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Research Article

On a New Hilbert-Type Intergral Inequality with the Intergral in Whole Plane

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By introducing some parameters and estimating the weight functions, we build a new Hilbert's inequality with the homogeneous kernel of 0 order and the integral in whole plane. The equivalent inequality and the reverse forms are considered. The best constant factor is calculated using Complex Analysis.

1. Introduction

If $f(x)$, $g(x) \geq 0$ and satisfy that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have [1]

$$\iint_0^\infty \frac{f(x)g(x)}{x+y}dx\,dy < \pi\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's integral inequality, which has been extended by Hardy-Riesz as [2].

If $p > 1$, $1/p + 1/q = 1$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \tag{1.2}$$

where the constant factor $\pi/\sin(\pi/p)$ also is the best possible.

Both of them are important in Mathematical Analysis and its applications [3]. It attracts some attention in recent years. Actually, inequalities (1.1) and (1.2) have many generalizations and variations. Equation (1.1) has been strengthened by Yang and others (including double series inequalities) [4–21].

In 2008, Xie and Zeng gave a new Hilbert-type Inequality [4] as follows.

If $a > 0$, $b > 0$, $c > 0$, $p > 1$, $1/p + 1/q = 1$, $f(x), g(x) \geq 0$ such that $0 < \int_0^\infty x^{-1-p/2} f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{-1-q/2} g^q(x)dx < \infty$, then

$$\begin{aligned}
&\iint_0^\infty \frac{f(x)g(y)}{(x+a^2y)(x+b^2y)(x+c^2y)} dx\,dy \\
&\quad < K\left\{\int_0^\infty x^{-1-p/2} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{-1-q/2} g^q(x)dx\right\}^{1/q},
\end{aligned} \tag{1.3}$$

where the constant factor $K = \pi/(a+b)(a+c)(b+c)$ is the best possible.

The main purpose of this paper is to build a new Hilbert-type inequality with homogeneous kernel of degree 0, by estimating the weight function. The equivalent inequality is considered.

In the following, we always suppose that: $1/p + 1/q = 1$, $p > 1$, $r \in (-1, 0)$, $0 < \alpha < \beta < \pi$.

2. Some Lemmas

We start by introducing some lemmas.

Lemma 2.1. *If* $k_1 := \int_0^\infty u^{-1+r} \ln((1+2u\cos\alpha+u^2)/(1+2u\cos\beta+u^2))du$, $k_2 := \int_0^\infty u^{-1+r} \ln((1-2u\cos\beta+u^2)/(1-2u\cos\alpha+u^2))du$, *then*

$$\begin{aligned}
k_1 &= \frac{4\pi \sin(r(\beta-\alpha)/2)\sin(r(\alpha+\beta)/2)}{r\sin r\pi}, \\
k_2 &= \frac{4\pi \sin(r(\beta-\alpha)/2)\sin(r\pi - r(\alpha+\beta)/2)}{r\sin r\pi}, \\
k &:= \int_{-\infty}^\infty |u|^{-1+r}\left|\ln\frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2}\right| du \\
&= k_1 + k_2 = \frac{4\pi\sin(r(\beta-\alpha)/2)\cos((r/2)(\pi-\alpha-\beta))}{r\cos(r\pi/2)}.
\end{aligned} \tag{2.1}$$

Proof. We have

$$\begin{aligned}
A &:= \int_0^\infty x^{r-1}\ln\left(x^2+2x\cos\alpha+1\right)dx = \frac{1}{r}x^r\ln\left(x^2+2x\cos\alpha+1\right)\Big|_0^\infty \\
&\quad - \frac{2}{r}\int_0^\infty \frac{x^r(x+\cos\alpha)}{x^2+2x\cos\alpha+1}dx \\
&:= -\frac{2}{r}B.
\end{aligned} \tag{2.2}$$

Setting $f(z) = z^r(z+\cos\alpha)/(z^2+2z\cos\alpha+1)$, $z_1 = -e^{i\alpha}$, $z_2 = -e^{-i\alpha}$, then

$$B = \frac{2\pi i}{1-e^{2\pi ri}}\left[Res(f,z_1)+Res(f,z_2)\right] = \frac{2\pi i}{1-e^{2\pi ri}}\left[\frac{z_1^r(z_1+\cos\alpha)}{z_1-z_2}+\frac{z_2^r(z_2+\cos\alpha)}{z_2-z_1}\right] = -\frac{\pi\cos r\alpha}{\sin r\pi} \tag{2.3}$$

we find that $A = -2B/r = 2\pi\cos r\alpha/r\sin r\pi$, then

$$\begin{aligned}
k_1 &:= \int_0^\infty u^{-1+r}\ln\frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2}du = \frac{4\pi\sin(r(\beta-\alpha)/2)\sin(r(\alpha+\beta)/2)}{r\sin r\pi},\\
k_2 &:= \int_0^\infty u^{-1+r}\ln\frac{1-2u\cos\beta+u^2}{1-2u\cos\alpha+u^2}du = \int_0^\infty u^{-1+r}\ln\frac{1+2u\cos(\pi-\beta)+u^2}{1+2u\cos(\pi-\alpha)+u^2}du\\
&= \frac{4\pi\sin(r(\beta-\alpha)/2)\sin((r/2)(2\pi-\alpha-\beta))}{r\sin r\pi},\\
k &= \int_{-\infty}^\infty |u|^{-1+r}\left|\ln\frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2}\right|du\\
&= \int_0^\infty u^{-1+r}\ln\frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2}du + \int_{-\infty}^0 (-u)^{-1+r}\ln\frac{1+2u\cos\beta+u^2}{1+2u\cos\alpha+u^2}du\\
&= k_1+k_2 = \frac{4\pi\sin(r(\beta-\alpha)/2)\cos((r/2)(\pi-\alpha-\beta))}{r\cos(r\pi/2)}.
\end{aligned} \tag{2.4}$$

The lemma is proved. □

Lemma 2.2. *Define the weight functions as follow:*

$$\begin{aligned}
w(x) &:= \int_{-\infty}^\infty \frac{|x|^{-r}}{|y|^{1-r}}\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dy,\\
\widetilde{w}(y) &:= \int_{-\infty}^\infty \frac{|y|^{r}}{|x|^{1+r}}\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx,
\end{aligned} \tag{2.5}$$

then $w(x) = \widetilde{w}(y) = k = (4\pi\sin(r(\beta-\alpha)/2)\cos((r/2)(\pi-\alpha-\beta)))/[r\cos(r\pi/2)]$.

Proof. We only prove that $w(x) = k$ for $x \in (-\infty, 0)$.

Using Lemma 2.1, setting $y = ux$ and$y = -ux$,

$$\begin{aligned}
w(x) &= \int_{-\infty}^0 \frac{(-x)^{-r}}{(-y)^{1-r}}\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}dy + \int_0^\infty \frac{(-x)^{-r}}{y^{1-r}}\ln\frac{x^2+2xy\cos\beta+y^2}{x^2+2xy\cos\alpha+y^2}dy\\
&= \int_0^\infty u^{-1+r}\ln\frac{1+2u\cos\alpha+u^2}{1+2u\cos\beta+u^2}du + \int_0^\infty u^{-1+r}\ln\frac{1-2u\cos\beta+u^2}{1-2u\cos\alpha+u^2}du = k_1+k_2 = k.
\end{aligned} \tag{2.6}$$

and the lemma is proved. □

Lemma 2.3. *For $\varepsilon > 0$, and $(r - \max\{2\varepsilon/p, 2\varepsilon/q\}) \in (-1,0)$, define both functions $\widetilde{f}, \widetilde{g}$ as follows:*

$$\widetilde{f}(x) = \begin{cases} x^{-r-1-2\varepsilon/p}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{-r-1-2\varepsilon/p}, & \text{if } x \in (-\infty,-1), \end{cases} \qquad \widetilde{g}(x) = \begin{cases} x^{r-1-2\varepsilon/q}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{r-1-2\varepsilon/q}, & \text{if } x \in (-\infty,-1), \end{cases} \tag{2.7}$$

then

$$\begin{gathered} I(\varepsilon) := \varepsilon\left\{\int_{-\infty}^{\infty} |x|^{p(r+1)-1}\widetilde{f}^p(x)\,dx\right\}^{1/p}\left\{\int_{-\infty}^{\infty} |x|^{q(r-1)-1}\widetilde{g}^q(x)\,dx\right\}^{1/q} = 1, \\ \widetilde{I}(\varepsilon) := \varepsilon\iint_{-\infty}^{\infty} \widetilde{f}(x)\widetilde{g}(y)\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\,dy \longrightarrow k \quad (\varepsilon \longrightarrow 0^+). \end{gathered} \tag{2.8}$$

Proof. Easily, we get the following:

$$I(\varepsilon) = \varepsilon\left\{2\int_1^{\infty} x^{-1}x^{-2\varepsilon}dx\right\}^{1/p}\left\{2\int_1^{\infty} x^{-1}x^{-2\varepsilon}dx\right\}^{1/q} = 1. \tag{2.9}$$

Let $y = -Y$, using $\widetilde{f}(-x) = \widetilde{f}(x), \widetilde{g}(-x) = \widetilde{g}(x)$ and

$$\widetilde{f}(-x)\int_{-\infty}^{\infty} \widetilde{g}(y)\left|\ln\frac{x^2-2xy\cos\alpha+y^2}{x^2-2xy\cos\beta+y^2}\right|dy = \widetilde{f}(x)\int_{-\infty}^{\infty} \widetilde{g}(Y)\left|\ln\frac{x^2+2xY\cos\alpha+Y^2}{x^2+2xY\cos\beta+Y^2}\right|dY, \tag{2.10}$$

we have that $\widetilde{f}(x)\int_{-\infty}^{\infty}\widetilde{g}(y)|\ln((x^2+2xy\cos\alpha+y^2)/(x^2+2xy\cos\beta+y^2))|dy$ is an even function on x, then

$$\begin{aligned} \widetilde{I}(\varepsilon) &= 2\varepsilon\int_0^{\infty} \widetilde{f}(x)\left(\int_{-\infty}^{\infty} \widetilde{g}(y)\left|\ln\frac{x^2+2xy\ \cos\ \alpha+y^2}{x^2+2xy\ \cos\ \beta+y^2}\right|dy\right)dx \\ &= 2\varepsilon\left[\int_1^{\infty} x^{-r-1-(2\varepsilon/p)}\left(\int_{-\infty}^{-1} (-y)^{r-1-(2\varepsilon/q)}\ln\frac{x^2+2xy\ \cos\ \beta+y^2}{x^2+2xy\ \cos\ \alpha+y^2}dy\right)dx\right. \\ &\quad\left.+\int_1^{\infty} x^{-r-1-(2\varepsilon/p)}\left(\int_1^{\infty} y^{r-1-(2\varepsilon/q)}\ln\frac{x^2+2xy\ \cos\ \alpha+y^2}{x^2+2xy\ \cos\ \beta+y^2}dy\right)dx\right] \\ &:= I_1 + I_2. \end{aligned} \tag{2.11}$$

Setting $y = tx$ then

$$
\begin{aligned}
I_1 &= 2\varepsilon\left[\int_1^{\infty} x^{-r-1-(2\varepsilon/p)}\left(\int_1^{\infty} y^{r-1-(2\varepsilon/q)} \ln\frac{x^2-2xy\cos\beta+y^2}{x^2-2xy\cos\alpha+y^2}dy\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{1/x}^{\infty} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_1^{\infty} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\right)dx\right.\\
&\qquad\left.+\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{1/x}^{1} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\right)dx\right]\\
&= \int_1^{\infty} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\\
&\quad+2\varepsilon\int_0^{1} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}\left(\int_{1/t}^{\infty} x^{-1-2\varepsilon}dx\right)dt\\
&= \int_1^{\infty} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt+\int_0^{1} t^{r-1+(2\varepsilon/p)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\\
&= \int_0^{\infty} t^{r-1-(2\varepsilon/q)} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt+\int_0^{1} \left(t^{(2\varepsilon/p)}-t^{-(2\varepsilon/q)}\right)t^{r-1} \ln\frac{1-2t\cos\beta+t^2}{1-2t\cos\alpha+t^2}dt\\
&= \frac{4\pi\sin((r-(2\varepsilon/q))(\beta-\alpha)/2)\sin((r-(2\varepsilon/q))(2\pi-\alpha-\beta)/2)}{(r-(2\varepsilon/q))\sin(r-(2\varepsilon/q))\pi}+\eta(\varepsilon),
\end{aligned}
\tag{2.12}
$$

where $\lim_{\varepsilon\to 0^+}\eta(\varepsilon)=0$, and we have $I_1 \to k_2$ $(\varepsilon \to 0^+)$.

Similarly, $I_2 \to k_1(\varepsilon \to 0^+)$. The lemma is proved. □

Lemma 2.4. *If $f(x)$ is a nonnegative measurable function and $0<\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx<\infty$, then*

$$
J := \int_{-\infty}^{\infty}|y|^{pr-1}\left(\int_{-\infty}^{\infty} f(x)\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\right)^p dy \le k^p\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx.
\tag{2.13}
$$

Proof. By Lemma 2.2, we find that

$$
\begin{aligned}
&\left(\int_{-\infty}^{\infty}\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\right)^p\\
&\quad=\left[\int_{-\infty}^{\infty}\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|\left(\frac{|x|^{(1+r)/q}}{|y|^{(1-r)/p}}f(x)\right)\left(\frac{|y|^{(1-r)/p}}{|x|^{(1+r)/q}}\right)dx\right]^p
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{\infty}\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| \frac{|x|^{(1+r)(p-1)}}{|y|^{1-r}} f^p(x)dx \\
&\quad\times\left(\int_{-\infty}^{\infty}\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| \frac{|y|^{(1-r)(q-1)}}{|x|^{1+r}} dx\right)^{p-1} \\
&= k^{p-1}|y|^{-rp+1}\int_{-\infty}^{\infty}\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| \frac{|x|^{(1+r)(p-1)}}{|y|^{1-r}} f^p(x)dx, \\
J &\leq k^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| \frac{|x|^{(1+r)(p-1)}}{|y|^{1-r}} f^p(x)dx\right]dy \\
&= k^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| \frac{|x|^{(1+r)(p-1)}}{|y|^{1-r}} dy\right] f^p(x)dx \\
&= k^p\int_{-\infty}^{\infty}|x|^{p(1+r)-1} f^p(x)dx.
\end{aligned}
\tag{2.14}
$$

□

3. Main Results

Theorem 3.1. *If both functions, $f(x)$ and $g(x)$, are nonnegative measurable functions and satisfy $0 < \int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx < \infty$ and $0 < \int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx < \infty$, then*

$$
\begin{aligned}
I^* &:= \iint_{-\infty}^{\infty} f(x)g(y)\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| dx\,dy \\
&< k\left(\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right)^{1/q},
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
J &= \int_{-\infty}^{\infty}|y|^{pr-1}\left(\int_{-\infty}^{\infty} f(x)\left|\ln \frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right| dx\right)^p dy \\
&< k^p\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx.
\end{aligned}
\tag{3.2}
$$

Inequalities (3.1) *and* (3.2) *are equivalent, and where the constant factors k and k^p are the best possibles.*

Proof. If (2.13) takes the form of equality for some $y \in (-\infty, 0) \cup (0, \infty)$, then there exists constants M and N, such that they are not all zero, and

$$
M\frac{|x|^{(1+r)(p-1)}}{|y|^{1-r}} f^p(x) = N\frac{|y|^{(1-r)(q-1)}}{|x|^{1+r}} \quad \text{a.e. in } (-\infty,\infty)\times(-\infty,\infty). \tag{3.3}
$$

Hence, there exists a constant C, such that

$$M|x|^{(1+r)p}f^p(x) = N|y|^{(1-r)q} = C \quad \text{a.e. in } (-\infty,\infty)\times(-\infty,\infty). \tag{3.4}$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $|x|^{p(1+r)-1}f^p(x) = C/(M|x|^{-1})$ a.e. in $(-\infty,\infty)$ which contradicts the fact that $0 < \int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx < \infty$. In the same way, we claim that $N = 0$. This is too a contradiction and hence by (2.13), we have (3.2).

By Hölder's inequality with weight [22] and (3.2), we have the following:

$$\begin{aligned} I^* &= \int_{-\infty}^{\infty}\left[|y|^{-1+r+(1/q)}\int_{-\infty}^{\infty}f(x)\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\right]\left[|y|^{1-r-(1/q)}g(y)\right]dy \\ &\le (J)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q(1-r)-1}g^q(y)dy\right)^{1/q}. \end{aligned} \tag{3.5}$$

Using (3.2), we have (3.1).

Setting $g(y) = |y|^{rp-1}(\int_{-\infty}^{\infty}f(x)|\ln((x^2+2xy\cos\alpha+y^2)/(x^2+2xy\cos\beta+y^2))|dx)^{p-1}$, then $J = \int_{-\infty}^{\infty}|y|^{q(1-r)-1}g^q(y)dy$ by (2.13), we have $J < \infty$. If $J = 0$ then (3.2) is proved. If $0 < J < \infty$, by (3.1), we obtain

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty}|y|^{q(1-r)-1}g^q(y)dy = J = I^* \\ &< k\left(\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right)^{1/q}, \\ &\left(\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right)^{1/p} = J^{1/p} < k\left(\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right)^{1/p}. \end{aligned} \tag{3.6}$$

Inequalities (3.1) and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with $h < k$), such that

$$\begin{aligned} &\iint_{-\infty}^{\infty}f(x)g(y)\left|\ln\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\,dy \\ &\quad< h\left(\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right)^{1/q}. \end{aligned} \tag{3.7}$$

For $\varepsilon > 0$, by (3.7), using Lemma 2.3, we have

$$k + o(1) < \varepsilon h\left(\int_{-\infty}^{\infty}|x|^{-1}\tilde{f}^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}\tilde{g}^q(x)dx\right)^{1/q} = k. \tag{3.8}$$

Hence, we find $k + o(1) < h$. For $\varepsilon \to 0^+$, it follows that $k \le h$, which contradicts the fact that $h < k$. Hence the constant k in (3.1) is the best possible.

Thus we complete the proof of the theorem. □

Remark 3.2. For $\alpha = \pi/4, \beta = \pi/3$ in (3.1), we have the following particular result:

$$\iint_{-\infty}^{\infty} f(x)g(y)\left|\ln\frac{x^2+\sqrt{2}xy+y^2}{x^2+xy+y^2}\right|dx\,dy < \frac{4\pi\sin(\pi r/24)\sin(5\pi r/24)}{r\sin(\pi r/2)}\left(\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right)^{1/q}. \quad (3.9)$$

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