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*Research Article*

# General Iterative Algorithm for Nonexpansive Semigroups and Variational Inequalities in Hilbert Spaces

**Jinlong Kang,[1,2] Yongfu Su,[1] and Xin Zhang[1]**

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*
[2] *Department of Foundation, Xian Communication Institute, Xian 710106, China*

Correspondence should be addressed to Yongfu Su, suyongfu@tjpu.edu.cn

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We introduce a general iterative method for finding the solution of the variational inequality problem over the fixed point set of a nonexpansive semigroup in a Hilbert space. We prove that the sequence converges strongly to a common element of the above two sets under some parameters controlling conditions. Our results improve and generalize many known corresponding results.

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. A mapping $T$ of $C$ into itself is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. A family $S = \{T(s) : 0 \leq s < \infty\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $x, y \in C$ and $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \{x \in C : T(s)x = x, 0 \leq s < \infty\}$. It is known that $F(\mathcal{S})$ is closed and convex.

Let $A$ be a strongly positive bounded linear operator on $H$: that is, there is a constant $\overline{\gamma}$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad \forall x \in H. \tag{1.1}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$:

$$\min_{x\in K}\frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.2}$$

where $K$ is the fixed point set of a nonexpansive mapping $T$ on $H$ and $b$ is a given point in $H$. In 2001, Yamada [1] presented the hybrid steepest descent method for problem (1.2). In 2003, Xu [2] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0 \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [3] introduced the viscosity approximation method for nonexpansive mappings (see [4] for further developments in both Hilbert and Banach spaces). Let $f$ be a contraction on $H$ such that $\|fx - fy\| \leq \alpha\|x - y\|$, where $\alpha \in [0,1)$ is a constant. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where $\{\beta_n\}$ is a sequence in $(0,1)$. It is proved [3, 4] that under certain appropriate conditions imposed on $\{\beta_n\}$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution $\tilde{x}$ in $K$ of the variational inequality

$$\left\langle (I - f)\tilde{x}, x - \tilde{x}\right\rangle \geq 0, \quad x \in K. \tag{1.5}$$

Recently, Marino and Xu [5] combine the iterative method (1.3) with the viscosity approximation (1.4) and consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0. \tag{1.6}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)\tilde{x}, x-\tilde{x}\rangle \geq 0, \quad x \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in K} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.8}$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f$ for $x \in H$).

Note that $I - f$ and $A - \gamma f$ in problems (1.5) and (1.7) are strongly monotone and Lipschitz continuous. Therefore, problems (1.5) and (1.7) can be solved by [1, 7, 8]. In [7, 8], algorithms to accelerate the hybrid steepest descent method have been proposed.

Quite recently, for the nonexpansive semigroups $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$, Plubtieng and Punpaeng [9] study the iteration process $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 0, \tag{1.9}$$

where $x_0 \in C$, $\{\alpha_n\}$,$\{\beta_n\}$ are two sequences in $(0,1)$, and $\{s_n\}$ is a positive real divergent real sequence and prove a strong convergence theorem.

In this paper, motivated and inspired by the above results, we prove a strong convergence of the iterative scheme in a real Hilbert space by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \tag{1.10}$$

Furthermore, we show that if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)\tilde{x}, x-\tilde{x}\rangle \geq 0, \quad x \in F(T), \tag{1.11}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.12}$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f$ for $x \in H$). The results of this paper extended and improved the results of Xu [2], Moudafi [3], Marino and Xu [5], and Plubtieng and Punpaeng [9].

## 2. Preliminaries

Recall that given a closed convex subset $K$ of a real Hilbert space $H$, the nearest point projection $P_K$ from $H$ onto $K$ assigns to each $x \in H$ its nearest point denoted by $P_K x$ in $K$ from $x$ to $K$; that is, $P_K x$ is the unique point in $K$ with the property

$$\|x - P_K x\| \leq \|x - y\| \quad \forall y \in K. \tag{2.1}$$

The following Lemmas 2.1 and 2.2 are well known.

**Lemma 2.1.** *Let $K$ be a closed convex subset of a real Hilbert space $H$.Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the following relation:*

$$\langle x - z, y - z \rangle \leq 0 \quad \forall y \in K. \tag{2.2}$$

**Lemma 2.2.** *Let $H$ be a real Hilbert space. There hold the following identities.*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \ \forall x, y \in H, t \in [0,1]$.

*Definition 2.3* (Opial's condition [10]). *A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{2.3}$$

It is well known that Hilbert spaces satisfy Opial's condition.

**Lemma 2.4** (Browder [6]). *Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$, and $T : K \to E$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

**Theorem 2.5** (Shimizu and Takahashi [11]). *Let $C$ be a nonempty closed convex bounded subset of a real Hilbert space $H$ and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $C$. For $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\limsup_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right)\right\| = 0. \tag{2.4}$$

**Theorem 2.6** (Marino and Xu [5]). *Assume that $A$ is a strong positive linear bounded operator on a Hilbert space $H$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

**Theorem 2.7** (Xu [12]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 0, \tag{2.5}$$

*where* $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbf{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\to\infty}\alpha_n = 0$.

## 3. Main Results

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a semigroup of nonexpansive mapping on $C$ such that $F(\mathcal{S})$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences of real numbers in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $f$ be a contraction of $C$ into itself with a coefficient $\alpha \in (0,1)$, $\{s_n\}$ be a positive real divergent sequence, and $A$ a strong positive bounded linear operator on $C$ with coefficient $\overline{\gamma} > 0$, and $0 < \gamma < \overline{\gamma}/\alpha$. Let the sequence $\{x_n\}$be defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \tag{3.1}$$

*Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $F(\mathcal{S})$ of the variational inequality*

$$\langle (A-\gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in F(\mathcal{S}) \tag{$*$}$$

*or equivalent to $\tilde{x} = P_{F(\mathcal{S})}(I - A + \gamma f)(\tilde{x})$, where $P$ is a metric projection mapping from $H$ onto $F(\mathcal{S})$.*

*Proof.* Since $\lim_{n\to\infty}\alpha_n = 0$ by the assumption, we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n$. From Theorem 2.6, we know that if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Note that $F(\mathcal{S})$ is a nonempty closed convex set. We first show that $\{x_n\}$ is bounded. Let $q \in F(\mathcal{S})$. Thus, we compute that

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\|\alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - q\right\| \\
&\leq \alpha_n \|\gamma f(x_n) - Aq\| + \beta_n\|x_n - q\| + \|(1-\beta_n)I - \alpha_n A\| \left\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - q\right\| \\
&\leq \alpha_n\left(\|\gamma f(x_n) - \gamma f(q)\| + \|\gamma f(q) - Aq\|\right) + \beta_n\|x_n - q\| \\
&\quad + \left((1-\beta_n) - \alpha_n\overline{\gamma}\right)\frac{1}{s_n}\int_0^{s_n}\|T(s)x_n - q\|ds \\
&\leq \alpha_n\gamma\alpha\|x_n - q\| + \alpha_n\|\gamma f(q) - Aq\| + (1 - \alpha_n\overline{\gamma})\|x_n - q\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\| + (\overline{\gamma} - \gamma\alpha)\alpha_n\left(\frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(q) - Aq\|\right) \\
&\leq \max\left\{\|x_n - q\|, \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(q) - Aq\|\right\}.
\end{aligned} \tag{3.2}$$

By induction, we get

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(q) - Aq\|\right\}, \quad n \ge 0. \tag{3.3}$$

Therefore, $\{x_n\}$ is bounded. $\{(1/s_n)\int_0^{s_n} T(s)x_n ds\}$ and $\{f(x_n)\}$ are also bounded. Put $z_0 = P_{F(\mathcal{S})}x_0$ and $D = \{z \in C : \|z - z_0\| \le \|x_0 - z_0\| + (1/(\overline{\gamma} - \gamma\alpha))\|\gamma f(z_0) - A(z_0)\|\}$. Then $D$ is a nonempty closed bounded convex subset of $C$. Since $T(s)$ is nonexpansive for any $s \in [0, +\infty)$, $D$ is $T(s)$-invariant for each $s \in [0, \infty)$ and contains $\{x_n\}$. Without loss of generality, we may assume that $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ is a nonexpansive semigroup on $D$. By Theorem 2.5, we get

$$\lim_{n\to\infty}\left\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right)\right\| = 0, \tag{3.4}$$

for every $h \in [0, \infty)$. Next we show $\|x_n - T(h)x_n\| \to 0$ as $n \to \infty$. Notice that

$$\begin{aligned}
\|x_{n+1} - T(h)x_{n+1}\| &\le \left\|x_{n+1} - \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right)\right\| \\
&\quad + \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right) - T(h)x_{n+1}\right\| \\
&\le 2\left\|x_{n+1} - \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right)\right\| \\
&\le 2\alpha_n\left\|\gamma f(x_n) - A\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right\| + 2\beta_n\left\|x_n - \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right\| \\
&\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds\right)\right\|.
\end{aligned} \tag{3.5}$$

From $\alpha_n \to 0$, $\beta_n \to 0$, and (3.4), we get $\|x_{n+1} - T(h)x_{n+1}\| \to 0$, and hence

$$\|x_n - T(h)x_n\| \longrightarrow 0. \tag{3.6}$$

Let $\tilde{x}$ be the unique solution of the variational inequality $(*)$; we show that

$$\limsup_{n\to\infty}\langle (A - rf)\tilde{x}, x_n - \tilde{x}\rangle \ge 0, \quad x \in F(\mathcal{S}). \tag{3.7}$$

Since $\{x_n\} \in D$ is bounded, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty}\langle (A - rf)\tilde{x}, x_{n_j} - \tilde{x}\rangle = \limsup_{n\to\infty}\langle (A - rf)\tilde{x}, x_n - \tilde{x}\rangle, \tag{3.8}$$

and $x_{n_j} \rightharpoonup \tilde{q}$. By Opial's condition, we have $\tilde{q} \in F(\mathcal{S})$. In fact, if $\tilde{q} \neq T(h)\tilde{q}$ for some $h \in [0,\infty)$, we have

$$\begin{aligned}
\liminf_{j\to\infty}\left\|x_{n_j}-\tilde{q}\right\| &< \liminf_{j\to\infty}\left\|x_{n_j}-T(h)\tilde{q}\right\| \\
&\leq \liminf_{j\to\infty}\left(\left\|x_{n_j}-T(h)x_{n_j}\right\|+\left\|T(h)x_{n_j}-T(h)\tilde{q}\right\|\right) \\
&\leq \liminf_{j\to\infty}\left\|x_{n_j}-\tilde{q}\right\|.
\end{aligned} \tag{3.9}$$

This is a contradiction. Therefore, we have $\tilde{q} = T(h)\tilde{q}$ for some $h \geq 0$, that is $\tilde{q} \in F(\mathcal{S})$. Hence, by $(*)$, we obtain

$$\limsup_{n\to\infty}\langle (A-rf)\tilde{x}, x_n-\tilde{x}\rangle = \langle (A-rf)\tilde{x}, \tilde{q}-\tilde{x}\rangle \geq 0 \tag{3.10}$$

as required. Finally we shall show that $x_n \to \tilde{x}$. For each $n \geq 0$, we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \left\|\alpha_n(\gamma f(x_n)-A\tilde{x})+\beta_n(x_n-\tilde{x})+((1-\beta_n)I-\alpha_nA)\left(\frac{1}{s_n}\int_0^{s_n}T(s)x_n ds-\tilde{x}\right)\right\|^2 \\
&\leq \left\|((1-\beta_n)I-\alpha_nA)\left(\frac{1}{s_n}\int_0^{s_n}T(s)x_n ds-\tilde{x}\right)+\beta_n(x_n-\tilde{x})\right\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(x_n)-A\tilde{x}, x_{n+1}-\tilde{x}\rangle \\
&= \left\|(1-\beta_n)I-\alpha_nA\right\|^2\left\|\frac{1}{s_n}\int_0^{s_n}T(s)x_n ds-\tilde{x}\right\|^2+\beta_n^2\|x_n-\tilde{x}\|^2 \\
&\quad + 2\beta_n\left\langle((1-\beta_n)I-\alpha_nA)\left(\frac{1}{s_n}\int_0^{s_n}T(s)x_n ds-\tilde{x}\right), x_n-\tilde{x}\right\rangle \\
&\quad + 2\alpha_n\gamma\langle f(x_n)-A(\tilde{x}), x_{n+1}-\tilde{x}\rangle \\
&\leq ((1-\beta_n)-\alpha_n\overline{\gamma})^2\frac{1}{s_n}\int_0^{s_n}\|T(s)x_n-\tilde{x}\|^2 ds+\beta_n^2\|x_n-\tilde{x}\|^2 \\
&\quad + 2\beta_n\left\|(1-\beta_n)I-\alpha_nA\right\|\|x_n-\tilde{x}\|^2+2\alpha_n\gamma\langle f(x_n)-f(\tilde{x}), x_{n+1}-\tilde{x}\rangle \\
&\quad + 2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x}, x_{n+1}-\tilde{x}\rangle \\
&\leq ((1-\beta_n)-\alpha_n\overline{\gamma})^2\|x_n-\tilde{x}\|^2+\beta_n^2\|x_n-\tilde{x}\|^2+2\beta_n((1-\beta_n)-\alpha_n\overline{\gamma})\|x_n-\tilde{x}\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x}, x_{n+1}-\tilde{x}\rangle \\
&\leq (1-\alpha_n\overline{\gamma})^2\|x_n-\tilde{x}\|^2+\alpha_n\gamma\alpha\left(\|x_n-\tilde{x}\|^2+\|x_{n+1}-\tilde{x}\|^2\right)+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x}, x_{n+1}-\tilde{x}\rangle \\
&= \left((1-\alpha_n\overline{\gamma})^2+\alpha_n\gamma\alpha\right)\|x_n-\tilde{x}\|^2+\alpha_n\gamma\alpha\|x_{n+1}-\tilde{x}\|^2+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x}, x_{n+1}-\tilde{x}\rangle,
\end{aligned} \tag{3.11}$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le \frac{1-2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&=\left[1-\frac{2(\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2+\frac{(\alpha_n\overline{\gamma})^2}{1-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&\le\left[1-\frac{2(\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2+\frac{2(\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\\
&\quad\times\left\{\frac{\alpha_n\overline{\gamma}^2M}{2(\overline{\gamma}-\gamma\alpha)}+\frac{1}{\overline{\gamma}-\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle\right\}\\
&=(1-\delta_n)\|x_n-\tilde{x}\|^2+\delta_nB_n,
\end{aligned}
\tag{3.12}
$$

where $M=\sup\{\|x_n-\tilde{x}\|^2 : n\in\mathbb{N}\}$, $\delta_n=2(\overline{\gamma}-\gamma\alpha)\alpha_n/(1-\alpha_n\gamma\alpha)$, and $B_n:=(\alpha_n\overline{\gamma}^2M)/2(\overline{\gamma}-\gamma\alpha)+(1/(\overline{\gamma}-\gamma\alpha))\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle$. It is easily to see that $\delta_n\to 0$, $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}B_n\le 0$ by (3.10). Finally by using Theorem 2.7, we can obtain that $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of $T$. This completes the proof. □

## 4. Applications

**Theorem 4.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\mathcal{S}=\{T(s) : 0\le s<\infty\}$ be a strongly continuous semigroup of nonexpansive mapping on $C$ such that $F(\mathcal{S})$ is nonempty. Let $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Let $f$ be a contraction of $C$ into itself with a coefficient $\alpha\in[0,1)$, $\{s_n\}$ a positive real divergent sequence, and $A$ a strong positive bounded linear operator on $C$ with coefficient $\overline{\gamma}>0$ and $0<\gamma<\overline{\gamma}/\alpha$. Let the sequences $\{x_n\}$ defined by $x_0\in C$ and*

$$
x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)\frac{1}{s_n}\int_0^{s_n}T(s)x_n\,ds,\quad n\ge 0. \tag{4.1}
$$

*Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $F(\mathcal{S})$ of the variational inequality*

$$
\langle(A-\gamma f)\tilde{x},x-\tilde{x}\rangle\ge 0,\quad x\in F(\mathcal{S}) \tag{4.2}
$$

*or equivalent to $\tilde{x}=P_{F(\mathcal{S})}(I-A+\gamma f)(\tilde{x})$, where $P$ is a metric projection mapping from $H$ onto $F(\mathcal{S})$.*

*Proof.* Taking $\beta_n = 0$ in Theorem 3.1, we get the desired conclusion easily. □

**Corollary 4.2** (Marino and Xu [5]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a nonexpansive mapping on $C$ such that $F(T)$ is nonempty. Let $\{\alpha_n\}$ be a sequence of real numbers satisfying $0 < \alpha_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $f$ be a contraction of $C$ into itself with a coefficient $\alpha \in [0,1)$ and $A$ be a strong positive bounded linear operator on $C$ with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Then the sequence $\{x_n\}$ defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T x_n, \quad n \geq 0. \tag{4.3}$$

*Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $F(\mathcal{S})$ of the variational inequality*

$$\langle (A - \gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in F(\mathcal{S}) \tag{4.4}$$

*or equivalent $\tilde{x} = P_{F(T)}(I - A + \gamma f)(\tilde{x})$, where $P$ is a metric projection mapping from $H$ into $F(T)$.*

*Proof.* Taking $\mathcal{S} = \{T(s) : 0 \leq s < \infty\} = \{T\}$ and $\beta_n = 0$ in the in Theorem 3.1, we get the desired conclusion easily. □

**Corollary 4.3** (Plubtieng and Punpaeng [9]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a strongly continuous semigroup of nonexpansive mapping on $C$ such that $F(\mathcal{S})$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences of real numbers in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $f$ be a contraction of $C$ into itself with a coefficient $\alpha \in [0,1)$ and let $\{s_n\}$ be a positive real divergent sequence. Let the sequence $\{x_n\}$ be defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \tag{4.5}$$

*Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $F(\mathcal{S})$ of the variational inequality*

$$\langle (I - f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in F(\mathcal{S}) \tag{4.6}$$

*or equivalent to $\tilde{x} = P_{F(\mathcal{S})}(f)(\tilde{x})$, where $P$ is a metric projection mapping from $H$ into $F(\mathcal{S})$.*

*Proof.* Taking $\gamma = 1$ and $A = I$ in Theorem 3.1, we get the desired conclusion easily. □

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