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Research Article

Existence Results for System of Variational Inequality Problems with Semimonotone Operators

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We introduce the system of variational inequality problems for semimonotone operators in reflexive Banach space. Using the Kakutani-Fan-Glicksberg fixed point theorem, we obtain some existence results for system of variational inequality problems for semimonotone with finite-dimensional continuous operators in real reflexive Banach spaces. The results presented in this paper extend and improve the corresponding results for variational inequality problems studied in recent years.

1. Introduction

Let E be a Banach space, let E^* be the dual space of E, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing of E^* and E. If E is a Hilbert space and K is a nonempty, closed, and convex subset of E, then let, K be a nonempty, closed, and convex subset of a Hilbert space H and let $A : K \to H$ be a mapping. *The classical variational inequality problem*, denoted by $\mathrm{VIP}(A, K)$, is to find $x^* \in K$ such that

$$\langle Ax^*, z - x^* \rangle \geq 0 \tag{1.1}$$

for all $z \in K$.

The variational inequality problem (VIP) has been recognized as suitable mathematical models for dealing with many problems arising in different fields, such as optimization theory, game theory, economic equilibrium, mechanics. In the last four decades, since the time of the celebrated Hartman Stampacchia theorem (see [1, 2]), solution existence of variational inequality and other related problems has become a basic research topic which continues to attract attention of researchers in applied mathematics (see, e.g., [3–14] and the references therein). Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the current interest in functional analysis. It is natural to consider a unified approach to these different problems; see, for example, [10, 11].

Let K be a nonempty, closed, and convex subset of E and let $A, B : K \to E^*$ be single valued. For the system of generalized variational inequality problem (SGVIP), find $(x^*, y^*) \in K \times K$ such that

$$\begin{aligned} &\langle Ay^*, z - x^* \rangle \geq 0, \quad \forall z \in K, \\ &\langle Bx^*, z - y^* \rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{1.2}$$

There are many kinds of mappings in the literature of recent years; see, for example, [12, 13, 15–18]. In 1999, Chen [19] introduced the concept of semimonotonicity for a single valued mapping, which occurred in the study of nonlinear partial differential equations of divergence type. Recently, Fang and Huang [20] introduced two classes of variational-like inequalities with generalized monotone mappings in Banach spaces. Using the KKM technique, they obtained the existence of solutions for variational-like inequalities with relaxed monotone mappings in reflexive Banach spaces. Moreover, they present the solvability of variational-like inequalities with relaxed semimonotone mappings in arbitrary Banach spaces by means of the Kakutani-Fan-Glicksberg fixed point theorem.

On the other hand, some interesting and important problems related to variational inequalities and complementarity problems were considered in recent papers. In 2004, Cho et al. [21], introduced, and studied a system of nonlinear variational inequalities. They proved the existence and uniqueness of solution for this problem and constructed an iterative algorithm for approximating the solution of system of nonlinear variational inequalities. In 2000, [22], systems of variational inequalities were introduced and an existence theorem was obtained by Ky Fan lemma. In 2002, Kassay et al. [23] introduced and studied Minty and Stampacchia variational inequality systems by the Kakutani-Fan-Glicksberg fixed point theorem. Very recently, Fang and Huang [20] introduced and studied systems of strong implicit vector variational inequalities by the same fixed point theorem. Zhao and Xia [24] introduced and established some existence results for systems of vector variational-like inequalities in Banach spaces by also using the Kakutani-Fan-Glicksberg fixed point theorem.

Let $A : E \times E \to E^*$ be a semimonotone mapping and $K \subset E$ a closed convex set. The *variational inequality problem* (VIP) is to find $u \in K$ such that

$$\langle A(u,u), z - u \rangle \geq 0, \quad \forall z \in K. \tag{1.3}$$

Let K be a nonempty closed convex subset of a real reflexive Banach space E with dual space E^*, and let $A, B : K \times K \to E^*$ be two semimonotone mappings. We consider the following as *the system of variational inequality problem* (SVIP). Find $(u, v) \in K \times K$ such that

$$\begin{aligned} &\langle A(u,v), z - u \rangle \geq 0, \quad \forall z \in K, \\ &\langle B(u,v), z - v \rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{1.4}$$

In particular, if setting $B : K \times K \to E^*$ by $B(u, v) = A(v, v)$ for all $u, v \in K$, then the system of variational inequality problem reduces to *the variational inequality problem* (VIP).

In this paper, we introduce the system of generalized variational inequality in real reflexive Banach space. By using the Kakutani-Fan-Glicksberg fixed point theorem, we obtain some existence results for system of generalized variational inequality for semimonotone and finite dimensional continuous in real reflexive Banach spaces. The results presented in this paper extend and improve the corresponding results of Chen [19] and many others.

2. Preliminaries

In this section, let E be a real Banach space, and let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if, for any $x, y \in S$,

$$x \neq y \quad \text{implies} \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in S$,

$$\|x-y\| \geq \varepsilon \quad \text{implies} \left\| \frac{x+y}{2} \right\| < 1-\delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex, and we define a function $\delta : [0,2] \to [0,1]$ called the *modulus of convexity of* E as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \ \|x\| = \|y\| = 1, \ \|x-y\| \geq \varepsilon \right\}. \tag{2.3}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.4}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in S$. We recall that E is uniformly convex if and only if E^* is uniformly smooth. It is well known that E is smooth if and only if E^* is strictly convex.

Definition 2.1 (KKM mapping). Let K be a nonempty subset of a linear space E. A set-valued mapping $G : K \to 2^E$ is said to be a KKM mapping if, for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of K, we have

$$\operatorname{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i), \tag{2.5}$$

where $\operatorname{co}\{y_1, y_2, \ldots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \ldots, y_n\}$.

Lemma 2.2 (Fan-KKM Theorem). *Let K be a nonempty convex subset of a Hausdorff topological vector space E, and let $G : K \to 2^E$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is a compact subset of K, then $\bigcap_{y\in K} G(y) \neq \emptyset$.*

Definition 2.3. Let X and Y be two topological vector spaces and K a nonempty convex subset of X. A set-valued mapping $F : K \to 2^Y$ is said to be properly C-quasiconvex if, for any $x, y \in K$ and $t \in [0,1]$, we have either $F(x) \subset F(tx+(1-t)y)+C$ or $F(y) \subset F(tx+(1-t)y)+C$.

Definition 2.4. Let X and Y be two topological vector spaces, and $T : X \to 2^Y$ be a set-valued mapping.

(i) T is said to be upper semicontinuous at $x \in X$ if, for any open set V containing $T(x)$, there exists an open set U containing x such that, for all $t \in U$, $T(t) \in V$; T is said to be upper semicontinuous on X if it is upper semicontinuous at all $x \in X$.

(ii) T is said to be lower semicontinuous at $x \in X$ if, for any open set V with $T(x) \cap V \neq \emptyset$, there exists an open set U containing x such that, for all $t \in U$, $T(x) \cap V \neq \emptyset$; T is said to be lower semicontinuous on X if it is lower semicontinuous at all $x \in X$.

(iii) T is said to be continuous on X if it is at the same time upper semicontinuous and lower semicontinuous on X.

(iv) T is said to be closed if the graph, Graph(T), of T, that is, $\operatorname{Graph}(T) = \{(x,y) : x \in X \text{ and } y \in T(x)\}$, is a closed set in $X \times Y$.

Lemma 2.5 (see [25]). *Let X and Y be two Hausdorff topological vector spaces, and let $T : X \to 2^Y$ be a set-valued mapping. Then the following properties hold.*

(i) *If T is closed and $\overline{T(X)}$ is compact, then T is upper semicontinuous, where $T(X) = \bigcup_{x\in X} T(x)$ and $\overline{E}$ denotes the closure of the set E.*

(ii) *If T is upper semicontinuous and, for any $x \in X$, $T(x)$ is closed, then T is closed.*

(iii) *T is lower semicontinuous at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$, $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.6 (see Kakutani-Fan-Glicksberg [26]). *Let K be a nonempty compact subset of locally convex Hausdorff vector topology space E. If $S : K \to 2^K$ is upper semicontinuous and, for any $x \in K$, $S(x)$ is nonempty convex and closed, then there exists an $x^* \in K$ such that $x^* \in S(x^*)$.*

Lemma 2.7 (see [27]). *For each $\lambda > 0$, φ_λ is a Fréchet differentiable convex function on H, and the Fréchet derivative $\partial_{\varphi_\lambda}$ of φ_λ is equal to Yosida approximation $(\partial_\varphi)_\lambda = (1/\lambda)(1 - J_\lambda)$ of ∂_φ. More precisely,*

$$0 \leq \varphi_\lambda(v) - \varphi_\lambda(u) - \left(\partial_{\varphi_\lambda(u)}, v-u\right) \leq \frac{1}{\lambda}\|v-u\|^2 \tag{2.6}$$

holds for $\lambda > 0$ and $u, v \in H$.

Let K be a nonempty closed convex subset of a real reflexive Banach space E with dual space E^*. A mapping $A : E \to E^*$ is said to be *monotone* if it satisfies

$$\langle A(v) - A(w), v-w\rangle \geq 0, \quad \forall u, v, w \in E. \tag{2.7}$$

Definition 2.8 (see [19]). A mapping $A : E\times E \to E^*$ is said to be *semimonotone* if it satisfies the following:

(a) for each $u \in E$, $A(u,\cdot)$ is monotone; that is, $\langle A(u,v) - A(u,w), v - w\rangle \geq 0$, for all $u, v, w \in E$;

(b) For each fixed $v \in E$, $A(\cdot, v)$ is completely continuous; that is, if $u_j \to u_0$ in weak topology of E, then $\{A(u_j, v)\}$ has a subsequence $A(u_{j_k}, v) \to A(u_0, v)$ in norm topology of E.

An operator $A : D(A) \subset E \to E^*$ is said to be *hemicontinuous at* $x_0 \in D(A)$, if, for any $y \in E$, $t \in (0,+\infty)$ with $x_0+ty \in E$, we have $\langle A(x_0+ty), z\rangle \to \langle Ax_0, z\rangle$ for all $z \in E$ at $t \to 0^+$.

Lemma 2.9. *Let $A : K \subseteq E \to E^*$ be a hemicontinuous monotone operator, let K be a convex subset, and let $f : E \to (-\infty,+\infty)$ be a convex function and $x_0 \in K$ a given point. Then*

$$\langle Ax_0, x - x_0\rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K, \tag{2.8}$$

if and only if

$$\langle Ax, x - x_0\rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K. \tag{2.9}$$

Proof. Let $x_0 \in K$ such that

$$\langle Ax_0, x - x_0\rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K. \tag{2.10}$$

By the monotonicity of A, we have

$$\langle Ax, x - x_0\rangle + f(x) - f(x_0) \geq \langle Ax_0, x - x_0\rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K. \tag{2.11}$$

On the other hand, suppose that

$$\langle Ax, x - x_0\rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K. \tag{2.12}$$

For any given $x \in K$ and any $t \in (0,1]$, taking $z = tx + (1-t)x_0 \in K$ since K is convex and Replacing x by z into the above inequality, one has

$$\langle Az, z - x_0\rangle + f(z) - f(x_0) \geq 0, \quad \forall z \in K. \tag{2.13}$$

It follows from the convexity of f on K that

$$\begin{aligned}
\langle Az, z - x_0\rangle + f(z) - f(x_0) &= \langle A(tx + (1-t)x_0), tx + (1-t)x_0 - x_0\rangle + f(tx + (1-t)x_0) - f(x_0)\\
&\leq \langle A(x_0 + t(x - x_0), t(x - x_0))\rangle + tf(x) + (1-t)f(x_0) - f(x_0)\\
&= t\big[\langle A(x_0 + t(x - x_0)), x - x_0\rangle + f(x) - f(x_0)\big]\\
&= \langle A(x_0 + t(x - x_0)), x - x_0\rangle + f(x) - f(x_0).
\end{aligned} \tag{2.14}$$

Letting $t \to 0^+$ and using the hemicontinuity of A, we have

$$\langle Ax_0, x - x_0 \rangle + f(x) - f(x_0) \geq 0, \quad \forall x \in K. \tag{2.15}$$

This completes the proof. □

3. The Existence of the System of Generalized Variational Inequality

In this section, we prove two existence theorems for system of variational inequality problems for semimonotone with finite dimensional continuous operators in real reflexive Banach spaces. First, we prove an existence theorem for system of variational inequality problems for continuous mappings as follows.

Theorem 3.1. *Let E be a reflexive Banach spac, let K be a compact convex subset of E, and let $A, B : K \to E^*$ be two continuous mappings. Then the problem* (1.2) *has a solution and the set of solutions of* (1.2) *is closed.*

Proof. Fix $x \in K$, for each $z \in K$, the sets $G_x(z)$ and $H_x(z)$ are defined as follows:

$$\begin{aligned} G_x(z) &:= \{y \in K : \langle Ax, z - y \rangle \geq 0\}, \\ H_x(z) &:= \{y \in K : \langle Bx, z - y \rangle \geq 0\}. \end{aligned} \tag{3.1}$$

Step 1 (Show that $G_x(z)$ and $H_x(z)$ are nonempty compact convex subsets of K). For any $z \in K$, we note that $z \in G_x(z)$ and $z \in H_x(z)$. Thus, $G_x(z)$ and $H_x(z)$ are nonempty subsets of K. Moreover, it follows from the definitions of $G_x(z)$ and $H_x(z)$ that both of them are compact convex subsets of K.

Step 2 (Show that $G_x(z)$ and $H_x(z)$ are KKM mappings). For any finite set $\{z_1, z_2, \ldots, z_n\} \in K$, we claim that $\operatorname{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{j=1}^n G_x(z_j)$ and $\operatorname{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{j=1}^n H_x(z_j)$. Let $z \in \{z_1, z_2, \ldots, z_n\}$. Then $z = \sum_{j=1}^n \lambda_j z_j$, where $\lambda_j \in [0,1]$ and $\sum_{j=1}^n \lambda_j = 1$. We observe that

$$\begin{aligned} \sum_{j=1}^n \lambda_j \langle Ax, z_j - z \rangle &\geq \left\langle Ax, \sum_{j=1}^n \lambda_j (z_j - z) \right\rangle = \left\langle Ax, \sum_{j=1}^n \lambda_j z_j - \sum_{j=1}^n \lambda_j z \right\rangle \\ &= \langle Ax, z - z \rangle = 0. \end{aligned} \tag{3.2}$$

So, there is at least one number $j = 1, 2, \ldots, n$ such that $\langle Ax, z_j - z \rangle \geq 0$. Therefore, $z \in G_x(z_j) \subset \bigcup_{j=1}^n G_x(z_j)$. Similarly, we obtain that $z \in H_x(z_j) \subset \bigcup_{j=1}^n H_x(z_j)$. Hence, we have $\operatorname{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{j=1}^n G_x(z_j)$ and $\operatorname{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{j=1}^n H_x(z_j)$. This implies that $G_x(\cdot)$ and $H_x(\cdot)$ are KKM mappings.

Step 3 (Show that $G_x(z)$ and $H_x(z)$ are closed for all $z \in K$). Let $\{x_n\}$ be a sequence in $G_x(z)$ such that $x_n \to x_0$. Then

$$\langle Ax, z - x_n \rangle \geq 0, \quad \forall z \in K. \tag{3.3}$$

Since $\langle \cdot, \cdot \rangle$ and A are continuous, we have $\langle Ax, z - x_0 \rangle \geq 0$ for all $z \in K$. Thus, we see that $x_0 \in G_x(z)$. This implies that $G_x(z)$ is closed for all $z \in K$. Similarly, we note that $H_x(z)$ is closed for all $z \in K$.

Step 4 (Show that $\bigcap_{z\in K} G_x(z) \neq \emptyset \neq \bigcap_{z\in K} H_x(z)$). Since $G_x(z)$ and $H_x(z)$ are closed subsets of K and K is compact, it follows that $G_x(z)$ and $H_x(z)$ are compact subsets of K. It follows from Lemma 2.2 that $\bigcap_{z\in K} G_x(z) \neq \emptyset \neq \bigcap_{z\in K} H_x(z)$. Moreover, we note that $\bigcap_{z\in K} G_y(z)$ and $\bigcap_{z\in K} H_x(z)$ are closed and convex.

Step 5 (Show that the problem (1.2) has a solution). Define the set-valued mapping $S : K \times K \to 2^{K\times K}$ by

$$S(x,y) = \left(\left\{\bigcap_{z\in K} G_x(z)\right\}, \left\{\bigcap_{z\in K} H_x(z)\right\}\right), \quad \forall (x,y) \in K \times K. \tag{3.4}$$

From Step 4, we note that $S(x,y)$ is nonempty closed convex subset of $K \times K$ for all $(x,y) \in K \times K$. Since $\bigcap_{z\in K} G_y(z), \bigcap_{z\in K} H_x(z) \subset K$, and K is compact, $\overline{\bigcap_{z\in K} G_y(z)}$ and $\overline{\bigcap_{z\in K} H_x(z)}$ are compact. It follows from Lemma 2.5(i) that S is upper semicontinuous. Hence, by the Kakutani-Fan-Glicksberg theorem, there exists a point $(x^*, y^*) \in S(x^*, y^*) = (\{\bigcap_{z\in K} G_x(z)\}, \{\bigcap_{z\in K} H_x(z)\})$; that is, $x^* \in G_{y^*}(z)$ and $y^* \in H_{x^*}(z)$ for all $z \in K$. By definition of $G_{y^*}(z)$ and $H_{x^*}(z)$, we get

$$\begin{aligned} &\langle Ay^*, z - x^* \rangle \geq 0, \quad \forall z \in K, \\ &\langle Bx^*, z - y^* \rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.5}$$

Hence, (x^*, y^*) are the solutions of problem (1.2).

Step 6 (Show that the set of solutions of problem (1.2) is closed). Let $\{(x_n, y_n)\}$ be a net in the set of solutions of problem (1.2) such that $(x_n, y_n) \to (x_0, y_0)$. By definition of the set of solutions of problem (1.2) we obtain that

$$\begin{aligned} &\langle Ay_n, z - x_n \rangle \geq 0, \quad \forall z \in K, \\ &\langle Bx_n, z - y_n \rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.6}$$

Since A, B are continuous and K is compact, it follows that $(x_0, y_0) \in K \times K$ and

$$\begin{aligned} &\langle Ay_0, z - x_0 \rangle \geq 0, \quad \forall z \in K, \\ &\langle Bx_0, z - y_0 \rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.7}$$

This mean that (x_0, y_0) belongs to the set of solution of problem (1.2). Hence, the set of solution of problem (1.2) is closed set. This completes the proof. □

Theorem 3.2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with dual space E^*. Suppose that $f : E \to (-\infty, +\infty)$ is a lower semicontinuous convex function with*

$K \subseteq D(f)$. Let $A : K \times K \to E^*$ and $B : K \times K \to E^*$ be two mappings satisfying the following conditions:

(i) *for each $z \in K$, $A(\cdot, z)$ and $B(z, \cdot)$ are monotone;*

(ii) *for each $z \in K$, $A(z, \cdot)$ and $B(\cdot, z)$ are completely continuous;*

(iii) *for any given $z \in K$, $A(\cdot, z) : K \to E^*$ and $B(z, \cdot) : K \to E^*$ are finite dimensional continuous; that is, for any finite dimensional subspace $M \subset E, A(\cdot, z)$ and $B(z, \cdot) : K \cap M \to E^*$ are continuous.*

Then, there exists $(u, v) \in K \times K$ such that

$$\begin{aligned} &\langle A(u, v), z - u\rangle + f(z) - f(u) \geq 0, \quad \forall z \in K, \\ &\langle B(u, v), z - v\rangle + f(z) - f(v) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.8}$$

Proof. Let ∂f be the subdifferential of f and $(\partial f)_\lambda$ the Yosida approximation of ∂f. Let F be a finite dimensional subspace of E with $F \cap K := K_F \neq \emptyset$. For any given $(u, v) \in K \times K$, we consider the following systems of variational inequalities $(\text{SGVIP})_F$. Find $(u_0, v_0) \in K_F \times K_F$ such that

$$\begin{aligned} &\langle A(u_0, v), z - u_0\rangle + \left((\partial f)_0 u_0, z - u_0\right) \geq 0, \quad \forall z \in K_F, \\ &\langle B(u, v_0), z - v_0\rangle + \left((\partial f)_0 v_0, z - v_0\right) \geq 0, \quad \forall z \in K_F. \end{aligned} \tag{3.9}$$

Since $K_F \times K_F \subset F \times F$ is nonempty bounded closed convex and $A(\cdot, v)$ and $B(v, \cdot)$ are continuous on $K_F \times K_F$ for each fixed $v \in K$, it follows from Theorem 3.1 that $(\text{SGVIP})_F$ has a solution $(u_0, v_0) \in K_F \times K_F$. By Lemma 2.9 and the Yosida approximation of ∂f, we have

$$\begin{aligned} &\langle A(z, v), z - u_0\rangle + f(z) - f(u_0) \geq 0, \quad \forall z \in K_F, \\ &\langle B(u, z), z - v_0\rangle + f(z) - f(v_0) \geq 0, \quad \forall z \in K_F. \end{aligned} \tag{3.10}$$

Now, we defined a mapping $T : K_F \times K_F \to 2^{K_F \times K_F}$ by the following:

$$T(u, v) = \left\{(u_0, v_0) \in K_F \times K_F : (u_0, v_0) \text{ solves problem } (\text{SGVIP})_F\right\}. \tag{3.11}$$

Next, we will show that this mapping has at least one fixed point in $K_F \times K_F$. To prove this, we need the following conditions.

(1) For all $(u, v) \in K_F \times K_F$, $T(u, v) \neq \emptyset$ as $(\text{SGVIP})_F$ has a solution.

(2) $T(u, v)$ is convex for all $(u, v) \in K_F \times K_F$.

Let $(u_1, v_1), (u_2, v_2) \in T(u, v)$ and $\lambda \in [0, 1]$. Thus, we have

$$\begin{aligned}
\lambda\left[\langle A(z,v), z-u_1\rangle + f(z) - f(u_1)\right] &\geq 0, \quad \forall z \in K_F, \\
\lambda\left[\langle B(u,z), z-v_1\rangle + f(z) - f(v_1)\right] &\geq 0, \quad \forall z \in K_F, \\
(1-\lambda)\left[\langle A(z,v), z-u_2\rangle + f(z) - f(u_2)\right] &\geq 0, \quad \forall z \in K_F, \\
(1-\lambda)\left[\langle B(u,z), z-v_2\rangle + f(z) - f(v_2)\right] &\geq 0, \quad \forall z \in K_F.
\end{aligned} \tag{3.12}$$

Adding (3.12), we get

$$\begin{aligned}
\langle A(z,v), z-(\lambda u_1 + (1-\lambda)u_2)\rangle + f(z) - \left[\lambda f(u_1) + (1-\lambda)f(u_2)\right] &\geq 0, \quad \forall z \in K_F, \\
\langle B(u,z), z-(\lambda v_1 + (1-\lambda)v_2)\rangle + f(z) - \left[\lambda f(v_1) + (1-\lambda)f(v_2)\right] &\geq 0, \quad \forall z \in K_F.
\end{aligned} \tag{3.13}$$

It implies that $T(u, v)$ is convex.

(3) $T(u, v)$ is closed for all $(u, v) \in K_F \times K_F$.

In fact, leting $\{(u_\alpha, v_\alpha)\} \in T(u, v)$ such that $(u_\alpha, v_\alpha) \to (u, v)$, we have

$$\begin{aligned}
\langle A(z,v), z-u_\alpha\rangle + f(z) - f(u_\alpha) &\geq 0, \quad \forall z \in K_F, \\
\langle B(u,z), z-v_\alpha\rangle + f(z) - f(v_\alpha) &\geq 0, \quad \forall z \in K_F.
\end{aligned} \tag{3.14}$$

Since $\langle \cdot, \cdot \rangle$, A, and B are continuous and f is lower semicontinuous, we get

$$\begin{aligned}
\langle A(z,v), z-u\rangle + f(z) - f(u) &\geq 0, \quad \forall z \in K_F, \\
\langle B(u,z), z-v\rangle + f(z) - f(v) &\geq 0, \quad \forall z \in K_F.
\end{aligned} \tag{3.15}$$

Thus $(u, v) \in T(u, v)$ implies that $T(u, v)$ is closed.

(4) $T(u, v)$ is bounded for $(u, v) \in K_F \times K_F$.

Since K_F is bounded, it is obvious that $T(u, v)$ is bounded.

(5) The mapping $T : K_F \times K_F \to 2^{K_F \times K_F}$ is upper semicontinuous.

By the completely continuity of $A(v, \cdot)$, $B(\cdot, v)$ and f being semicontinuous, we note that T is upper semicontinuous.

Hence, by the Kakutani-Fan-Glicksberg fixed point theorem, T has a fixed point; that is, there exists $(u_\alpha, v_\alpha) \in K_F \times K_F$ such that $(u_\alpha, v_\alpha) \in T(u_\alpha, v_\alpha)$. Thus, we have

$$\begin{aligned}
\langle A(u_\alpha, v_\alpha), z-u_\alpha\rangle + f(z) - f(u_\alpha) &\geq 0, \quad \forall z \in K_F, \\
\langle B(u_\alpha, v_\alpha), z-v_\alpha\rangle + f(z) - f(v_\alpha) &\geq 0, \quad \forall z \in K_F.
\end{aligned} \tag{3.16}$$

Let $\mathfrak{I} = \{F \subset E \mid F$ is finite dimensional and $F \cap K \neq \emptyset\}$. For each $F \in \mathfrak{I}$, we let S_F be the set of all solutions of the following problem. Find $(u', v') \in K \times K$ such that

$$\begin{aligned} &\langle A(z, v'), z - u' \rangle + f(z) - f(u') \geq 0, \quad \forall z \in K_F, \\ &\langle B(u', z), z - v' \rangle + f(z) - f(v') \geq 0, \quad \forall z \in K_F. \end{aligned} \tag{3.17}$$

By (3.10), we know that S_F is nonempty bounded. We observe that $\overline{S_F} \subset K \times K$, where $\overline{S_F}$ is the weak* closure of S_F in $E \times E$. Since E is reflexive, it follows that $\overline{S_F}$ is weak* compact. For any $F_1, F_2, \ldots, F_N \in \mathfrak{I}$, we note that $S_{\bigcup_{i=1}^{n} F_i} \subset \bigcap_{i=1}^{N} S_{F_i}$. So $\{\overline{S_F} : F \in \mathfrak{I}\}$ has the finite intersection property. Therefore, it follows that $\bigcap_{F \in \mathfrak{I}} \overline{S_F} \neq \emptyset$. Let $(u^*, v^*) \in \bigcap_{F \in \mathfrak{I}} \overline{S_F}$, then we have $(u^*, v^*) \in K \times K$.

Next, we claim that

$$\begin{aligned} &\langle A(u^*, v^*), z - u^* \rangle + f(z) - f(u^*) \geq 0, \quad \forall z \in K, \\ &\langle B(u^*, v^*), z - v^* \rangle + f(z) - f(v^*) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.18}$$

Indeed, for each $z \in K$, we choose $F \in \mathfrak{I}$ such that $z, u^*, v^* \in K_F \subset K$. Since $(u^*, v^*) \in \overline{S_F}$, there exists a sequence $(u_j, v_j)_{j=1}^{\infty} \subseteq S_F$ such that (u_j, v_j) converge weakly to (u^*, v^*). This implies that

$$\begin{aligned} &\langle A(z, v_j), z - u_j \rangle + f(z) - f(u_j) \geq 0, \\ &\langle B(u_j, z), z - v_j \rangle + f(z) - f(v_j) \geq 0 \end{aligned} \tag{3.19}$$

for all $j \geq 1$. Since f is lower semicontinuous, it follows that f is weakly lower semicontinuous. Therefore, by using the completely continuity of $A(z, \cdot)$ and $B(\cdot, z)$, respectively, and letting $j \to \infty$, we have

$$\begin{aligned} &\langle A(z, v^*), z - u^* \rangle + f(z) - f(u^*) \geq 0, \\ &\langle B(u^*, z), z - v^* \rangle + f(z) - f(v^*) \geq 0. \end{aligned} \tag{3.20}$$

By Lemma 2.9, we have

$$\begin{aligned} &\langle A(u^*, v^*), z - u^* \rangle + f(z) - f(u^*) \geq 0, \quad \forall z \in K, \\ &\langle B(u^*, v^*), z - v^* \rangle + f(z) - f(v^*) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.21}$$

This completes the proof. □

Setting $f := 0$ in Theorem 3.2, we have the following result.

Corollary 3.3. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with dual space E^*. Let $A, B : K \times K \to E^*$ be two mappings satisfying the following conditions:*

(i) *for each $z \in K$, $A(\cdot, z)$ and $B(z, \cdot)$ are monotone;*

(ii) *for each* $z \in K$, $A(z,\cdot)$ *and* $B(\cdot,z)$ *are completely continuous;*

(iii) *for any given* $z \in K$, $A(\cdot,z) : K \to E^*$ *and* $B(z,\cdot) : K \to E^*$ *are finite dimensional continuous; that is, for any finite dimensional subspace* $M \subset E$,$A(\cdot,z)$ *and* $B(z,\cdot) : K \cap M \to E^*$ *are continuous.*

Then, there exists $(u,v) \in K \times K$ *such that*

$$\begin{aligned} &\langle A(u,v), z-u\rangle \geq 0, \quad \forall z \in K,\\ &\langle B(u,v), z-v\rangle \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.22}$$

Corollary 3.4 (see [19]). *Let* E *be a real reflexive Banach space and* $K \subset E$ *a bounded closed convex subset. Suppose that* $f : E(-\infty,+\infty]$ *is a lower semicontinuous convex function with* $K \subseteq D(f)$, $B : K \times K \to E^*$ *is semimonotone, and* $B(u,\cdot)$ *is finite dimensional continuous for each* $u \in K$. *Then there exists* $w_0 \in K$ *such that*

$$(B(w_0,w_0), u-w_0) + f(u) - f(w_0) \geq 0, \quad \forall u \in K. \tag{3.23}$$

Proof. Define a mapping $A : K \times K \to E^*$ by $A(u,v) = B(v,v)$ for all $u,v \in K$. We observe that $A(\cdot,z)$ is monotone and $A(z,\cdot)$ is completely continuous for all $z \in K$. Moreover, A is finite dimensional continuous. Therefore, by Theorem 3.2, there exists $v \in K$ such that

$$(B(v,v), z-v) + f(z) - f(v) \geq 0, \quad \forall z \in K. \tag{3.24}$$

□

Next, we consider the system of generalized variational inequality in which K is unbounded. We have the following result.

Theorem 3.5. *Let* E *be a real reflexive Banach space with dual space* E^*, *and let* $K \subset E$ *be a nonempty unbounded closed convex subset with* $0 \in E$. *Suppose that* $f : E \to (-\infty,+\infty]$ *is a lower semicontinuous convex function with* $K \subseteq D(f)$. *Let* $A : K \times K \to E^*$, $B : K \times K \to E^*$ *be two mappings satisfying the following conditions:*

(i) *for each* $z \in K$, $A(\cdot,z)$ *and* $B(z,\cdot)$ *are monotone;*

(ii) *for each* $z \in K$, $A(z,\cdot)$ *and* $B(\cdot,z)$ *are completely continuous;*

(iii) *for any given* $z \in K$, $A(\cdot,z) : K \to E^*$ *and* $B(z,\cdot) : K \to E^*$ *are finite dimensional continuous;*

(iv) $\liminf_{u\to\infty} [\langle A(u,v),u\rangle + f(u)] > f(0)$ *and* $\liminf_{v\to\infty} [\langle B(u,v),v\rangle + f(v)] > f(0)$.

Then, there exists $(u^*,v^*) \in K \times K$ *such that*

$$\begin{aligned} &\langle A(u^*,v^*), z-u^*\rangle + f(z) - f(u^*) \geq 0, \quad \forall z \in K,\\ &\langle B(u^*,v^*), z-v^*\rangle + f(z) - f(v^*) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.25}$$

Proof. Let $\overline{B(0,r)}$ be the closed ball in E at center zero with radius r such that $K_r = \overline{B(0,r)} \cap K \neq \emptyset$. By Theorem 3.2, there exists $(u_r, v_r) \in K_r \times K_r$ such that

$$\begin{aligned} &\langle A(u_r, v_r), z - u_r \rangle + f(z) - f(u_r) \geq 0, \quad \forall z \in K_r, \\ &\langle B(u_r, v_r), z - v_r \rangle + f(z) - f(v_r) \geq 0, \quad \forall z \in K_r. \end{aligned} \tag{3.26}$$

Leting $z = 0 \in K_r$ in (3.26), we get

$$\begin{aligned} &\langle A(u_r, v_r), -u_r \rangle + f(0) - f(u_r) \geq 0, \\ &\langle B(u_r, v_r), -v_r \rangle + f(0) - f(v_r) \geq 0, \end{aligned} \tag{3.27}$$

which implies that

$$\begin{aligned} &\langle A(u_r, v_r), u_r \rangle + f(u_r) \leq f(0), \\ &\langle B(u_r, v_r), v_r \rangle + f(v_r) \leq f(0). \end{aligned} \tag{3.28}$$

By condition (iii), we know that $\{(u_r, v_r)\}_{r>0}$ is bounded. So, we may assume that (u_r, v_r) converge weakly to $(u^*, v^*) \in K \times K$ as $r \to \infty$. From (3.26), it follows by Lemma 2.9 that

$$\begin{aligned} &\langle A(z, v_r), z - u_r \rangle + f(z) - f(u_r) \geq 0, \quad \forall u_r \in K, \\ &\langle B(u_r, z), z - v_r \rangle + f(z) - f(v_r) \geq 0, \quad \forall v_r \in K. \end{aligned} \tag{3.29}$$

Since $A(z, \cdot), B(\cdot, z)$ are complete continuous and f is weakly lower semicontinuous, it follows by letting $r \to \infty$ that

$$\begin{aligned} &\langle A(z, v^*), z - u^* \rangle + f(z) - f(u^*) \geq 0, \quad \forall z \in K, \\ &\langle B(u^*, z), z - v^* \rangle + f(z) - f(v^*) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.30}$$

Using Lemma 2.9 again, we obtain

$$\begin{aligned} &\langle A(u^*, v^*), z - u^* \rangle + f(z) - f(u^*) \geq 0, \quad \forall z \in K, \\ &\langle B(u^*, v^*), z - v^* \rangle + f(z) - f(v^*) \geq 0, \quad \forall z \in K. \end{aligned} \tag{3.31}$$

This completes the proof. □

Setting $f := 0$ in Theorem 3.5, we have the following result.

Corollary 3.6. *Let E be a real reflexive Banach space with dual space E^*, and let $K \subset E$ be a nonempty unbounded closed convex subset with $0 \in K$. Let $A, B : K \times K \to E^*$ be two mappings satisfying the following conditions:*

(i) *for each $z \in K$, $A(\cdot, z)$ and $B(z, \cdot)$ are monotone;*

(ii) *for each* $z \in K$, $A(z,\cdot)$ *and* $B(\cdot,z)$ *are completely continuous;*

(iii) *for any given* $z \in K$, $A(\cdot,z) : K \to E^*$ *and* $B(z,\cdot) : K \to E^*$ *are finite dimensional continuous; that is, for any finite dimensional subspace* $M \subset E, A(\cdot,z)$ *and* $B(z,\cdot) : K \cap M \to E^*$ *are continuous;*

(iv) $\liminf_{u,v\in K,\, u\to\infty}\langle A(u,v),u\rangle > 0$ *and* $\liminf_{u,v\in K,\, v\to\infty}\langle B(u,v),v\rangle > 0$

Then, there exists $(u,v) \in K \times K$ *such that*

$$\begin{aligned}\langle A(u,v), z-u\rangle &\geq 0, \quad \forall z \in K,\\ \langle B(u,v), z-v\rangle &\geq 0, \quad \forall z \in K.\end{aligned} \tag{3.32}$$

Similarly as in the proof of Corollary 3.4, we have the following result.

Corollary 3.7 (see [19]). *Let* E *be a real reflexive Banach space and* $K \subset E$ *an unbounded closed convex subset with* $0 \in K$. *Suppose that* $f : E(-\infty,+\infty]$ *is a lower semicontinuous convex function with* $K \subseteq D(f)$, $B : K \times K \to E^*$ *is semimonotone, and* $B(u,\cdot)$ *is finite dimensional continuous for each* $u \in K$. *Assume that the following condition holds:*

$$\lim\inf_{v\to\infty}\left[\langle B(u,v),v\rangle + f(v)\right] > f(0). \tag{3.33}$$

Then there exists $w_0 \in K$, *such that*

$$(B(w_0,w_0), u-w_0) + f(u) - f(w_0) \geq 0, \quad \forall u \in K. \tag{3.34}$$

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