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Research Article

Poincaré Inequalities with Luxemburg Norms in $L^{\varphi}(m)$-Averaging Domains

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We prove both local and global Poincaré inequalities with Luxemburg norms for differential forms in $L^{\varphi}(m)$-averaging domains, which can be considered as generalizations of the existing versions of Poincaré inequalities.

1. Introduction

The Poincaré-type inequality has been playing a crucial role in analysis and related fields during the last several decades. Many versions of the Poincaré inequality have been developed and used in different areas of mathematics, including PDEs and analysis. For example, in 1989, Staples in [1] proved the following Poincaré inequality for Sobolev functions in L^s-averaging domains. If D is an L^p-averaging domain, $p \geq n$, then there exists a constant C, such that

$$\left(\frac{1}{|D|}\int_D |u - u_D|^p dm\right)^{1/p} \leq C|D|^{1/n}\left(\frac{1}{|D|}\int_D |\nabla u|^p dm\right)^{1/p} \tag{1.1}$$

for each Sobolev function u defined in D. In [2], a global Poincaré inequality for solutions of the A-harmonic equation was proved over the John domains; see [3–7] for more results about the Poincaré inequality. However, most of these inequalities are developed with the L^p-norms. In this paper, we will establish the Poincaré inequalities with the Luxemburg norms in a relative large class of domains, the $L^{\varphi}(m)$-averaging domain, so that many existing versions of the Poincaré inequality are special cases of our new results.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let B and σB be the balls with the same center and $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$ throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. For a function u, we denote the average of

u over B by $u_B = (1/|B|)\int_B u\,dm$. All integrals involved in this paper are the Lebesgue integrals. Differential forms are generalizations of differentiable functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^{n} u_i(x_1, x_2, \ldots, x_n)dx_i$, where the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable. Similarly, a differential k-form $u(x)$ can be expressed as

$$u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{1.2}$$

where $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and let $L^p(\Omega, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by d and the Hodge star operator by $\star$. The Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d \star$, $l = 1, 2, \ldots, n$. We consider here the nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.3}$$

which is called nonhomogeneous A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions $|A(x, \xi)| \le a|\xi|^{p-1}$, $A(x, \xi) \cdot \xi \ge |\xi|^p$ and $|B(x, \xi)| \le b|\xi|^{p-1}$ for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.3). A solution to (1.3) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that $\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0$ for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, (1.3) reduces to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{1.4}$$

If the operator $B = 0$, (1.3) becomes

$$d^\star A(x, du) = 0 \tag{1.5}$$

which is called the (homogeneous) A-harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and (1.5) becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. See [8–12] for recent results on the A-harmonic equations and related topics.

2. Local Poincaré Inequalities

In this section, we establish the local Poincaré inequalities for the differential forms in any bounded domain. A continuously increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ is called an Orlicz function. The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on

Ω such that $\int_\Omega \varphi(|f|/\lambda)dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^\varphi(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{\varphi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right)dx \le 1\right\}. \tag{2.1}$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_\varphi$ defines a norm in $L^\varphi(\Omega)$, which is called the Luxemburg norm.

Definition 2.1 (see [13]). We say that a Young function φ lies in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, if (i) $1/C \le \varphi(t^{1/p})/g(t) \le C$ and (ii) $1/C \le \varphi(t^{1/q})/h(t) \le C$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0,\infty)$.

From [13], each of φ, g and h in above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \le h^{-1}(\varphi(t)) \le C_2 t^q, \qquad C_1 t^p \le g^{-1}(\varphi(t)) \le C_2 t^p, \tag{2.2}$$

where C_1 and C_2 are constants. Also, for all $1 \le p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\varphi(t) = t^p \log_+^\alpha t$ belongs to $G(p_1,p_2,C)$ for some constant $C = C(p,\alpha,p_1,p_2)$. Here $\log_+(t)$ is defined by $\log_+(t) = 1$ for $t \le e$; and $\log_+(t) = \log(t)$ for $t > e$. Particularly, if $\alpha = 0$, we see that $\varphi(t) = t^p$ lies in $G(p_1,p_2,C)$, $1 \le p_1 < p < p_2$. We will need the following Reverse Hölder inequality.

Lemma 2.2 (see [8]). *Let u be a solution of the nonhomogeneous A-harmonic equation* (1.3) *in a domain Ω an d let $0 < s,t < \infty$. Then, there exists a constant C, independent of u, such that $\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B}$ for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.*

We first prove the following generalized Poincaré inequality that will be used to establish the global inequality.

Theorem 2.3. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and Ω be a bounded domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega,m)$ and u is a solution of the nonhomogeneous A-harmonic equation* (1.3) *in Ω, $\varphi(|du|) \in L^1_{loc}(\Omega,m)$. Then for any ball B with $\sigma B \subset \Omega$, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|u-u_B|)dm \le C\int_{\sigma B} \varphi(|du|)dm. \tag{2.3}$$

Proof. From [7, (3.5)], we have

$$\|u-u_B\|_{s,B} \le C_1|B|^{1+1/n}\|du\|_{s,B} \tag{2.4}$$

for any $s > 0$. Note that if u is a solution of the nonhomogeneous A-harmonic equation (1.3), then $u-u_B$ is also a solution of (1.3) since u_B is a closed form. From Lemma 2.2, it follows that

$$\left(\int_B |u-u_B|^q dm\right)^{1/q} \le C_2 |B|^{(p-q)/pq} \left(\int_{\sigma B} |u-u_B|^p dm\right)^{1/p} \tag{2.5}$$

for any positive numbers p and q. From (2.5), (i) in Definition 2.1, and using the fact that φ is an increasing function, Jensen's inequality, and noticing that φ and g are doubling, we have

$$\begin{aligned}
\varphi\left(\left(\int_B |u-u_B|^q dm\right)^{1/q}\right) &\le \varphi\left(C_2 |B|^{(p-q)/pq} \left(\int_{\sigma B} |u-u_B|^p dm\right)^{1/p}\right)\\
&\le \varphi\left(C_3 |B|^{1+1/n+(p-q)/pq} \left(\int_{\sigma B} |du|^p dm\right)^{1/p}\right)\\
&\le \varphi\left(\left(C_3^p |B|^{p(1+1/n)+(p-q)/q} \int_{\sigma B} |du|^p dm\right)^{1/p}\right)\\
&\le C_4 g\left(C_3^p |B|^{p(1+1/n)+(p-q)/q} \int_{\sigma B} |du|^p dm\right)\\
&= C_4 g\left(\int_{\sigma B} C_3^p |B|^{p(1+1/n)+(p-q)/q} |du|^p dm\right)\\
&\le C_4 \int_{\sigma B} g\left(C_3^p |B|^{p(1+1/n)+(p-q)/q} |du|^p\right) dm.
\end{aligned} \tag{2.6}$$

Since $p \ge 1$, then $1 + 1/n + (p-q)/pq > 0$. Hence, we have $|B|^{1+1/n+(p-q)/pq} \le |\Omega|^{1+1/n+(p-q)/pq} \le C_5$. From (i) in Definition 2.1, we find that $g(t) \le C_6 \varphi(t^{1/p})$. Thus,

$$\begin{aligned}
\int_{\sigma B} g\left(C_3^p |B|^{p(1+1/n)+(p-q)/q} |du|^p\right) dm &\le C_6 \int_{\sigma B} \varphi\left(C_3 |B|^{1+1/n+(p-q)/pq} |du|\right) dm\\
&\le C_6 \int_{\sigma B} \varphi(C_7 |du|) dm.
\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$\varphi\left(\left(\int_B |u-u_B|^q dm\right)^{1/q}\right) \le C_8 \int_{\sigma B} \varphi(C_7 |du|) dm. \tag{2.8}$$

Using Jensen's inequality for h^{-1}, (2.2), and noticing that φ and h are doubling, we obtain

$$\begin{aligned}
\int_B \varphi(|u-u_B|)dm &= h\left(h^{-1}\left(\int_B \varphi(|u-u_B|)dm\right)\right) \\
&\le h\left(\int_B h^{-1}(\varphi(|u-u_B|))dm\right) \\
&\le h\left(C_9\int_B |u-u_B|^q dm\right) \\
&\le C_{10}\varphi\left(\left(C_9\int_B |u-u_B|^q dm\right)^{1/q}\right) \\
&\le C_{11}\varphi\left(\left(\int_B |u-u_B|^q dm\right)^{1/q}\right).
\end{aligned} \tag{2.9}$$

Substituting (2.8) into (2.9) and noticing that φ is doubling, we have

$$\int_B \varphi(|u-u_B|)dm \le C_{12}\int_{\sigma B}\varphi(C_7|du|)dm \le C_{13}\int_{\sigma B}\varphi(|du|)dm. \tag{2.10}$$

We have completed the proof of Theorem 2.3. □

Since each of φ, g, and h in Definition 2.1 is doubling, from the proof of Theorem 2.3 or directly from (2.3), we have

$$\int_B \varphi\left(\frac{|u-u_B|}{\lambda}\right)dm \le C\int_{\sigma B}\varphi\left(\frac{|du|}{\lambda}\right)dm \tag{2.11}$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From (2.1) and (2.11), the following Poincaré inequality with the Luxemburg norm

$$\|u-u_B\|_{\varphi(B)} \le C\|du\|_{\varphi(\sigma B)} \tag{2.12}$$

holds under the conditions described in Theorem 2.3.

Theorem 2.4. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, let Ω be a bounded domain and $q(n-p) < np$. Assume that $u \in D'(\Omega, \wedge^l)$ is any differential l-form, $l = 0,1,\dots,n-1$, $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega, m)$ and $\varphi(|du|) \in L^1_{\mathrm{loc}}(\Omega, m)$. Then for any ball $B \subset \Omega$, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|u-u_B|)dm \le C\int_B \varphi(|du|)dm. \tag{2.13}$$

Proof. From (2.9), it follows that

$$\int_B \varphi(|u-u_B|)dm \le C_1\varphi\left(\left(\int_B |u-u_B|^q dm\right)^{1/q}\right). \tag{2.14}$$

If $1 < p < n$, by assumption, we have $q < np/(n-p)$. Using the Poincaré-type inequality for differential forms

$$\left(\int_B |u-u_B|^{np/(n-p)} dm\right)^{(n-p)/np} \le C_2\left(\int_B |du|^p dm\right)^{1/p}, \tag{2.15}$$

we find that

$$\left(\int_B |u-u_B|^q dm\right)^{1/q} \le C_3\left(\int_B |du|^p dm\right)^{1/p}. \tag{2.16}$$

Note that the L^p-norm of $|u-u_B|$ increases with p and $np/(n-p) \to \infty$ as $p \to n$, and it follows that (2.16) still holds when $p \ge n$. Since φ is increasing, from (2.14) and (2.16), we obtain

$$\int_B \varphi(|u-u_B|)dm \le C_1\varphi\left(C_3\left(\int_B |du|^p dm\right)^{1/p}\right). \tag{2.17}$$

Applying (2.17), (i) in Definition 2.1, Jensen's inequality, and noticing that φ and g are doubling, we have

$$\begin{aligned}\int_B \varphi(|u-u_B|)dm &\le C_1\varphi\left(C_3\left(\int_B |du|^p dm\right)^{1/p}\right)\\ &\le C_1 g\left(C_4\left(\int_B |du|^p dm\right)\right)\\ &\le C_5\int_B g(|du|^p)dm.\end{aligned} \tag{2.18}$$

Using (i) in Definition 2.1 again yields

$$\int_B g(|du|^p)dm \le C_6\int_B \varphi(|du|)dm. \tag{2.19}$$

Combining (2.18) and (2.19), we obtain

$$\int_B \varphi(|u-u_B|)dm \le C_7 \int_B \varphi(|du|)dm. \tag{2.20}$$

The proof of Theorem 2.4 has been completed. □

Similar to (2.12), from (2.1) and (2.13), the following Luxemburg norm Poincaré inequality

$$\|u-u_B\|_{\varphi(B)} \le C\|du\|_{\varphi(B)} \tag{2.21}$$

holds if all conditions of Theorem 2.4 are satisfied.

3. Global Poincaré Inequalities

In this section, we extend the local Poincaré inequalities into the global cases in the following $L^{\varphi}(m)$-averaging domains.

Definition 3.1 (see [14]). Let φ be an increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^{\varphi}(m)$-averaging domain, if $m(\Omega) < \infty$ and there exists a constant C such that

$$\int_\Omega \varphi(\tau|u-u_{B_0}|)dm \le C \sup_{B\subset\Omega} \int_B \varphi(\sigma|u-u_B|)dm \tag{3.1}$$

for some ball $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{\text{loc}}(\Omega, m)$, where τ, σ are constants with $0<\tau<\infty$, $0<\sigma<\infty$ and the supremum is over all balls $B \subset \Omega$.

From above definition we see that L^s-averaging domains and $L^s(m)$-averaging domains are special $L^{\varphi}(m)$-averaging domains when $\varphi(t) = t^s$ in Definition 3.1. Also, uniform domains and John domains are very special $L^{\varphi}(m)$-averaging domains; see [15–18] for more results about domains.

Theorem 3.2. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, an d let Ω be any bounded $L^{\varphi}(m)$-averaging domain. Assume that $\varphi(|u|) \in L^1(\Omega, m)$ and u is a solution of the nonhomogeneous A-harmonic equation* (1.4) *in Ω, $\varphi(|du|) \in L^1(\Omega, m)$. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi(|u-u_{B_0}|)dm \le C \int_\Omega \varphi(|du|)dm, \tag{3.2}$$

where $B_0 \subset \Omega$ is some fixed ball.

Proof. From Definition 3.1, (2.3) and noticing that φ is doubling, we have

$$\begin{aligned}\int_\Omega \varphi(|u-u_{B_0}|)dm &\le C_1\sup_{B\subset\Omega}\int_B \varphi(|u-u_B|)dm \\ &\le C_1\sup_{B\subset\Omega}\left(C_2\int_{\sigma B}\varphi(|du|)dm\right) \\ &\le C_1\sup_{B\subset\Omega}\left(C_2\int_{\Omega}\varphi(|du|)dm\right) \\ &\le C_3\int_\Omega \varphi(|du|)dm.\end{aligned} \tag{3.3}$$

We have completed the proof of Theorem 3.2. □

Similar to the local case, the following global Poincaré inequality with the Luxemburg norm

$$\|u-u_B\|_{\varphi(\Omega)} \le C\|du\|_{\varphi(\Omega)} \tag{3.4}$$

holds if all conditions in Theorem 3.2 are satisfied. Also, by the same way, we can extend Theorem 2.4 into the following global result in $L^\varphi(m)$-averaging domains.

Theorem 3.3. *Let φ be a Young function in the class $G(p,q,C)$, $1\le p<q<\infty$, $C\ge 1$, Ω be a bounded $L^\varphi(m)$-averaging domain and $q(n-p)<np$. Assume that $u\in D'(\Omega,\wedge^0)$ and $\varphi(|u|)\in L^1(\Omega,m)$ and $\varphi(|du|)\in L^1(\Omega,m)$. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi(|u-u_{B_0}|)dm \le C\int_\Omega \varphi(|du|)dm, \tag{3.5}$$

where $B_0\subset\Omega$ is some fixed ball.

Note that (3.5) can be written as

$$\|u-u_{B_0}\|_{\varphi(\Omega)} \le C\|du\|_{\varphi(\Omega)}. \tag{3.6}$$

It has been proved that any John domain is a special $L^\varphi(m)$-averaging domain. Hence, we have the following results.

Corollary 3.4. *Let φ be a Young function in the class $G(p,q,C)$, $1\le p<q<\infty$, $C\ge 1$, and Ω be a bounded John domain. Assume that $\varphi(|u|)\in L^1(\Omega,m)$ and u is a solution of the nonhomogeneous A-harmonic equation* (1.4) *in Ω, $\varphi(|du|)\in L^1(\Omega,m)$. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi(|u-u_{B_0}|)dm \le C\int_\Omega \varphi(|du|)dm, \tag{3.7}$$

where $B_0\subset\Omega$ is some fixed ball.

Choosing $\varphi(t) = t^p \log_+^\alpha t$ in Theorems 3.2 and 3.3, respectively, we obtain the following Poincaré inequalities with the $L^p(\log_+^\alpha L)$-norms.

Corollary 3.5. *Let $\varphi(t) = t^p \log_+^\alpha t$, $p \geq 1$ and $\alpha \in \mathbb{R}$. Assume that $\varphi(|u|) \in L^1(\Omega, m)$ and u is a solution of the nonhomogeneous A-harmonic equation* (1.4)*, $\varphi(|du|) \in L^1(\Omega, m)$. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega |u - u_{B_0}|^p \log_+^\alpha(|u - u_{B_0}|) dm \leq C \int_\Omega |du|^p \log_+^\alpha(|du|) dm \tag{3.8}$$

for any bounded $L^\varphi(m)$-averaging domain Ω and $B_0 \subset \Omega$ is some fixed ball.

Note that (3.8) can be written as the following version with the Luxemburg norm:

$$\|u - u_{B_0}\|_{L^p(\log_+^\alpha L)(\Omega)} \leq C \|du\|_{L^p(\log_+^\alpha L)(\Omega)}, \tag{3.9}$$

provided that the conditions in Corollary 3.5 are satisfied.

Corollary 3.6. *Let $\varphi(t) = t^p \log_+^\alpha t$, $1 \leq p_1 < p < p_2$ and $\alpha \in \mathbb{R}$ and Ω be a bounded $L^\varphi(m)$-averaging domain and $p_2(n - p_1) < np_1$. Assume that $u \in D'(\Omega, \wedge^0)$, $\varphi(|u|) \in L^1(\Omega, m)$, and $\varphi(|du|) \in L^1(\Omega, m)$ Then, there exists a constant C, independent of u, such that*

$$\int_\Omega |u - u_{B_0}|^p \log_+^\alpha(|u - u_{B_0}|) dm \leq C \int_\Omega |du|^p \log_+^\alpha(|du|) dm, \tag{3.10}$$

where $B_0 \subset \Omega$ is some fixed ball.

4. Applications

Choose u to be a 0-form (a function) in the homogeneous A-harmonic equation (1.5). Then, (1.5) reduces to the following A-harmonic equation:

$$\operatorname{div} A(x, \nabla u) = 0 \tag{4.1}$$

for functions. If the above operator $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ is defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$, then, A satisfies the required conditions and (4.1) becomes the usual p-harmonic equation

$$\operatorname{div}\left(\nabla u |\nabla u|^{p-2}\right) = 0 \tag{4.2}$$

for functions. Thus, from Theorem 3.2, we have the following example.

Example 4.1. Let u be a solution of the usual A-harmonic equation (4.1) or the p-harmonic equation (4.2), let φ be a Young function in the class $G(p, q, C)$, $1 \leq p < q < \infty$, $C \geq 1$, and Ω

be any bounded $L^{\varphi}(m)$-averaging domain. If $\varphi(|u|) \in L^1(\Omega, m)$ and $\varphi(|du|) \in L^1(\Omega, m)$, then there exists a constant C, independent of u, such that

$$\int_{\Omega} \varphi(|u - u_{B_0}|)dm \leq C \int_{\Omega} \varphi(|du|)dm, \tag{4.3}$$

where $B_0 \subset \Omega$ is some fixed ball.

Example 4.2. For any locally L^s-integrable form $u(y)$, $1 \leq s < \infty$, the Hardy-Littlewood maximal operator $\mathbb{M}_s$ is defined by $\mathbb{M}_s(u) = \sup_{r>0}((1/|B(x,r)|)\int_{B(x,r)}|u(y)|^s dy)^{1/s}$ and the sharp maximal operator $\mathbb{M}_s^{\#}$ by $\mathbb{M}_s^{\#}(u) = \sup_{r>0}((1/|B(x,r)|)\int_{B(x,r)}|u(y) - u_{B(x,r)}|^s dy)^{1/s}$, where $B(x,r)$ is the ball of radius r, centered at x. Under the conditions of Theorem 3.3, we have

$$\begin{gathered} \int_{\Omega} \varphi\left(\left|\mathbb{M}_s(u) - (\mathbb{M}_s(u))_{B_0}\right|\right)dm \leq C \int_{\Omega} \varphi(|d(\mathbb{M}_s(u))|)dm, \\ \int_{\Omega} \varphi\left(\left|\mathbb{M}_s^{\#}(u) - \left(\mathbb{M}_s^{\#}(u)\right)_{B_0}\right|\right)dm \leq C \int_{\Omega} \varphi\left(\left|d\left(\mathbb{M}_s^{\#}(u)\right)\right|\right)dm, \end{gathered} \tag{4.4}$$

where $B_0 \subset \Omega$ is some fixed ball.

Remark 4.3. (i) We know that the L^s-averaging domains are the special $L^{\varphi}(m)$-averaging domains. Thus, Theorems 3.2 and 3.3 also hold for the L^s-averaging domain. (ii) In Theorems 2.4 and 3.3, u does not need to be a solution of any version of the A-harmonic equation.

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