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Research Article

On Linear Maps Preserving g-Majorization from $\mathbb{F}^n$ to $\mathbb{F}^m$

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Let $\mathbb{F}^n$ and $\mathbb{F}_m$ be the usual spaces of n-dimensional column and m-dimensional row vectors on $\mathbb{F}$, respectively, where $\mathbb{F}$ is the field of real or complex numbers. In this paper, the relations gs-majorization, lgw-majorization, and rgw-majorization are considered on $\mathbb{F}^n$ and $\mathbb{F}_m$. Then linear maps $T : \mathbb{F}^n \to \mathbb{F}^m$ preserving lgw-majorization or gs-majorization and linear maps $S : \mathbb{F}_n \to \mathbb{F}_m$, preserving rgw-majorization are characterized.

1. Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics. If x and y are n-vectors of real numbers such that $x = Dy$ for some doubly stochastic matrix D, then we say that x is (vector) majorized by y; see [1]. Marshall and Olkin's text [2] is the standard general reference for majorization. Some kinds of majorization such as multivariate or matrix majorization were motivated by the concepts of vector majorization and were introduced in [3]. Let V and W be two vector spaces over a field $\mathbb{F}$, and let $\sim$ be a relation on both V and W. We say that a linear map $T : V \to W$, preserves the relation $\sim$ if

$$Tx \sim Ty \quad \text{whenever } x \sim y. \tag{1.1}$$

The problem of describing these preserving linear maps is one of the most studied linear preserver problems. A lot of effort has been done in [4–9] and [10–12] to characterize the structure of majorization preserving linear maps on certain spaces of matrices. A complex $n \times m$ matrix R is said to be g-row (or g-column) stochastic, if $Re = e$ (or $R^t e = e$), where $e = (1, \ldots, 1)^t \in \mathbb{F}^n$ (or $e = (1, \ldots, 1)^t \in \mathbb{F}^m$). A complex $n \times n$ matrix D is said to be g-doubly stochastic if it is both g-row and g-column stochastic. The notaions of generalized majorization (g-majorization) were motivated by the matrix majorization and were introduced in [4–6] as follows.

Definition 1.1. Let x and y be two vectors in $\mathbb{F}^n$. It is said that

(1) x is gs-majorized by y if there exists an $n \times n$ g-doubly stochastic matrix D such that $x = Dy$, and denoted by $y \succ_{\mathrm{gs}} x$;

(2) x is lgw-majorized by y if there exists an $n \times n$ g-row stochastic matrix R such that $x = Ry$, and denoted by $y \succ_{\mathrm{lgw}} x$;

(3) x^t is rgw-majorized by y^t if there exists an $n \times n$ g-row stochastic matrix R such that $x^t = y^t R$, and denoted by $y^t \succ_{\mathrm{rgw}} x^t$ (here z^t is the transpose of z).

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ that preserve left matrix majorization or weak majorization were already characterized in [10, 11]. In this paper we characterize all linear maps preserving $\succ_{\mathrm{rgw}}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$ and all linear maps preserving $\succ_{\mathrm{lgw}}$ or $\succ_{\mathrm{gs}}$ from $\mathbb{F}^n$ to $\mathbb{F}^m$.

Throughout this paper, the standard bases of $\mathbb{F}^n$ and $\mathbb{F}_m$ are denoted by $\{e_1, \ldots, e_n\}$ and $\{\epsilon_1, \ldots, \epsilon_m\}$, respectively. The notation $\mathrm{tr}(x)$ is used for the sum of the components of a vector $x \in \mathbb{F}^n$ or $x \in \mathbb{F}_n$. The vector space of all $n \times m$ complex matrices is denoted by $\mathbf{M}_{n,m}$. The notations $[x_1/x_2/\cdots/x_n]$ and $[y_1 \mid y_2 \mid \cdots \mid y_m]$ are used for the $n \times m$ matrix with rows $x_1, x_2, \ldots, x_n \in \mathbb{F}_m$ and columns $y_1, y_2, \ldots, y_m \in \mathbb{F}^n$. The sets of g-row and g-column stochastic $m \times n$ matrices are denoted by $\mathbf{GR}_{m,n}$ and $\mathbf{GC}_{m,n}$, respectively. The set of g-doubly stochastic $n \times n$ matrices is denoted by $\mathbf{GD}_n$. The symbol $\mathbf{J}_n$ is used for the $n \times n$ matrix with all entries equal to one. The notation $[T]$ is used for the matrix representation of the linear map $T : V \to W$ with respect to the standard bases of V and W where $V, W \in \{\mathbb{F}^n, \mathbb{F}^m, \mathbb{F}_n, \mathbb{F}_m\}$.

2. Main Results

In this section we state some preliminary lemmas to describe the linear maps preserving $\succ_{\mathrm{rgw}}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$ and the linear maps preserving $\succ_{\mathrm{lgw}}$ or $\succ_{\mathrm{gs}}$ from $\mathbb{F}^n$ to $\mathbb{F}^m$.

Lemma 2.1. *Let $T : \mathbb{F}_n \to \mathbb{F}_m$ be a linear map. Then T preserves the subspace $\{x \in \mathbb{F}_n : \mathrm{tr}(x) = 0\}$ if and only if $[T] \in \mathbf{GR}_{m,n}$.*

Proof. Let $B = [b_{ij}] := [T]$. Assume that $Be = \lambda e$ for some $\lambda \in \mathbb{F}$. If $x \in \mathbb{F}_n$ and $\mathrm{tr}(x) = 0$, then $0 = xe = x(\lambda e) = x(Be) = (xB)e = \mathrm{tr}(xB) = \mathrm{tr}(Tx)$, so T preserves the subspace $\{x \in \mathbb{F}_n : \mathrm{tr}(x) = 0\}$. Conversely, assume that T preserves the subspace $\{x \in \mathbb{F}_n : \mathrm{tr}(x) = 0\}$. Then $\mathrm{tr}(T(\epsilon_1 - \epsilon_i)) = \mathrm{tr}((\epsilon_1 - \epsilon_i)B) = 0$ for every i $(1 \leq i \leq n)$. Therefore $Be = \lambda e$ where $\lambda = \sum_{k=1}^{n} b_{1k} = \sum_{k=1}^{n} b_{ik}$ for every i $(1 \leq i \leq n)$. □

The following lemma gives an equivalent condition for $\succ_{\mathrm{rgw}}$ on $\mathbb{F}_m$.

Lemma 2.2 (see [4, Lemma 2.2]). *Let $x, y \in \mathbb{F}_n$ and let $x \neq 0$. Then $x \succ_{rgw} y$ if and only if* $\mathrm{tr}(x) = \mathrm{tr}(y)$.

The following theorem characterizes all linear maps which preserve $\succ_{\mathrm{rgw}}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$. It is clear that every $T : \mathbb{F}_1 \to \mathbb{F}_m$ preserves $\succ_{\mathrm{rgw}}$, so assume that $n \geq 2$.

Theorem 2.3. *A nonzero linear map $T : \mathbb{F}_n \to \mathbb{F}_m$ preserves $\succ_{\mathrm{rgw}}$ if and only if $[T] \in \mathbf{GR}_{m,n}$ and $\{x \in \mathbb{F}_n : x[T] = 0\} = \{x \in \mathbb{F}_n : \mathrm{tr}(x) = 0\}$ or $\{0\}$.*

Proof. Put $B := [T]$. Let $Be = \lambda e$ for some $\lambda \in \mathbb{F}$. If $\{x \in \mathbb{F}_n : xB = 0\} = \{x \in \mathbb{F}_n : \operatorname{tr}(x) = 0\}$ it is clear that T preserves $\succ_{\mathrm{rgw}}$. If $\{x \in \mathbb{F}_n : xB = 0\} = \{0\}$, $x \succ_{\mathrm{rgw}} y$ and $x \neq 0$ then $Tx \neq 0$ and by Lemma 2.2, $\operatorname{tr}(x) = \operatorname{tr}(y)$. So $\operatorname{tr}(x-y) = 0$ and hence $\operatorname{tr}(T(x-y)) = 0$ by Lemma 2.1. Therefore $Tx \succ_{\mathrm{rgw}} Ty$ by Lemma 2.2 and so T preserves $\succ_{\mathrm{rgw}}$. Now, we prove the necessity of the conditions. Let $T : \mathbb{F}_n \to \mathbb{F}_m$ be a linear preserver of $\succ_{\mathrm{rgw}}$. If $\operatorname{tr}(x) = 0$, then $x \succ_{\mathrm{rgw}} 0$ by Lemma 2.2. So $Tx \succ_{\mathrm{rgw}} T0 = 0$ and hence $\operatorname{tr}(Tx) = 0$ by Lemma 2.2. Therefore T preserves the subspace $\{x \in \mathbb{F}_n : \operatorname{tr}(x) = 0\}$ and so $B \in \mathbf{GR}_{m,n}$ by Lemma 2.1. If $\{x \in \mathbb{F}_n : xB = 0\} \neq \{0\}$, then there exists a nonzero vector $a \in \mathbb{F}_n$ such that $Ta = aB = 0$. If $\operatorname{tr}(a) = \delta \neq 0$ then $a \succ_{\mathrm{rgw}} \delta\epsilon_j$ for every j $(1 \leq j \leq n)$, by Lemma 2.2. Then $Ta = 0 \succ_{\mathrm{rgw}} \delta T\epsilon_j$ for every j $(1 \leq j \leq n)$ and hence $T = 0$ which is a contradiction. Therefore $\operatorname{tr}(a) = 0$ and hence $a \succ_{\mathrm{rgw}} (\epsilon_1 - \epsilon_j)$ for every j $(1 \leq j \leq n)$, by Lemma 2.2. Then $Ta = 0 \succ_{\mathrm{rgw}} T(\epsilon_1 - \epsilon_j)$ and so $T\epsilon_1 = T\epsilon_j$ for every j $(1 \leq j \leq n)$. Put $b := T\epsilon_1 = \epsilon_1 B$. Thus $B = [b/\cdots/b]$ and hence $\{x \in \mathbb{F}_n : xB = 0\} = \{x \in \mathbb{F}_n : \operatorname{tr}(x) = 0\}$. □

We use the following lemmas to find the structure of linear preservers of lgw-majorization.

Remark 2.4 (see [7, Lemma 2.2]). If $x \notin \operatorname{Span}\{e\}$, then $x \succ_{\mathrm{lgw}} y$, for all $y \in \mathbb{F}^n$.

Lemma 2.5. *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. If $x \notin \operatorname{Span}\{e\}$ implies $Tx \notin \operatorname{Span}\{e\}$, then T preserves $\succ_{\mathrm{lgw}}$.*

Proof. Let $x, y \in \mathbb{F}^n$ and $x \succ_{\mathrm{lgw}} y$. If $x \in \operatorname{Span}\{e\}$ then $y = x$ and it is clear that $Tx \succ_{\mathrm{lgw}} Ty$. If $x \notin \operatorname{Span}\{e\}$ so $Tx \notin \operatorname{Span}\{e\}$ by the hypothesis and hence $Tx \succ_{\mathrm{lgw}} Ty$, by Remark 2.4. Therefore T preserves $\succ_{\mathrm{lgw}}$. □

Lemma 2.6. *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a nonzero singular linear map. Then T preserves $\succ_{\mathrm{lgw}}$ if and only if $\operatorname{Ker}(T) = \operatorname{Span}\{e\}$ and $e \notin \operatorname{Im}(T)$.*

Proof. Let T be a linear preserver of $\succ_{\mathrm{lgw}}$. If $x \in \operatorname{Ker}(T)$ and $x \notin \operatorname{Span}\{e\}$, then $Tx = 0$ and $x \succ_{\mathrm{lgw}} y$, for all $y \in \mathbb{F}^n$ by Remark 2.4. So $Ty = 0$, for all $y \in \mathbb{F}^n$, which is a contradiction. Therefore $\operatorname{Ker}(T) \subset \operatorname{Span}\{e\}$ and since $\operatorname{Ker}(T) \neq \{0\}$, $\operatorname{Ker}(T) = \operatorname{Span}\{e\}$. If $e \in \operatorname{Im}(T)$, then there exists $x \in \mathbb{F}^n$ such that $Tx = e$ and $x \notin \operatorname{Span}\{e\}$. Therefore $x \succ_{\mathrm{lgw}} y$, for all $y \in \mathbb{F}^n$, and hence $Ty = e$ for all $y \in \mathbb{F}^n$, which is a contradiction. So $e \notin \operatorname{Im}(T)$. The converse follows from Lemma 2.5. □

Proposition 2.7. *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a nonzero linear preserver of $\succ_{\mathrm{lgw}}$. Then $n \leq m$.*

Proof. If T is injective, then $n \leq m$. If T is not injective, we obtain $\operatorname{Ker}(T) = \operatorname{Span}\{e\}$ by Lemma 2.6 and $e \notin \operatorname{Im}(T)$. Therefore $n \leq m$, by the rank and nullity theorem. □

Theorem 2.8. *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a nonzero linear map and $A := [T]$. Then T preserves $\succ_{\mathrm{lgw}}$ if and only if one of the following holds:*

(i) $\{x : Ax \in \operatorname{Span}\{e\}\} = \{0\}$,

(ii) $A \in \operatorname{Span}\{\mathbf{GR}_{n,m}\}$ *and* $\{x : Ax \in \operatorname{Span}\{e\}\} = \operatorname{Span}\{e\}$.

Proof. If (i) or (ii) holds, it is easy to show that T preserves $\succ_{\mathrm{lgw}}$ by Lemmas 2.5 and 2.6. Conversely, assume that T preserves $\succ_{\mathrm{lgw}}$. If (i) does not hold, we show that (ii) holds. Since (i) does not hold, there exists a nonzero vector $b \in \mathbb{F}^n$ such that $Tb = Ab = \mu e$ for some $\mu \in \mathbb{F}$. If $b \notin \operatorname{Span}\{e\}$, then $b \succ_{\mathrm{lgw}} x$, for all $x \in \mathbb{F}^n$ by Remark 2.4. So $Tb \succ_{\mathrm{lgw}} Tx$, for all $x \in \mathbb{F}^n$

and hence $T = 0$, which is a contradiction. Then $b = \lambda e$ for some nonzero $\lambda \in \mathbb{F}$, and hence $Ae = (\mu/\lambda)e$. Therefore, $A \in \mathrm{Span}\{\mathbf{GR}_{n,m}\}$ and $\{x : Ax \in \mathrm{Span}\{e\}\} = \mathrm{Span}\{e\}$. □

The following examples show that Proposition 2.7 does not hold for $\succ_{\mathrm{gs}}$ or $\succ_{\mathrm{rgw}}$.

Example 2.9. For any positive integer n, the linear map $T : \mathbb{F}^n \to \mathbb{F}$ defined by $Tx = \mathrm{tr}(x)$, preserves $\succ_{\mathrm{gs}}$.

Example 2.10. The linear map $T : \mathbb{F}_3 \to \mathbb{F}_2$ defined by $Tx = xB$, where $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}^t$, preserves rgw-majorization.

We use the following statements to find the structure of linear preservers of gs-majorization.

Lemma 2.11 (see [6, Proposition 2.1]). *Let x and y be two distinct vectors in $\mathbb{F}^n$. Then $y \succ_{\mathrm{gs}} x$ if and only if $y \notin \mathrm{Span}\{e\}$ and $\mathrm{tr}(x) = \mathrm{tr}(y)$.*

Lemma 2.12. *If a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ preserves $\succ_{gs}$, then $[T] \in \mathrm{Span}\{\mathbf{GC}_{m,n}\}$.*

Proof. Let $A := [T]$. For every i, j $(1 \le i \ne j \le n)$, it is clear that $(e_i - e_j) \succ_{\mathrm{gs}} 0$ by Lemma 2.11. Then $A(e_i - e_j) \succ_{\mathrm{gs}} 0$ and hence there exists $D \in \mathbf{GD}_m$ such that $DA(e_i - e_j) = 0$. So $\mathbf{J}_m A(e_i - e_j) = \mathbf{J}_m D(Ae_i - Ae_j) = 0$ and therefore $A \in \mathrm{Span}\{\mathbf{GC}_{m,n}\}$. □

Theorem 2.13. *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Then T preserves $\succ_{\mathrm{gs}}$ if and only if one of the following holds:*

(i) *there exists some $a \in \mathbb{F}^m$ such that $Tx = \mathrm{tr}(x)a$, for all $x \in \mathbb{F}^n$,*

(ii) *$\lambda[T] \in \mathbf{GR}_{m,n} \cap \mathrm{Span}\{\mathbf{GC}_{m,n}\}$ for some $0 \ne \lambda \in \mathbb{F}$ and $\mathrm{Ker}(T) \subset \mathrm{Span}\{e\}$,*

(iii) *$[T] \in \mathrm{Span}\{\mathbf{GC}_{m,n}\}$ and $e \notin \mathrm{Im}([T])$.*

Proof. Let $A := [T]$. Assume that T preserves $\succ_{\mathrm{gs}}$. So $A \in \mathrm{Span}\{\mathbf{GC}_{m,n}\}$ by Lemma 2.12. Now, we consider two cases.

Case 1. Suppose there exists $b \in \mathbb{F}^n \setminus \mathrm{Span}\{e\}$ such that $Tb = Ab = \lambda e$ for some $\lambda \in \mathbb{F}$. If $\mathrm{tr}(b) = 0$, then $0 = \mathrm{tr}(b)e = J_m b = (J_m A)b = J_m(Ab) = J_m(Tb) = J_m(\lambda e)$. So $\lambda = 0$ and hence $Ab = 0$. For every i, j $(1 \le i \ne j \le n)$, $b \succ_{\mathrm{gs}} (e_i - e_j)$ by Lemma 2.11. Then $0 = Ab \succ_{\mathrm{gs}} A(e_i - ej)$ and hence $Ae_i = Ae_j$, for all i, j $(1 \le i, j \le n)$. Then $A = [a \mid \cdots \mid a]$, for some $a \in \mathbb{F}^m$ and hence $T(x) = \mathrm{tr}(x)a$ for all $x \in \mathbb{F}^n$. If $\mathrm{tr}(b) = \delta \ne 0$, consider the basis $\{\delta e_1, \ldots, \delta e_n\}$ for $\mathbb{F}^n$. For every i $(1 \le i \le n)$, $b \succ_{\mathrm{gs}} (\delta e_i)$, by Lemma 2.11. Consequently $Te_i = (\lambda/\delta)e$ for every i $(1 \le i \le n)$ and hence $Tx = \mathrm{tr}(x)a$ for all $x \in \mathbb{F}^n$, where $a = (\lambda/\delta)e$. Therefore, (i) holds in this case.

Case 2. Assume that $x \notin \mathrm{Span}\{e\}$ implies $Tx \notin \mathrm{Span}\{e\}$. Since $e_1 \succ_{\mathrm{gs}} e_i$, we have $T(e_1) \succ_{\mathrm{gs}} T(e_i)$ for every i $(1 \le i \le n)$. Thus it follows that $\mathrm{tr}(A_i) = \mathrm{tr}(Te_i) = \mathrm{tr}(Te_1) = \mathrm{tr}(A_1)$ for every i $(1 \le i \le n)$, where A_i is the ith column of A and hence $A \in \mathrm{Span}\{\mathbf{GC}_{m,n}\}$. If $e \in \mathrm{Im}(A)$, then there exists $0 \ne \lambda \in \mathbb{F}$ such that $A(\lambda e) = e$ and hence $\lambda A \in \mathbf{GR}_{m,n} \cap \mathrm{Span}\{\mathbf{GC}_{m,n}\}$. By the hypothesis of this case, $\mathrm{Ker}(T) \subset \mathrm{Span}\{e\}$. Then (ii) holds. If $e \notin \mathrm{Im}(A)$ it is clear (iii) holds.

Conversly, if (i) or (iii) holds it is easy to show that T preserves gs-majorization. Suppose that (ii) holds. Then there exists $z \in \operatorname{Span}\{e\}$ such that $Tz = e$. Assume that $x \succ_{\text{gs}} y$. If $Tx \notin \operatorname{Span}\{e\}$ then $Tx \succ_{\text{gs}} Ty$ by Lemma 2.11. If $Tx \in \operatorname{Span}\{e\}$, then there exists $\mu \in \mathbb{F}$ such that $Tx = \mu e$ and hence $T(x - \mu z) = 0$. Therefore, $x - \mu z \in \operatorname{Span}\{e\}$, and hence $x \in \operatorname{Span}\{e\}$. Then $x = y$ and hence T preserves gs-majorization. □

Corollary 2.14. *If* $T : \mathbb{F}^n \to \mathbb{F}^m$ *preserves* $\succ_{\text{gs}}$ *and* $\operatorname{rank}(T) > 1$ *then* $n \leq m$.

Proof. If T is injective it is clear that $n \leq m$. Assume that T is not injective, so there exists a nonzero vector $b \in \mathbb{F}^n$ such that $Tb = 0$. If $b \notin \operatorname{Span}\{e\}$, then by Case 1 in the proof of Theorem 2.13, $Tx = \operatorname{tr}(x)a$ for some $a \in \mathbb{F}^m$. Therefore, $\operatorname{rank}(T) \leq 1$, which is a contradiction. So $b \in \operatorname{Span}\{e\}$ and hence $\operatorname{Ker}(T) = \operatorname{Span}\{e\}$. It is clear that $e \notin \operatorname{Im}(T)$, from which and the rank and nullity theorem, we obtain $n \leq m$, completing the proof. □

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