha f(x)|^q dx \leq C\int_a^b |f(y)|^q dy, \tag{2.21}$$

$$\int_a^b \left|I_{b_-}^\alpha f(x)\right|^q dx \leq C\int_a^b |f(y)|^q dy \tag{2.22}$$

hold, where $C = (b-a)^{q\alpha}/(\Gamma(\alpha))^q q\alpha(p(\alpha-1)+1)^{q-1}$.

Proof. We will prove only inequality (2.21), since the proof of (2.22) is analogous. We have

$$\left|(I_{a_+}^{\alpha} f)(x)\right| \leq \frac{1}{\Gamma(\alpha)} \int_a^x \left|f(t)\right| (x-t)^{\alpha-1} dt. \tag{2.23}$$

Then by the Hölder inequality, the right-hand side of the above inequality is

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{p(\alpha-1)} dt\right)^{1/p} \left(\int_a^x |f(t)|^q dt\right)^{1/q} \\
&= \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{(\alpha-1)+1/p}}{(p(\alpha-1)+1)^{1/p}} \left(\int_a^x |f(t)|^q dt\right)^{1/q} \\
&\leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{(\alpha-1)+1/p}}{(p(\alpha-1)+1)^{1/p}} \left(\int_a^b |f(t)|^q dt\right)^{1/q}.
\end{aligned} \tag{2.24}$$

Thus, we have

$$\left|(I_{a_+}^{\alpha} f)(x)\right| \leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{(\alpha-1)+1/p}}{(p(\alpha-1)+1)^{1/p}} \left(\int_a^b |f(t)|^q dt\right)^{1/q}, \quad \text{for every } x \in [a,b]. \tag{2.25}$$

Consequently, we find

$$\left|(I_{a_+}^{\alpha} f)(x)\right|^q \leq \frac{1}{(\Gamma(\alpha))^q} \frac{(x-a)^{q(\alpha-1)+q/p}}{(p(\alpha-1)+1)^{q/p}} \left(\int_a^b \left|f(t)\right|^q dt\right), \tag{2.26}$$

and we obtain

$$\int_a^b \left|I_{a_+}^{\alpha} f(x)\right|^q dx \leq \frac{(b-a)^{q(\alpha-1)+q/p+1}}{(\Gamma(\alpha))^q (q(\alpha-1)+q/p+1)(p(\alpha-1)+1)^{q/p}} \int_a^b \left|f(t)\right|^q dt. \tag{2.27}$$

□

Remark 2.7. For $\alpha \geq 1$, inequalities (2.21) and (2.22) are refinements of (1.3) since

$$q\alpha(p(\alpha-1)+1)^{q-1} \geq q\alpha^q > \alpha^q, \quad \text{so } C < \left(\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}\right)^q. \tag{2.28}$$

We proved that Theorem 2.6 is a refinement of (1.3), and Corollaries 2.2 and 2.4 are generalizations of (1.3).

Next, we give results with respect to the *generalized Riemann-Liouville fractional derivative*. Let us recall the definition, for details see [1, page 448].

We define the generalized Riemann-Liouville fractional derivative of f of order $\alpha > 0$ by

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\,dy, \tag{2.29}$$

where $n = [\alpha] + 1,\ x \in [a,b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\alpha} f$ $(\alpha > 0)$ in $[a,b]$, if and only if

(1) $D_a^{\alpha-k} f \in C([a,b])$, $k = 1, \dots, n = [\alpha] + 1$,

(2) $D_a^{\alpha-1} f \in AC([a,b])$,

(3) $D_a^{\alpha} \in L_\infty(a,b)$.

Next, lemma is very useful in the upcoming corollary (see [1, page 449] and [2]).

Lemma 2.8. *Let $\beta > \alpha \geq 0$ and let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a,b]$ and let*

$$D_a^{\beta-k} f(a) = 0, \quad k = 1, \dots, [\beta] + 1, \tag{2.30}$$

then

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\,dy, \tag{2.31}$$

for all $a \leq x \leq b$.

Corollary 2.9. *Let u be a weight function on (a,b), and let assumptions in Lemma 2.8 be satisfied. Define v on (a,b) by*

$$v(y) := (\beta-\alpha) \int_y^b u(x) \frac{(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}}\,dx < \infty. \tag{2.32}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} \left|D_a^{\alpha} f(x)\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|D_a^{\beta} f(y)\right|\right) dy \tag{2.33}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq y \leq x, \\ 0, & x < y \leq b, \end{cases} \tag{2.34}$$

we get that $K(x) = (x-a)^{\beta-\alpha}/\Gamma(\beta-\alpha+1)$. Replace f by $D_a^{\beta} f$. Then, by Lemma 2.8, $g(x) = (D_a^{\alpha} f)(x)$ and we get (2.33). □

Remark 2.10. In particular for the weight function $u(x) = (x-a)^{\beta-\alpha}$, $x \in (a,b)$ in Corollary 2.9, we obtain the inequality

$$\int_a^b (x-a)^{\beta-\alpha} \phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} \left|D_a^{\alpha} f(x)\right|\right) dx \le \int_a^b (b-y)^{\beta-\alpha} \phi\left(\left|D_a^{\beta} f(y)\right|\right) dy. \tag{2.35}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then after some calculation, we obtain

$$\int_a^b \left|D_a^{\alpha} f(x)\right|^q dx \le \left(\frac{(b-a)^{(\beta-\alpha)}}{\Gamma(\beta-\alpha+1)}\right)^q \int_a^b \left|D_a^{\beta} f(y)\right|^q dy. \tag{2.36}$$

Next, we define *Canavati-type fractional derivative* (ν*-fractional derivative of* f), for details see [1, page 446]. We consider

$$C^{\nu}([a,b]) = \left\{f \in C^n([a,b]) : I_{a+}^{n-\nu+1} f^{(n)} \in C^1([a,b])\right\}, \tag{2.37}$$

$\nu > 0$, $n = [\nu]$. Let $f \in C^{\nu}([a,b])$. We define the generalized ν-fractional derivative of f over $[a,b]$ as

$$D_a^{\nu} f = \left(I_{a+}^{n-\nu+1} f^{(n)}\right)', \tag{2.38}$$

the derivative with respect to x.

Lemma 2.11. *Let $\nu \ge \gamma+1$, where $\gamma \ge 0$ and $f \in C^{\nu}([a,b])$. Assume that $f^{(i)}(a) = 0$, $i = 0,1,\ldots,[\nu]-1$, then*

$$\left(D_a^{\gamma} f\right)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^{\nu} f)(t) dt, \tag{2.39}$$

for all $x \in [a,b]$.

Corollary 2.12. *Let u be a weight function on (a,b), and let assumptions in Lemma 2.11 be satisfied. Define v on (a,b) by*

$$v(y) := (\nu-\gamma) \int_y^b u(x) \frac{(x-y)^{\nu-\gamma-1}}{(x-x_0)^{\nu-\gamma}} dx < \infty. \tag{2.40}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}\left|D_a^{\gamma}f(x)\right|\right)dx \le \int_a^b v(y)\phi\left(\left|D_a^{\nu}f(y)\right|\right)dy \tag{2.41}$$

holds.

Proof. Similar to the proof of Corollary 2.9. □

Remark 2.13. In particular for the weight function $u(x) = (x-a)^{\nu-\gamma}$, $x \in (a,b)$ in Corollary 2.12, we obtain the inequality

$$\int_a^b (x-a)^{\nu-\gamma}\phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}\left|D_a^{\gamma}f(x)\right|\right)dx \le \int_a^b (b-y)^{\nu-\gamma}\phi\left(\left|D_a^{\nu}f(y)\right|\right)dy. \tag{2.42}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (2.42) reduces to

$$(\Gamma(\nu-\gamma+1))^q\int_a^b (x-a)^{(\nu-\gamma)(1-q)}\left|D_a^{\gamma}f(x)\right|^q dx \le \int_a^b (b-y)^{\nu-\gamma}\left|D_a^{\nu}f(y)\right|^q dy. \tag{2.43}$$

Since $x \in [a,b]$ and $(\nu-\gamma)(1-q) \le 0$, then we obtain

$$\int_a^b \left|D_a^{\gamma}f(x)\right|^q dx \le \left(\frac{(b-a)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right)^q \int_a^b \left|D_a^{\nu}f(y)\right|^q dy. \tag{2.44}$$

Taking power $1/q$ on both sides of (2.44), we obtain

$$\|D_a^{\gamma}f(x)\|_q \le \frac{(b-a)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\|D_a^{\nu}f(y)\|_q. \tag{2.45}$$

When $\gamma = 0$, we find that

$$(\Gamma(\nu+1))^q\int_a^b (x-a)^{\nu(1-q)}|f(x)|^q dx \le \int_a^b (b-y)^{\nu}\left|D_a^{\nu}f(y)\right|^q dy, \tag{2.46}$$

that is,

$$\|f\|_q \le \frac{(b-a)^{\nu}}{\Gamma(\nu+1)}\|D_a^{\nu}f(y)\|_q. \tag{2.47}$$

In the next corollary, we give results with respect to the *Caputo fractional derivative*. Let us recall the definition, for details see [1, page 449].

Let $\alpha \geq 0$, $n = \lceil \alpha \rceil$, $g \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}} dy, \tag{2.48}$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

Corollary 2.14. *Let u be a weight function on (a,b) and $\alpha > 0$. $D_{*a}^{\alpha} g$ denotes the Caputo fractional derivative of g. Define v on (a,b) by*

$$v(y) := (n-\alpha) \int_y^b u(x) \frac{(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}} dx < \infty. \tag{2.49}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x) \phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} \left| D_{*a}^{\alpha} g(x) \right| \right) dx \leq \int_a^b v(y) \phi\left(\left| g^{(n)}(y) \right| \right) dy \tag{2.50}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a \leq y \leq x, \\ 0, & x < y \leq b, \end{cases} \tag{2.51}$$

we get that $K(x) = (x-a)^{n-\alpha}/\Gamma(n-\alpha+1)$. Replace f by $g^{(n)}$, so g becomes $D_{*a}^{\alpha} g$ and (2.50) follows. □

Remark 2.15. In particular for the weight function $u(x) = (x-a)^{n-\alpha}$, $x \in (a,b)$ in Corollary 2.14, we obtain the inequality

$$\int_a^b (x-a)^{n-\alpha} \phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} \left| D_{*a}^{\alpha} g(x) \right| \right) dx \leq \int_a^b (b-y)^{n-\alpha} \phi\left(\left| g^{(n)}(y) \right| \right) dy. \tag{2.52}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then after some calculation, we obtain

$$\int_a^b \left| D_{*a}^{\alpha} g(x) \right|^q dx \leq \left(\frac{(b-a)^{(n-\alpha)}}{\Gamma(n-\alpha+1)} \right)^q \int_a^b \left| g^{(n)}(y) \right|^q dy. \tag{2.53}$$

Taking power $1/q$ on both sides, we obtain

$$\| D_{*a}^{\alpha} g(x) \|_q \leq \frac{(b-a)^{(n-\alpha)}}{\Gamma(n-\alpha+1)} \| g^{(n)}(y) \|_q. \tag{2.54}$$

Theorem 2.16. *Let $p, q > 1$, $1/p + 1/q = 1$, $n - \alpha > 1/q$, $D^{\alpha}_{*a}f(x)$ denotes the Caputo fractional derivative of f, then the following inequality*

$$\int_a^b \left|D^{\alpha}_{*a}f(x)\right|^q dx \leq \frac{(b-a)^{q(n-\alpha)}}{(\Gamma(n-\alpha))^q (p(n-\alpha-1)+1)^{q/p} q(n-\alpha)} \int_a^b \left|f^{(n)}(y)\right|^q dy \tag{2.55}$$

holds.

Proof. Similar to the proof of Theorem 2.6. □

The following result is given [1, page 450].

Lemma 2.17. *Let $\alpha \geq \gamma + 1$, $\gamma > 0$, and $n = [\alpha]$. Assume that $f \in AC^n([a,b])$ such that $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, and $D^{\alpha}_{*a}f \in L_\infty(a,b)$, then $D^{\gamma}_{*a}f \in C([a,b])$, and*

$$D^{\gamma}_{*a}f(x) = \frac{1}{\Gamma(\alpha-\gamma)} \int_a^x (x-y)^{\alpha-\gamma-1} D^{\alpha}_{*a}f(y)\,dy, \tag{2.56}$$

for all $a \leq x \leq b$.

Corollary 2.18. *Let u be a weight function on (a,b) and $\alpha > 0$. $D^{\alpha}_{*a}f$ denotes the Caputo fractional derivative of f, and assumptions in Lemma 2.17 are satisfied. Define v on (a,b) by*

$$v(y) := (\alpha-\gamma) \int_y^b u(x) \frac{(x-y)^{\alpha-\gamma-1}}{(x-a)^{\alpha-\gamma}} dx < \infty. \tag{2.57}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha-\gamma+1)}{(x-a)^{\alpha-\gamma}} \left|D^{\gamma}_{*a}f(x)\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|D^{\alpha}_{*a}f(y)\right|\right) dy \tag{2.58}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{(x-y)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}, & a \leq y \leq x, \\ 0, & x < y \leq b, \end{cases} \tag{2.59}$$

we get that $K(x) = (x-a)^{\alpha-\gamma}/\Gamma(\alpha-\gamma+1)$. Replace f by $D^{\alpha}_{*a}f$, so g becomes $D^{\gamma}_{*a}f$ and (2.58) follows. □

Remark 2.19. In particular for the weight function $u(x) = (x-a)^{\alpha-\gamma}$, $x \in (a,b)$ in Corollary 2.18, we obtain the inequality

$$\int_a^b (x-a)^{\alpha-\gamma} \phi\left(\frac{\Gamma(\alpha-\gamma+1)}{(x-a)^{\alpha-\gamma}} \left|D_{*a}^{\gamma} f(x)\right|\right) dx \le \int_a^b (b-y)^{\alpha-\gamma} \phi\left(\left|D_{*a}^{\alpha} f(y)\right|\right) dy. \tag{2.60}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then after some calculation, we obtain

$$\int_a^b \left|D_{*a}^{\gamma} f(x)\right|^q dx \le \left(\frac{(b-a)^{(\alpha-\gamma)}}{\Gamma(\alpha-\gamma+1)}\right)^q \int_a^b \left|D_{*a}^{\alpha} f(y)\right|^q dy. \tag{2.61}$$

For $\gamma = 0$, we obtain

$$\int_a^b |f(x)|^q dx \le \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b \left|D_{*a}^{\alpha} f(y)\right|^q dy. \tag{2.62}$$

We continue with definitions and some properties of the *fractional integrals of a function f with respect to given function g*. For details see, for example, [3, page 99].

Let (a,b), $-\infty \le a < b \le \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a,b]$ and g' a continuous function on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x > a, \tag{2.63}$$

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(t)-g(x)]^{1-\alpha}}, \quad x < b, \tag{2.64}$$

respectively.

Corollary 2.20. *Let u be a weight function on (a,b), and let g be an increasing function on $(a,b]$, such that g' is a continuous function on (a,b) and $\alpha > 0$. $I_{a_+;g}^{\alpha} f$ denotes the left-sided fractional integral of a function f with respect to another function g in $[a,b]$. Define v on (a,b) by*

$$v(y) := \alpha g'(y) \int_y^b u(x) \frac{(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} dx < \infty. \tag{2.65}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x) \phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} \left|I_{a_+;g}^{\alpha} f(x)\right|\right) dx \le \int_a^b v(y) \phi\left(\left|f(y)\right|\right) dy \tag{2.66}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \dfrac{g'(y)}{(g(x)-g(y))^{1-\alpha}}, & a \le y \le x, \\ 0, & x < y \le b, \end{cases} \tag{2.67}$$

we get that $K(x) = (1/\Gamma(\alpha+1))(g(x)-g(a))^{\alpha}$, so (2.66) follows. □

Remark 2.21. In particular for the weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, $x \in (a,b)$ in Corollary 2.20, we obtain the inequality

$$\begin{aligned} &\int_a^b g'(x)(g(x)-g(a))^{\alpha} \phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} \left|I^{\alpha}_{a_+;g} f(x)\right|\right) dx \\ &\quad\le \int_a^b g'(y)(g(b)-g(y))^{\alpha} \phi(|f(y)|) dy. \end{aligned} \tag{2.68}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (2.68) reduces to

$$\begin{aligned} &(\Gamma(\alpha+1))^q \int_a^b g'(x)(g(x)-g(a))^{\alpha(1-q)} \left|I^{\alpha}_{a_+;g} f(x)\right|^q dx \\ &\quad\le \int_a^b g'(y)(g(b)-g(y))^{\alpha} |f(y)|^q dy. \end{aligned} \tag{2.69}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, g is increasing, then $(g(x)-g(a))^{\alpha(1-q)} > (g(b)-g(a))^{\alpha(1-q)}$ and $(g(b)-g(y))^{\alpha} < (g(b)-g(a))^{\alpha}$ and we obtain

$$\int_a^b g'(x)\left|I^{\alpha}_{a_+;g} f(x)\right|^q dx \le \left(\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b g'(y)|f(y)|^q dy. \tag{2.70}$$

Remark 2.22. If $g(x) = x$, then $I^{\alpha}_{a_+;x} f(x)$ reduces to $I^{\alpha}_{a_+} f(x)$ Riemann-Liouville fractional integral and (2.70) becomes (2.13).

Analogous to Corollary 2.20, we obtain the following result.

Corollary 2.23. *Let u be a weight function on (a,b), and let g be an increasing function on $(a,b]$, such that g' is a continuous function on (a,b) and $\alpha > 0$. $I^{\alpha}_{b_-;g} f$ denotes the right-sided fractional integral of a function f with respect to another function g in $[a,b]$. Define v on (a,b) by*

$$v(y) := \alpha g'(y) \int_a^y u(x) \frac{(g(y)-g(x))^{\alpha-1}}{(g(b)-g(x))^{\alpha}} dx < \infty. \tag{2.71}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{(g(b)-g(x))^{\alpha}}\left|I^{\alpha}_{b_-;g}f(x)\right|\right)dx \le \int_a^b v(y)\phi(|f(y)|)dy \tag{2.72}$$

holds.

Remark 2.24. In particular for the weight function $u(x) = g'(x)(g(b)-g(x))^{\alpha}$, $x \in (a,b)$ and for function $\phi(x) = x^q$, $q > 1$, we obtain after some calculation

$$\int_a^b g'(x)\left|I^{\alpha}_{b_-;g}f(x)\right|^q dx \le \left(\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b g'(y)|f(y)|^q dy. \tag{2.73}$$

Remark 2.25. If $g(x) = x$, then $I^{\alpha}_{b_-;x}f(x)$ reduces to $I^{\alpha}_{b_-}f(x)$ Riemann-Liouville fractional integral and (2.73) becomes (2.20).

The refinements of (2.70) and (2.73) for $\alpha > 1/q$ are given in the following theorem.

Theorem 2.26. *Let $p,q > 1$, $1/p + 1/q = 1$, $\alpha > 1/q$, $I^{\alpha}_{a_+;g}f$ and $I^{\alpha}_{b_-;g}f$ denote the left-sided and right-sided fractional integral of a function f with respect to another function g in $[a,b]$, then the following inequalities:*

$$\begin{aligned}
\int_a^b \left|I^{\alpha}_{a_+;g}f(x)\right|^q g'(x)dx &\le \frac{(g(b)-g(a))^{\alpha q}}{\alpha q(\Gamma(\alpha))^q(p(\alpha-1)+1)^{q/p}}\int_a^b |f(y)|^q g'(y)dy, \\
\int_a^b \left|I^{\alpha}_{b_-;g}f(x)\right|^q g'(x)dx &\le \frac{(g(b)-g(a))^{\alpha q}}{\alpha q(\Gamma(\alpha))^q(p(\alpha-1)+1)^{q/p}}\int_a^b |f(y)|^q g'(y)dy
\end{aligned} \tag{2.74}$$

hold.

We continue by defining *Hadamard type fractional integrals*.

Let (a,b), $0 \le a < b \le \infty$ be a finite or infinite interval of the half-axis $\mathbb{R}_+$ and $\alpha > 0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$(J^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y}, \quad x > a, \tag{2.75}$$

$$(J^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\log\frac{y}{x}\right)^{\alpha-1}\frac{f(y)dy}{y}, \quad x < b, \tag{2.76}$$

respectively.

Notice that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$ in $[a,b]$, where $0 \le a < b \le \infty$, so (2.70) reduces to

$$\int_a^b \left|(J_{a+}^{\alpha} f)(x)\right|^q \frac{dx}{x} \le \left(\frac{(\log(b/a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b |f(y)|^q \frac{dy}{y}, \tag{2.77}$$

and (2.73) becomes

$$\int_a^b \left|(J_{b-}^{\alpha} f)(x)\right|^q \frac{dx}{x} \le \left(\frac{(\log(b/a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b |f(y)|^q \frac{dy}{y}. \tag{2.78}$$

Also, from Theorem 2.26 we obtain refinements of (2.77) and (2.78), for $\alpha > 1/q$,

$$\begin{aligned} \int_a^b \left|(J_{a+}^{\alpha} f)(x)\right|^q \frac{dx}{x} &\le \frac{(\log(b/a))^{q\alpha}}{q\alpha(\Gamma(\alpha))^q (p(\alpha-1)+1)^{q/p}} \int_a^b |f(y)|^q \frac{dy}{y}, \\ \int_a^b \left|(J_{b-}^{\alpha} f)(x)\right|^q \frac{dx}{x} &\le \frac{(\log(b/a))^{q\alpha}}{q\alpha(\Gamma(\alpha))^q (p(\alpha-1)+1)^{q/p}} \int_a^b |f(y)|^q \frac{dy}{y}. \end{aligned} \tag{2.79}$$

Some results involving Hadamard type fractional integrals are given in [3, page 110]. Here, we mention the following result that can not be compared with our result.

Let $\alpha > 0$, $1 \le p \le \infty$, and $0 \le a < b \le \infty$, then the operators $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ are bounded in $L_p(a,b)$ as follows:

$$\|J_{a+}^{\alpha} f\|_p \le K_1 \|f\|_p, \qquad \|J_{b-}^{\alpha} f\|_p \le K_2 \|f\|_p, \tag{2.80}$$

where

$$K_1 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha-1} e^{t/p} dt, \qquad K_2 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha-1} e^{-t/p} dt. \tag{2.81}$$

Now we present the definitions and some properties of the *Erdélyi-Kober type fractional integrals*. Some of these definitions and results were presented by Samko et al. in [4].

Let (a,b), $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0$, $\sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$\left(I_{a_+;\sigma;\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}}, \tag{2.82}$$

$$\left(I_{b_-;\sigma;\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)dt}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}, \tag{2.83}$$

respectively. Integrals (2.82) and (2.83) are called the Erdélyi-Kober type fractional integrals.

Corollary 2.27. *Let u be a weight function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function, and $I^{\alpha}_{a_+;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional left-sided integral. Define v by*

$$v(y) := \alpha\sigma y^{\sigma\eta+\sigma-1}\int_y^b u(x)\frac{x^{-\sigma\eta}\left(x^{\sigma}-y^{\sigma}\right)^{\alpha-1}}{\left(x^{\sigma}-a^{\sigma}\right)^{\alpha}\ {}_2F_1\left(\alpha,-\eta;\alpha+1;1-(a/x)^{\sigma}\right)}dx<\infty. \tag{2.84}$$

If $\phi:(0,\infty)\to\mathbb{R}$ is convex and increasing function, then the inequality

$$\begin{aligned}\int_a^b u(x)\phi\Bigg(&\frac{\Gamma(\alpha+1)}{\left(1-(a/x)^{\sigma}\right)^{\alpha}\ {}_2F_1\left(\alpha,-\eta;\alpha+1;1-(a/x)^{\sigma}\right)}\left|I^{\alpha}_{a_+;\sigma;\eta}f(x)\right|\Bigg)dx\\ &\le\int_a^b v(y)\phi(|f(y)|)dy\end{aligned} \tag{2.85}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$,

$$k(x,y)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\dfrac{\sigma x^{-\sigma(\alpha+\eta)}}{\left(x^{\sigma}-y^{\sigma}\right)^{1-\alpha}}y^{\sigma\eta+\sigma-1}, & a\le y\le x,\\ 0, & x<y\le b,\end{cases} \tag{2.86}$$

we get that $K(x)=(1/\Gamma(\alpha+1))(1-(a/x)^{\sigma})^{\alpha}\ {}_2F_1(\alpha,-\eta;\alpha+1;1-(a/x)^{\sigma})$, so (2.85) follows. □

Remark 2.28. In particular for the weight function $u(x)=x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha}\ {}_2F_1(x)$ where $({}_2F_1(x)={}_2F_1(\alpha,-\eta;\alpha+1;1-(a/x)^{\sigma})$ in Corollary 2.27, we obtain the inequality

$$\begin{aligned}\int_a^b x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha}\ {}_2F_1(x)\phi\Bigg(&\frac{\Gamma(\alpha+1)}{\left(1-(a/x)^{\sigma}\right)^{\alpha}\ {}_2F_1(x)}\left|I^{\alpha}_{a_+;\sigma;\eta}f(x)\right|\Bigg)dx\\ &\le\int_a^b y^{\sigma-1}(b^{\sigma}-y^{\sigma})^{\alpha}\ {}_2F_1(y)\phi(|f(y)|)dy,\end{aligned} \tag{2.87}$$

where ${}_2F_1(y)={}_2F_1(\alpha,\eta;\alpha+1;1-(a/y)^{\sigma})$.

Corollary 2.29. *Let u be a weight function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function, and $I^{\alpha}_{b_-;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional right-sided integral. Define v by*

$$v(y) := \alpha\sigma y^{\sigma(1-\alpha-\eta)-1}\int_a^y u(x)\frac{x^{\sigma(\eta+\alpha)}\left(y^{\sigma}-x^{\sigma}\right)^{\alpha-1}}{\left(b^{\sigma}-x^{\sigma}\right)^{\alpha}\ {}_2F_1\left(\alpha,\alpha+\eta;\alpha+1;1-(b/x)^{\sigma}\right)}dx<\infty. \tag{2.88}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\begin{aligned}&\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{((b/x)^\sigma-1)^\alpha\ {}_2F_1(\alpha,\alpha+\eta;\alpha+1;1-(b/x)^\sigma)}\left|I^\alpha_{b_-;\sigma;\eta}f(x)\right|\right)dx\\&\quad\le\int_a^b v(y)\phi(|f(y)|)dy\end{aligned}\tag{2.89}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$,

$$k(x,y)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\dfrac{\sigma x^{\sigma\eta}}{(y^\sigma-x^\sigma)^{1-\alpha}}y^{\sigma(1-\alpha-\eta)-1}, & x<y\le b,\\ 0, & a\le y\le x,\end{cases}\tag{2.90}$$

we get that $K(x)=(1/\Gamma(\alpha+1))((b/x)^\sigma-1)^\alpha\ {}_2F_1(\alpha,\alpha+\eta;\alpha+1;1-(b/x)^\sigma)$, so (2.89) follows. □

Remark 2.30. In particular for the weight function $u(x) = x^{\sigma-1}(b^\sigma - x^\sigma)^\alpha\,{}_2F_1(x)$ where $({}_2F_1(x) = {}_2F_1(\alpha,\alpha+\eta;\alpha+1;1-(b/x)^\sigma)$ in Corollary 2.29, we obtain the inequality

$$\begin{aligned}&\int_a^b x^{\sigma-1}(b^\sigma-x^\sigma)^\alpha\ {}_2F_1(x)\phi\left(\frac{\Gamma(\alpha+1)}{((b/x)^\sigma-1)^\alpha\ {}_2F_1(x)}\left|I^\alpha_{b_-;\sigma;\eta}f(x)\right|\right)dx\\&\quad\le\int_a^b y^{\sigma-1}(y^\sigma-a^\sigma)^\alpha\ {}_2F_1(y)\phi(|f(y)|)dy,\end{aligned}\tag{2.91}$$

where $({}_2F_1(y) = {}_2F_1(\alpha,-\alpha-\eta;\alpha+1;1-(b/y)^\sigma)$.

In the next corollary, we give some results related to the *Caputo radial fractional derivative*. Let us recall the following definition, see [1, page 463].

Let $f : \overline{A} \to \mathbb{R}$, $\nu \ge 0$, $n := \lceil\nu\rceil$, such that $f(\cdot\omega) \in AC^n([R_1,R_2])$, for all $\omega \in S^{N-1}$, where $\overline{A} = [R_1,R_2]\times S^{N-1}$ for $N\in\mathbb{N}$ and $S^{N-1} := \{x\in\mathbb{R}^N : |x| = 1\}$. We call the Caputo radial fractional derivative as the following function:

$$\frac{\partial^\nu_{*R_1}f(x)}{\partial r^\nu}:=\frac{1}{\Gamma(n-\nu)}\int_{R_1}^r(r-t)^{n-\nu-1}\frac{\partial^n f(t\omega)}{\partial r^n}dt,\tag{2.92}$$

where $x\in\overline{A}$, that is, $x=r\omega$, $r\in[R_1,R_2]$, $\omega\in S^{N-1}$.

Clearly,

$$\begin{aligned}\frac{\partial^0_{*R_1}f(x)}{\partial r^0}&=f(x),\\ \frac{\partial^\nu_{*R_1}f(x)}{\partial r^\nu}=\frac{\partial^\nu f(x)}{\partial r^\nu}&\quad\text{if }\nu\in\mathbb{N},\ \text{the usual radial derivative.}\end{aligned}\tag{2.93}$$

Corollary 2.31. *Let u be a weight function on (R_1, R_2), and $\partial^{\nu}_{*R_1} f(x)/\partial r^{\nu}$ denotes the Caputo radial fractional derivative of f. Define v on (R_1, R_2) by*

$$v(t) := (n-\nu)\int_t^{R_2} u(r)\frac{(r-t)^{n-\nu-1}}{(r-R_1)^{n-\nu}}dr < \infty. \tag{2.94}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_{R_1}^{R_2} u(r)\phi\left(\frac{\Gamma(n-\nu+1)}{(r-R_1)^{n-\nu}}\left|\frac{\partial^{\nu}_{*R_1} f(x)}{\partial r^{\nu}}\right|\right)dr \le \int_{R_1}^{R_2} v(t)\phi\left(\left|\frac{\partial^{n} f(t\omega)}{\partial r^{n}}\right|\right)dt \tag{2.95}$$

holds.

Proof. Apply Theorem 2.1 with $\Omega_1 = \Omega_2 = (R_1, R_2)$, $d\mu_1(x) = dr$, $d\mu_2(y) = dt$, and

$$k(r,t) = \begin{cases} \dfrac{(r-t)^{n-\nu-1}}{\Gamma(n-\nu)}, & R_1 \le t \le r, \\ 0, & r < t \le R_2. \end{cases} \tag{2.96}$$

Then replace $f(x)$ by $\partial^n f(t\omega)/\partial r^n$, so (2.95) follows. □

Remark 2.32. In particular for the weight function $u(r) = (r-R_1)^{n-\nu}$, $r \in (R_1, R_2)$, we obtain the following inequality:

$$\int_{R_1}^{R_2} (r-R_1)^{n-\nu}\phi\left(\frac{\Gamma(n-\nu+1)}{(r-R_1)^{(n-\nu)}}\left|\frac{\partial^{\nu}_{*R_1} f(x)}{\partial r^{\nu}}\right|\right)dr \le \int_{R_1}^{R_2} (R_2-t)^{n-\nu}\phi\left(\left|\frac{\partial^{n} f(t\omega)}{\partial r^{n}}\right|\right)dt. \tag{2.97}$$

Let $q > 1$ and $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (2.97) becomes

$$(\Gamma(n-\nu+1))^q \int_{R_1}^{R_2} (r-R_1)^{(n-\nu)(1-q)}\left|\frac{\partial^{\nu}_{*R_1} f(x)}{\partial r^{\nu}}\right|^q dr \le \int_{R_1}^{R_2} (R_2-t)^{n-\nu}\left|\frac{\partial^{n} f(t\omega)}{\partial r^{n}}\right|^q dt. \tag{2.98}$$

Since $r \in (R_1, R_2)$ and $(1-q)(n-\nu) \le 0$, we obtain

$$\int_{R_1}^{R_2} \left|\frac{\partial^{\nu}_{*R_1} f(x)}{\partial r^{\nu}}\right|^q dr \le \left(\frac{(R_2-R_1)^{n-\nu}}{\Gamma(n-\nu+1)}\right)^q \int_{R_1}^{R_2} \left|\frac{\partial^{n} f(t\omega)}{\partial r^{n}}\right|^q dt. \tag{2.99}$$

Taking power $1/q$ on both sides, we get

$$\left\|\frac{\partial^{\nu}_{*R_1} f(x)}{\partial r^{\nu}}\right\|_q \le \frac{(R_2-R_1)^{n-\nu}}{\Gamma(n-\nu+1)}\left\|\frac{\partial^{n} f(t\omega)}{\partial r^{n}}\right\|_q. \tag{2.100}$$

If $\nu = 0$, then

$$\|f(x)\|_q \leq \frac{(R_2 - R_1)^n}{\Gamma(n+1)} \left\| \frac{\partial^n f(t\omega)}{\partial r^n} \right\|_q. \tag{2.101}$$

If $\nu \in \mathbb{N}$, then

$$\left\| \frac{\partial^\nu f(x)}{\partial r^\nu} \right\|_q \leq \frac{(R_2 - R_1)^{n-\nu}}{\Gamma(n-\nu+1)} \left\| \frac{\partial^n f(t\omega)}{\partial r^n} \right\|_q. \tag{2.102}$$

Now, we continue with the *Riemann-Liouville radial fractional derivative of f of order β*, but first we need to define the following: let $\mathcal{B}_X$ stand for the Borel class on space X and define the measure R_N on $((0,\infty), \mathcal{B}_{(0,\infty)})$ by

$$R_N(\Gamma) = \int_\Gamma r^{N-1} dr, \quad \text{any } \Gamma \in \mathcal{B}_{(0,\infty)}. \tag{2.103}$$

Now, let $f \in L_1(A) = L_1([R_1, R_2] \times S^{N-1})$.

For a fixed $\omega \in S^{N-1}$, we define

$$g_\omega(r) := f(r\omega) = f(x), \tag{2.104}$$

where

$$\begin{gathered} x \in A := B(0, R_2) - \overline{B(0, R_1)}, \\ 0 < R_1 \leq r \leq R_2, \quad r = |x|, \qquad \omega = \frac{x}{r} \in S^{N-1}. \end{gathered} \tag{2.105}$$

The above led to the following definition of Riemann-Liouville radial fractional derivative. For details see [1, page 466]. Let $\beta > 0$, $m := [\beta] + 1$, $f \in L_1(A)$, and A is the spherical shell. We define

$$\begin{aligned} \frac{\partial^\beta_{R_1} f(x)}{\partial r^\beta} &= D^\beta_{R_1} f(r\omega) \\ &= \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \left(\dfrac{\partial}{\partial r}\right)^m \displaystyle\int_{R_1}^r (r-t)^{m-\beta-1} f(t\omega) dt, & \omega \in S^{N-1} - K(f), \\ 0, & \omega \in K(f), \end{cases} \end{aligned} \tag{2.106}$$

where

$$\begin{gathered} x = r\omega \in A, \quad r \in [R_1, R_2], \ \omega \in S^{N-1}, \\ K(f) =: \left\{\omega \in S^{N-1} : f(\cdot\omega) \notin L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right)\right\}. \end{gathered} \tag{2.107}$$

If $\beta = 0$, define

$$\frac{\partial^{\beta}_{R_1} f(x)}{\partial r^{\beta}} := f(x). \tag{2.108}$$

We call $\partial^{\beta}_{R_1} f(x)/\partial r^{\beta}$ the Riemann-Liouville radial fractional derivative of f of order β.

The following result is given in [1, page 466].

Lemma 2.33. *Let $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := \lceil \nu \rceil$, $f : \overline{A} \to \mathbb{R}$ with $f \in L_1(A)$. Assume that $f(\cdot\omega) \in AC^n([R_1, R_2])$, for every $\omega \in S^{N-1}$, and that $\partial^{\nu}_{R_1} f(\cdot\omega)/\partial r^{\nu}$ is measurable on $[R_1, R_2]$ for every $\omega \in S^{N-1}$. Also assume that there exists $\partial^{\nu}_{R_1} f(r\omega)/\partial r^{\nu} \in \mathbb{R}$ for every $r \in [R_1, R_2]$ and for every $\omega \in S^{N-1}$, and $\partial^{\nu}_{R_1} f(\omega)/\partial r^{\nu}$ is measurable on $\overline{A}$. Suppose that there exists $M_1 > 0$,*

$$\left| \frac{\partial^{\nu}_{R_1} f(\omega)}{\partial r^{\nu}} \right| \leq M_1, \quad \text{for every } (r, \omega) \in [R_1, R_2] \times S^{N-1}. \tag{2.109}$$

We suppose that $\partial^j f(R_1\omega)/\partial r^j = 0$, $j = 0, 1, \ldots, n-1$, for every $\omega \in S^{N-1}$, then

$$\frac{\partial^{\gamma}_{R_1} f(x)}{\partial r^{\gamma}} = D^{\gamma}_{R_1} f(r\omega) = \frac{1}{\Gamma(\nu - \gamma)} \int_{R_1}^{r} (r - t)^{\nu - \gamma - 1} \left(D^{\nu}_{R_1} f \right)(t\omega) dt \tag{2.110}$$

is valid for every $x \in \overline{A}$, that is, true for every $r \in [R_1, R_2]$ and for every $\omega \in S^{N-1}$, $\gamma > 0$.

Corollary 2.34. *Let u be a weight function on (R_1, R_2). Let the assumption of the Lemma 2.33 be satisfied, and $D^{\gamma}_{R_1} f(r\omega)$ denotes the Riemann-Liouville radial fractional derivative of f. Define v on (R_1, R_2) by*

$$v(t) := (\nu - \gamma) \int_{t}^{R_2} u(r) \frac{(r - t)^{\nu - \gamma - 1}}{(r - R_1)^{\nu - \gamma}} dr < \infty. \tag{2.111}$$

If $\phi : (0, \infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_{R_1}^{R_2} u(r) \phi\left(\frac{\Gamma(\nu - \gamma + 1)}{(r - R_1)^{\nu - \gamma}} \left| D^{\gamma}_{R_1} f(r\omega) \right| \right) dr \leq \int_{R_1}^{R_2} v(t) \phi\left(\left| \left(D^{\nu}_{R_1} f \right)(t\omega) \right| \right) dt \tag{2.112}$$

holds.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (R_1, R_2)$,

$$k(r, t) = \begin{cases} \dfrac{(r - t)^{\nu - \gamma - 1}}{\Gamma(\nu - \gamma)}, & R_1 \leq t \leq r, \\ 0, & r < t \leq R_2, \end{cases} \tag{2.113}$$

we get that $K(r) = (r - R_1)^{\nu-\gamma}/\Gamma(\nu-\gamma+1)$. Replace $f(\cdot)$ by $D^{\nu}_{R_1} f(\cdot\omega)$, and then from the above Lemma 2.33, we get $g(r) = D^{\gamma}_{R_1} f(r\omega)$. This will give us (2.112). □

Remark 2.35. In particular for the weight function $u(r) = (r - R_1)^{\nu-\gamma}$, $r \in (R_1, R_2)$ in above Corollary 2.34 and for $\phi(x) = x^q$, $q > 1$ we obtain, after some calculation, the following inequality:

$$\left\| D^{\gamma}_{R_1} f(r\omega) \right\|_q \leq \frac{(R_2 - R_1)^{\nu-\gamma}}{\Gamma(\nu - \gamma + 1)} \left\| (D^{\nu}_{R_1} f)(t\omega) \right\|_q. \tag{2.114}$$

If $\gamma = 0$, then

$$\| f(r\omega) \|_q \leq \frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu + 1)} \left\| (D^{\nu}_{R_1} f)(t\omega) \right\|_q. \tag{2.115}$$

In the previous corollaries, we derived only inequalities over some subsets of $\mathbb{R}$. However, Theorem 2.1 covers much more general situations. We conclude this paper with multidimensional fractional integrals. Such operations of fractional integration in the n-dimensional Euclidean space $\mathbb{R}^n$, $(n \in \mathbb{N})$ are natural generalizations of the corresponding one-dimensional fractional integrals and fractional derivatives, being taken with respect to one or several variables.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, we use the following notations:

$$\Gamma(\alpha) = (\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)), \qquad [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n], \tag{2.116}$$

and by $\mathbf{x} > \mathbf{a}$, we mean $x_1 > a_1, \ldots, x_n > a_n$.

The *partial Riemann-Liouville fractional integrals* of order $\alpha_k > 0$ with respect to the kth variable x_k are defined by

$$\left(I^{\alpha_k}_{\mathbf{a}_{k+}} f\right)(\mathbf{x}) = \frac{1}{\Gamma(\alpha_k)} \int_{a_k}^{x_k} f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n)(x_k - t_k)^{\alpha_k - 1} dt_k, \quad (x_k > a_k), \tag{2.117}$$

$$\left(I^{\alpha_k}_{\mathbf{b}_{k-}} f\right)(\mathbf{x}) = \frac{1}{\Gamma(\alpha_k)} \int_{x_k}^{b_k} f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n)(t_k - x_k)^{\alpha_k - 1} dt_k, \quad (x_k < b_k), \tag{2.118}$$

respectively. These definitions are valid for functions $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ defined for $x_k > a_k$ and $x_k < b_k$, respectively.

Next, we define the *mixed Riemann-Liouville fractional integrals* of order $\alpha > 0$ as

$$\begin{aligned} \left(I^{\alpha}_{\mathbf{a}_+} f\right)(\mathbf{x}) &= \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x} - \mathbf{t})^{\alpha - 1} d\mathbf{t}, \qquad (\mathbf{x} > \mathbf{a}), \\ \left(I^{\alpha}_{\mathbf{b}_-} f\right)(\mathbf{x}) &= \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f(\mathbf{t})(\mathbf{t} - \mathbf{x})^{\alpha - 1} d\mathbf{t}, \qquad (\mathbf{x} < \mathbf{b}). \end{aligned} \tag{2.119}$$

Corollary 2.36. *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{a}_+} f$ denotes the mixed partial Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}) := \alpha \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} u(\mathbf{x}) \frac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{(\mathbf{x}-\mathbf{a})^{\alpha}} d\mathbf{x} < \infty. \tag{2.120}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\mathbf{x}) \phi\left(\frac{\Gamma(\alpha+1)}{(\mathbf{x}-\mathbf{a})^{\alpha}} \left| I^{\alpha}_{\mathbf{a}_+} f(\mathbf{x}) \right| \right) d\mathbf{x} \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(\mathbf{y}) \phi(|f(\mathbf{y})|) d\mathbf{y} \tag{2.121}$$

holds for all measurable functions $f : (\mathbf{a},\mathbf{b}) \to \mathbb{R}$.

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (\mathbf{a},\mathbf{b})$,

$$k(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{\Gamma(\alpha)}, & \mathbf{a} \leq \mathbf{y} \leq \mathbf{x}, \\ 0, & \mathbf{x} < \mathbf{y} \leq \mathbf{b}, \end{cases} \tag{2.122}$$

we get that $K(\mathbf{x}) = (\mathbf{x}-\mathbf{a})^{\alpha}/\Gamma(\alpha+1)$ and $g(\mathbf{x}) = I^{\alpha}_{\mathbf{a}_+} f(\mathbf{x})$, so (2.121) follows. □

Corollary 2.37. *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{b}_-} f$ denotes the mixed partial Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}) := \alpha \int_{a_1}^{y_1} \cdots \int_{a_n}^{y_n} u(\mathbf{x}) \frac{(\mathbf{y}-\mathbf{x})^{\alpha-1}}{(\mathbf{b}-\mathbf{x})^{\alpha}} d\mathbf{x} < \infty. \tag{2.123}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\mathbf{x}) \phi\left(\frac{\Gamma(\alpha+1)}{(\mathbf{b}-\mathbf{x})^{\alpha}} \left| I^{\alpha}_{\mathbf{b}_-} f(\mathbf{x}) \right| \right) d\mathbf{x} \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(\mathbf{y}) \phi(|f(\mathbf{y})|) d\mathbf{y} \tag{2.124}$$

holds for all measurable functions $f : (\mathbf{a},\mathbf{b}) \to \mathbb{R}$.

Remark 2.38. Analogous to Remarks 2.3 and 2.5, we obtain multidimensional version of inequality (1.3) for $q > 1$ as follows:

$$\begin{gathered} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| I^{\alpha}_{\mathbf{a}_+} f(\mathbf{x}) \right|^{g} d\mathbf{x} \leq \left(\frac{(\mathbf{b}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)} \right)^{q} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\mathbf{y})|^{q} d\mathbf{y}, \\ \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| I^{\alpha}_{\mathbf{b}_-} f(\mathbf{x}) \right|^{g} d\mathbf{x} \leq \left(\frac{(\mathbf{b}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)} \right)^{q} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\mathbf{y})|^{q} d\mathbf{y}. \end{gathered} \tag{2.125}$$

Acknowledgments

The authors would like to express their gratitude to professor S. G. Samko for his help and suggestions and also to the careful referee whose valuable advice improved the final version of this paper. The research of the second and third authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant no. 117-1170889-0888.

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