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Research Article

Global Existence, Uniqueness, and Asymptotic Behavior of Solution for p-Laplacian Type Wave Equation

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We study the global existence and uniqueness of a solution to an initial boundary value problem for the nonlinear wave equation with the p-Laplacian operator $u_{tt}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)-\Delta u_t+g(x,u)=f(x)$. Further, the asymptotic behavior of solution is established. The nonlinear term g likes $g(x,u)=a(x)|u|^{\alpha-1}u-b(x)|u|^{\beta-1}u$ with appropriate functions $a(x)$ and $b(x)$, where $\alpha>\beta\geq 1$.

1. Introduction

This paper is concerned with the global existence, uniqueness, and asymptotic behavior of solution for the nonlinear wave equation with the p-Laplacian operator

$$u_{tt}-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)-\Delta u_t+g(x,u)=f(x), \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \text{in } \Omega; \qquad u(x,t)=0, \quad \text{on } \partial\Omega\times[0,\infty), \tag{1.2}$$

where $2\leq p<n$ and Ω is a boundary domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. The assumptions on f,g,u_0 and u_1 will be made in the sequel.

Recently, Ma and Soriano in [1] investigated the global existence of solution $u(t)$ for the problem (1.1)-(1.2) under the assumptions

$$p=n, \qquad g(u)u\geq 0, \quad \left|g(u)\right|\leq C_\beta\exp\left(\beta|u|^{n/(n-1)}\right), \quad u\in\mathbf{R}. \tag{1.3}$$

Moreover, if $f = 0$ and $ug(u) \geq G(u)$, then there exist positive constants c and γ such that

$$E(t) \leq c \exp(-\gamma t), \quad t \geq 0, \text{ if } n = 2, \tag{1.4}$$

$$E(t) \leq c(1+t)^{-n/(n-2)}, \quad t \geq 0, \text{ if } n \geq 3, \tag{1.5}$$

where

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{n}\|\nabla u(t)\|_n^n + \int_\Omega G(x, u(t))dx \tag{1.6}$$

with $G(x, u) = \int_0^u f(x, s)ds$.

Gao and Ma in [2] also considered the global existence of solution for (1.1)-(1.2). In Theorem 3.1 of [2], the similar results to (1.4)-(1.5) for asymptotic behavior of solution were obtained if the nonlinear function $g(x, u) = g(u)$ satisfies

$$|g(u)| \leq a|u|^{\sigma-1} + b, \quad ug(u) \geq \rho G(u) \geq 0, \quad \text{in } \ \Omega \times \mathbf{R}, \tag{1.7}$$

where $a, b > 0$, $\rho > 0$, $1 < \sigma < np/(n-p)$ if $1 < p < n$ and $1 < \sigma < \infty$ if $n \leq p$.

More precisely, they obtained that the global existence of solution for (1.1)-(1.2) if one of the following assumptions was satisfied:

(i) $1 < \sigma < p$, the initial data $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$;

(ii) $p < \sigma$, the initial data $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ is small.

Similar consideration can be found in [3–5]. In [6], Yang obtained the uniqueness of solution of the Laplacian wave equation (1.1)-(1.2) for $n = 1$. To the best of our knowledge, there are few information on the uniqueness of solution of (1.1)-(1.2) for $n > 1$ and $p > 2$.

In this paper, we are interested in the global existence, the uniqueness, the continuity and the asymptotic behavior of solution for (1.1)-(1.2). The nonlinear term g in (1.1) likes $g(x, u) = a(x)|u|^{\alpha-1}u - b(x)|u|^{\beta-1}u$ with $\alpha > \beta \geq 1$ and $a, b \geq 0$. Obviously, the sign condition $ug(u) \geq 0$ fails to hold for this type of function.

For these purposes, we must establish the global existence of solution for (1.1)-(1.2). Several methods have been used to study the existence of solutions to nonlinear wave equation. Notable among them is the variational approach through the use of Faedo-Galerkin approximation combined with the method of compactness and monotonicity, see [7]. To prove the uniqueness, we need to derive the various estimates for assumed solution $u(t)$. For the decay property, like (1.5), we use the method recently introduced by Martinez [8] to study the decay rate of solution to the wave equation $u_{tt} - \Delta u + g(u_t) = 0$ in $\Omega \times \mathbf{R}^+$, where Ω is a bounded domain of $\mathbf{R}^n$.

This paper is organized as follows. In Section 2, some assumptions and the main results are stated. In Section 3, we use Faedo-Galerkin approximation together with a combination of the compactness and the monotonicity methods to prove the global existence of solution to problem (1.1)-(1.2). Further, we establish the uniqueness of solution by some a priori estimate to assumed solutions. The proof of asymptotic behavior of solution is given in Section 4.

2. Assumptions and Main Results

We first give some notations and definitions. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. We denote the space L^p and $W_0^{1,p}$ for $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ and relevant norms by $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, respectively. In general, $\|\cdot\|_X$ denotes the norm of Banach space X. We also denote by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ the inner product of $L^2(\Omega)$ and the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. As usual, we write $u(t)$ instead $u(x,t)$. Sometimes, let $u'(t)$ represent for $u_t(t)$ and so on.

If $T > 0$ is given and X is a Banach space, we denote by $L^p(0,T;X)$ the space of functions which are L^p over $(0,T)$ and which take their values in X. In this space, we consider the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_X. \tag{2.1}$$

Let us state our assumptions on f and g.

(A_1) $f \in L^{p'}$ with $p' = p/(p-1)$, $p > 1$.

(A_2) Let $g(x,u) \in C^1(\Omega \times \mathbf{R})$ and satisfy

$$ug(x,u) + h_1(x)|u| \ge k_0(G(x,u) + h_1(x)|u|) \ge 0, \quad \text{in } \Omega \times \mathbf{R} \tag{2.2}$$

and growth condition

$$\left|g(x,u)\right| \le k_1(|u|^\alpha + h_2(x)), \quad \left|g_u(x,u)\right| \le k_1\left(|u|^{\alpha-1} + h_3(x)\right), \quad \text{in } \Omega \times \mathbf{R} \tag{2.3}$$

with some $k_0, k_1 > 0$ and the nonnegative functions $h_1(x) \in L^{p'}$, $h_2 \in L^2 \cap L^{(\alpha+1)/\alpha}$, $h_3 \in L^2 \cap L^{(\alpha+1)/(\alpha-1)}$, where $1 \le \alpha \le np/(n-p) - 1$, $G(x,u) = \int_0^u g(x,s)ds$.

A typical function g is $g(x,u) = a(x)|u|^{\alpha-1}u - b(x)|u|^{\beta-1}u$ with the appropriate nonnegative functions $a(x)$ and $b(x)$, where $\alpha > \beta \ge 1$.

Definition 2.1 (see [7]). A measurable function $u = u(x,t)$ on $\Omega \times \mathbf{R}^+$ is said to be a (weak) solution of (1.1)-(1.2) if all $T > 0$, $u \in L^\infty(0,T;W_0^{1,p})$, $u_t \in L^2(0,T;W_0^{1,2})$, $u_{tt} \in L^2(0,T;W^{-1,p'})$, and u satisfies (1.2) with $(u_0,u_1) \in W_0^{1,p}$ and the integral identity

$$\int_\Omega \left(u_{tt}\phi + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi + \nabla u_t \cdot \nabla\phi + g\phi - f\phi\right)dx = 0 \tag{2.4}$$

for all $\phi \in C_0^\infty(\Omega)$.

Now we are in a position to state our results.

Theorem 2.2. *Assume* (A_1)-(A_2) *hold and* $(u_0, u_1) \in W_0^{1,p} \times L^2$. *Then the problem* (1.1)-(1.2) *admits a solution* $u(t)$ *satisfying*

$$u \in C\Big([0,\infty);, W_0^{1,2}\Big) \cap L^\infty\Big([0,\infty);, W_0^{1,p}\Big),$$
$$u_t \in L^2\Big([0,\infty);, W_0^{1,2}\Big), \qquad u_{tt} \in L^2_{\mathrm{loc}}\Big([0,\infty);, W^{-1,p'}\Big), \tag{2.5}$$

and the following estimates

$$\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p + \int_0^t \|\nabla u_t(s)\|_2^2 ds \le C_1(A+B), \quad \forall t \ge 0, \tag{2.6}$$

where

$$A = \|u_0\|_p^p + \|\nabla u_0\|_p^{\alpha+1} + \|u_1\|_2^2, \qquad B = H_1 + H_2 + H_3 + F, \tag{2.7}$$

with $F = \|f\|_{p'}^{p'}$, $H_i = \|h_i\|_{p'}^{p'}$, $i = 1, 2$, $H_3 = \|h_3\|_{\lambda_1}^{\lambda_1}$, $\lambda_1 = n/2$.

Further, if $1 \le \alpha \le (n+p)/(n-p)$ and $2 \le p \le 4$, the solution satisfying (2.5)-(2.6) is unique.

Theorem 2.3. *Let* u *be a solution of* (1.1)-(1.2) *with* $f = 0$. *In addition, let* $2 < p < n$ *and*

$$g(x,u)u \ge pG(x,u) \ge 0, \quad \text{in } \Omega \times \mathbf{R}. \tag{2.8}$$

Then there exists $C_0 = C_0(u_0, u_1)$, *such that*

$$\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p + \int_\Omega G(x, u(x,t))dx \le C_0(1+t)^{-p/(p-2)}, \quad \forall t \ge 0. \tag{2.9}$$

The following theorem shows that the asymptotic estimate (2.9) can be also derived if assumption (2.8) fails to hold.

Theorem 2.4. *Let* u *be a solution of* (1.1)-(1.2) *with* $f = 0$. *In addition, let* $2 < p < n$ *and*

$$g(x,u) = \lambda |u|^{\alpha-1}u - |u|^{\beta-1}u, \quad \text{in } \Omega \times \mathbf{R} \tag{2.10}$$

with $p < \beta+1 < 2p$, $\beta < \alpha < np/(n-p)$. *Then there exists* $C_0 = C_0(u_0, u_1) > 0$ *and* $\lambda_2 = \lambda_2(\alpha, \beta) > 0$, *such that* $\lambda > \lambda_2$, *the solution* $u(t)$ *satisfies*

$$\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p + \|u(t)\|_{\alpha+1}^{\alpha+1} \le C_0(1+t)^{-p/(p-2)}, \quad \forall t \ge 0. \tag{2.11}$$

3. Proof of Theorem 2.2

In this section, we assume that all assumptions in Theorem 2.2 are satisfied. We first prove the global existence of a solution to problem (1.1)-(1.2) with the Faedo-Galerkin method as in [1, 2, 7, 9].

Let r be an integer for which the embedding $H_0^r(\Omega) = W_0^{r,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous. Let $w_j (j = 1, 2, \ldots)$ be eigenfunctions of the spectral problem

$$(w_j, v)_{H_0^r} = \lambda_j (w_j, v), \quad \forall v \in H_0^r(\Omega), \tag{3.1}$$

where $(\cdot,\cdot)_{H_0^r}$ represents the inner product in $H_0^r(\Omega)$. Then the family $\{w_1, w_2, \ldots, w_m, \ldots\}$ yields a basis for both $H_0^r(\Omega)$ and $L^2(\Omega)$. For each integer m, let $V_m = \operatorname{span}\{w_1, w_2, \ldots, w_m\}$. We look for an approximate solution to problem (1.1)-(1.2) in the form

$$u_m(t) = \sum_{j=1}^{m} T_{jm}(t) w_j, \tag{3.2}$$

where $T_{jm}(t)$ are the solution of the nonlinear ODE system in the variant t:

$$(u_m'', w_j) - (\Delta_p u_m, w_j) - (\Delta u_m', w_j) + (g, w_j) = (f, w_j), \qquad j = 1, 2, \ldots m \tag{3.3}$$

with the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and the initial conditions

$$u_m(0) = u_{0m}, \qquad u_m'(0) = u_{1m}, \tag{3.4}$$

where u_{0m} and u_{1m} are chosen in V_m so that

$$u_{0m} \longrightarrow u_0 \quad \text{in } W_0^{1,p}, \qquad u_{1m} \longrightarrow u_1 \quad \text{in } L^2. \tag{3.5}$$

As it is well known, the system (3.3)-(3.4) has a local solution $u_m(t)$ on some interval $[0, t_m)$. We claim that for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$ by using the first a priori estimate below. We denote by C_k the constant which is independent of m and the initial data u_0 and u_1.

Multiplying (3.3) by $T_{jm}'(t)$ and summing the resulting equations over j, we get after integration by parts

$$E_m'(t) + \|\nabla u_m'(t)\|_2^2 = 0, \quad \forall t \geq 0, \tag{3.6}$$

where

$$E_m(t) = \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{p}\|\nabla u_m(t)\|_p^p + \int_\Omega G(x, u_m) dx - \int_\Omega f(x) u_m dx. \tag{3.7}$$

By (2.2) and Young inequality, we have

$$\int_{\Omega} G(x, u_m)dx \geq -\int_{\Omega} h_1(x)|u_m|dx \geq -\varepsilon\|\nabla u_m\|_p^p - C_\varepsilon\|h_1\|_{p'}^{p'},$$
$$\int_{\Omega} f(x)u_m dx \geq -\varepsilon\|\nabla u_m\|_p^p - C_\varepsilon\|f\|_{p'}^{p'}. \tag{3.8}$$

Let $\varepsilon > 0$ be so small that $2p^{-1} - 4\varepsilon \geq p^{-1}$. Then

$$E_m(t) \geq \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2p}\|\nabla u_m(t)\|_p^p - C_1(H_1 + F), \tag{3.9}$$

or

$$\|u_m'(t)\|_2^2 + \|\nabla u_m(t)\|_p^p \leq C_1(E_m(t) + H_1 + F_1) \tag{3.10}$$

for some $C_1 > 0$.

Thus, it follows from (3.6) and (3.10) that, for any $m = 1, 2, \ldots,$ and $t \geq 0$

$$\|u_m'(t)\|_2^2 + \|\nabla u_m(t)\|_p^p + \int_0^t \|\nabla u_m(s)\|_2^2 ds \leq C_2(E_m(0) + H_1 + F_1). \tag{3.11}$$

By assumption (A_2), we obtain that $\alpha + 1 \leq np/(n-p)$ and

$$\begin{aligned}\left|\int_{\Omega} G(x, u_m)dx\right| &\leq k_1\left(\|u_m\|_{\alpha+1}^{\alpha+1} + \int_{\Omega}|h_2||u_m|dx\right)\\ &\leq C_2\left(\|\nabla u_m\|_p^{\alpha+1} + \|u_m\|_p^p + \|h_2\|_{p'}^{p'}\right)\\ &\leq C_2\left(\|\nabla u_m\|_p^{\alpha+1} + \|\nabla u_m\|_p^p + H_2\right).\end{aligned} \tag{3.12}$$

Then it follows (3.5) and (3.6) that

$$\begin{aligned}E_m(t) \leq E_m(0) &= \frac{1}{2}\|u_{1m}'\|_2^2 + \frac{1}{p}\|\nabla u_{0m}\|_p^p + \int_{\Omega} G(x, u_{0m})\ dx - \int_{\Omega} f(x)u_{0m}dx\\ &\leq C_2\left(\|u_1\|_2^2 + \|\nabla u_0\|_p^p + \|\nabla u_0\|_p^\alpha + H_1 + H_2 + F\right)\\ &\leq C_2(A + B).\end{aligned} \tag{3.13}$$

Hence, for any $t \geq 0$ and $m = 1, 2, \ldots,$ we have from (3.11) and (3.13) that

$$\|u_m'(t)\|_2^2 + \|\nabla u_m(t)\|_p^p + \int_0^t \|\nabla u_m'(s)\|_2^2 ds \leq C_2(A + B), \quad \forall t \geq 0. \tag{3.14}$$

With this estimate we can extend the approximate solution $u_m(t)$ to the interval $[0,T]$ and we have that

$$\{u_m(t)\} \text{ is bounded in } L^\infty\left(0,T;W_0^{1,p}\right), \tag{3.15}$$

$$\{u_m'(t)\} \text{ is bounded in } L^\infty(0,T;L^2), \tag{3.16}$$

$$\{u_m'(t)\} \text{ is bounded in } L^2\left(0,T;W_0^{1,2}\right). \tag{3.17}$$

Now we recall that operator $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is bounded, monotone, and hemicontinuous from $W_0^{1,p}$ to $W^{-1,p'}$ with $p \geq 2$. Then we have

$$\{-\Delta_p u_m(t)\} \text{ is bounded } L^\infty\left(0,T;W^{-1,p'}\right). \tag{3.18}$$

By the standard projection argument as in [1], we can get from the approximate equation (3.3) and the estimates (3.15)–(3.17) that

$$\{u_m''(t)\} \text{ is bounded in } L^2(0,T;\mathrm{H}^{-r}(\Omega)). \tag{3.19}$$

From (3.15)-(3.16), going to a subsequence if necessary, there exists u such that

$$u_m \rightharpoonup u \text{ weakly star in } L^\infty\left(0,T;W_0^{1,p}\right), \tag{3.20}$$

$$u_m' \rightharpoonup u' \text{ weakly star in } L^\infty\left(0,T;L^2\right), \tag{3.21}$$

$$u_m' \rightharpoonup u' \text{ weakly in } L^2\left(0,T;L^2\right), \tag{3.22}$$

and in view of (3.18), there exists $\chi(t)$ such that

$$-\Delta_p u_m(t) \rightharpoonup \chi(t) \text{ weakly star in } L^\infty\left(0,T;W^{-1,p'}\right). \tag{3.23}$$

By applying the Lions-Aubin compactness Lemma in [7], we get, from (3.15) and (3.16),

$$u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;L^2\right), \tag{3.24}$$

and $u_m \to u$ a.e. in $\Omega \times (0,T)$.

Since the embedding $W_0^{1,2} \hookrightarrow L^2$ is compact, we get, from (3.18) and (3.19),

$$u_m' \longrightarrow u' \text{ strongly in } L^2\left(0,T;L^2\right). \tag{3.25}$$

Using the growth condition (2.3) and (3.25), we see that

$$\int_0^T \int_\Omega \left|g(x,u_m(x,t))\right|^{(\alpha+1)/\alpha} dx\,dt \tag{3.26}$$

is bounded and

$$g(x,u_m) \longrightarrow g(x,u) \quad \text{a.e. in } (\Omega \times T). \tag{3.27}$$

Therefore, from [7, Chapter 1, Lemma 1.3], we infer that

$$g(x,u_m) \rightharpoonup g(x,u) \text{ weakly in } L^{(\alpha+1)/\alpha}\left(0,T;L^{(\alpha+1)/\alpha}\right). \tag{3.28}$$

With these convergences, we can pass to the limit in the approximate equation and then

$$\frac{d}{dt}\left(u'(t),v\right) + \langle \chi(t),v\rangle + \left(\nabla u',\nabla v\right) + (g,v) = (f,v), \quad \forall v \in W_0^{1,p}. \tag{3.29}$$

Obviously, u satisfies the estimates (2.5)-(2.6). Finally, using the standard monotonicity argument as done in [1, 7], we get that $\chi(t) = -\Delta_p u(t)$. This completes the proof of existence of solution $u(t)$.

To prove the uniqueness, we assume that $u(t)$ and $v(t)$ are two solutions which satisfy (2.5)-(2.6) and $u(0) = v(0), u_t(0) = v_t(0)$. Setting $U(t) = u_t(t), V(t) = v_t(t)$, and $W(t) = U(t) - V(t)$. We see from (1.1) and (1.2) that

$$W_t - \Delta W - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) = g(x,v) - g(x,u). \tag{3.30}$$

Multiplying (3.30) by W and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|W(t)\|_2^2 + \|\nabla W(t)\|_2^2 + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla W dx = \int_\Omega (g(x,v) - g(x,u))W dx,\\
&\|W(t)\|_2^2 + 2\int_0^t \|\nabla W(s)\|_2^2 ds + 2\int_0^t \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla W\,dx\,d\tau\\
&\quad = 2\int_0^t \int_\Omega (g(x,v) - g(x,u))W\,dx\,ds
\end{aligned} \tag{3.31}$$

Now setting $U_\epsilon = \epsilon u + (1-\epsilon)v$, $0 \le \epsilon \le 1$, then

$$\begin{aligned}
&\int_0^t \int_\Omega \left| \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla W \right| dx\, d\tau \\
&\quad \le \int_0^t \int_\Omega \left| \int_0^1 \frac{d}{d\epsilon} \left(|\nabla U_\epsilon|^{p-2} \nabla U_\epsilon \right) d\epsilon \right| |\nabla W| dx\, d\tau \\
&\quad \le (p-1) \int_0^t \int_\Omega \int_0^1 |\nabla U_\epsilon|^{p-2} |\nabla (u(\tau) - v(\tau))| |\nabla W| d\epsilon\, dx\, d\tau \equiv I.
\end{aligned} \tag{3.32}$$

Note that

$$\begin{gathered}
|\nabla U_\epsilon(\tau)| \le |\nabla u(\tau)| + |\nabla v(\tau)|, \\
|\nabla (u(\tau) - v(\tau))| \le \int_0^\tau |\nabla (u_s(s) - v_s(s))| ds = \int_0^\tau |\nabla W(s)| ds.
\end{gathered} \tag{3.33}$$

Then, by the estimates (2.6) and $2 \le p \le 4$, we have

$$\begin{aligned}
I &\le C_1 \int_0^t \int_\Omega \int_0^\tau \left(|\nabla u(\tau)|^{p-2} + |\nabla v(\tau)|^{p-2} \right) |\nabla W(s)| |\nabla W(\tau)| dx\, ds\, d\tau \\
&\le C_1 \int_0^t \int_0^\tau \left(\|\nabla u(\tau)\|_p^{p-2} + \|\nabla v(\tau)\|_p^{p-2} \right) \|\nabla W(s)\|_2 \|\nabla W(\tau)\|_2 ds\, d\tau \\
&\le C_1 (A+B)^{(p-2)/p} \int_0^t \int_0^\tau \|\nabla W(s)\|_2 \|\nabla W(\tau)\|_2 ds\, d\tau \\
&\le C_1 (A+B)^{(p-2)/p} \left(\int_0^t \|\nabla W(s)\|_2 ds \right)^2 \le C_2 t \int_0^t \|\nabla W(s)\|_2^2 ds
\end{aligned} \tag{3.34}$$

with $C_2 = C_1 (A+B)^{(p-2)/p}$.

For the term of the right side to (3.31), we have

$$\begin{aligned}
G_1 &= \int_0^t \int_\Omega |g(x,v) - g(x,u)| |W| dx\, d\tau = \int_0^t \int_\Omega \left| \int_0^1 \frac{d}{d\epsilon} g(x, U_\epsilon) d\epsilon \right| |W| dx\, d\tau \\
&\le \int_0^t \int_\Omega \int_0^1 \left| g_u(x, U_\epsilon) \| u(\tau) - v(\tau) \| W(\tau) \right| d\epsilon\, dx d\tau \\
&\le \int_0^t \int_0^\tau \int_0^1 \left\| g_u(x, U_\epsilon) \right\|_{\lambda_1} d\epsilon \|u_s(s) - v_s(s)\|_{\lambda_2} \|W(\tau)\|_{\lambda_2} d\epsilon\, ds\, d\tau
\end{aligned} \tag{3.35}$$

with $\lambda_1 = n/2$, $\lambda_2 = 2n/(n-2)$.

By the assumption (A_2) and $1 \le \alpha \le (n+p)/(n-p)$, we see that

$$\begin{aligned}\|g_u(x,U_\epsilon)\|_{\lambda_1}^{\lambda_1} &\le k_1 \int_\Omega \left(|u(\tau)|^{\alpha-1} + |v(\tau)|^{\alpha-1} + |h_3|\right)^{n/2} dx \\ &\le C_3 \int_\Omega \left(|u(\tau)|^{n(\alpha-1)/2} + |v(\tau)|^{n(\alpha-1)/2} + |h_3|^{n/2}\right) dx \\ &\le C_3 \left(\|\nabla u(\tau)\|_{\mathrm{p}}^{n(\alpha-1)/2} + \|\nabla v(\tau)\|_p^{n(\alpha-1)/2} + H_3\right).\end{aligned} \tag{3.36}$$

By the estimate (2.6), we have

$$\|\nabla u(t)\|_p, \quad \|v(t)\|_p \le C_2(A+B)^{1/p}, \qquad \forall t \ge 0. \tag{3.37}$$

Therefore, there exists $C_4 > 0$, depending u_0, v_0, f, h_i such that

$$\|g_u(x,U_\epsilon)\|_{\lambda_1} \le C_4, \quad \forall t \ge 0. \tag{3.38}$$

Since $u, v \in W_0^{1,p} \subset W_0^{1,2}$, $u_t, v_t \in W_0^{1,2}$, we get

$$\begin{gathered}\|u_s(s) - v_s(s)\|_{\lambda_2} \le C_0 \|\nabla(u_s(s) - v_s(s))\|_2 = C_0 \|\nabla W(s)\|_2, \\ \|W(\tau)\|_2 \le C_0 \|\nabla W(\tau)\|_2\ .\end{gathered} \tag{3.39}$$

Then (3.35) becomes

$$G_1 \le C_4 \int_0^t \int_0^\tau \|W(s)\|_{\lambda_2} \|W(\tau)\|_{\lambda_2} ds d\tau \le C_4 \left(\int_0^t \|\nabla W(s)\|_2 ds\right)^2 \le C_4 t \int_0^t \|\nabla W(s)\|_2^2 ds. \tag{3.40}$$

Therefore, it follows from (3.31), (3.34), and (3.40) that

$$\|W(t)\|_2^2 + 2\int_0^t \|\nabla W(s)\|_2^2 ds \le (C_2 + C_4) t \int_0^t \|\nabla W(s)\|_2^2. \tag{3.41}$$

The integral inequality (3.41) shows that there exists $T_1 > 0$, such that

$$W(t) = 0, \qquad 0 \le t \le T_1. \tag{3.42}$$

Consequently, $u(t) - v(t) = u(0) - v(0) = 0,\ 0 \le t \le T_1$.

Repeating the above procedure, we conduce that $u(t) = v(t)$ on $[T_1, 2T_1], [2T_1, 3T_1], \ldots$ and $u(t) = v(t)$ on $[0, \infty)$. This ends the proof of uniqueness.

Next, we prove that $u \in C([0, \infty); W_0^{1,2})$. Let $t > s \geq 0$, we have

$$\begin{aligned} \|\nabla(u(t) - u(s))\|_2^2 &= \int_\Omega \left| \int_s^t \nabla u_\tau(\tau) d\tau \right|^2 dx \leq \int_\Omega \int_s^t |\nabla u_\tau(\tau)|^2 ds\, dx (t-s) \\ &= (t-s) \int_s^t \|\nabla u_\tau(\tau)\|_2^2 d\tau \longrightarrow 0, \quad \text{as } t \longrightarrow s. \end{aligned} \tag{3.43}$$

This shows that $u(t) \in C([0, \infty); W_0^{1,2})$. We complete the proof of Theorem 2.2.

4. Proof of Theorem 2.3

Let us first state a well-known lemma that will be needed later.

Lemma 4.1 (see [10]). *Let $E : \mathbf{R}^+ \to \mathbf{R}^+$ be a nonincreasing function and assume that there are constants $q \geq 0$ and $\gamma > 0$, such that*

$$\int_S^\infty E^{q+1}(t) dt \leq \gamma^{-1} E^q(0) E(S), \quad \forall S \geq 0. \tag{4.1}$$

Then, we have

$$\begin{aligned} E(t) &\leq E(0) \left(\frac{1+q}{1+q\gamma t} \right)^{1/q}, \quad \forall t \geq 0, \ \textit{if } q > 0, \\ E(t) &\leq E(0) e^{1-\gamma t}, \quad \forall t \geq 0, \ \textit{if } q = 0. \end{aligned} \tag{4.2}$$

4.1. The Proof of Theorem 2.3

Let

$$E(t) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{p} \|\nabla u(t)\|_p^p + \int_\Omega G(x, u) dx, \quad t \geq 0. \tag{4.3}$$

Then, we have from (1.1) that

$$E'(t) + \|\nabla u_t(t)\|_2^2 = 0, \quad \forall t \geq 0. \tag{4.4}$$

This shows that $E(t)$ is nonincreasing in $[0, \infty)$.

Multiplying (1.1) by $E^q(\mathrm{t})u(t)$ with $q = (p-2)/p > 0$, we get

$$\int_S^T E^q(t) \int_\Omega u\big(u_{tt} - \Delta_p u - \Delta u_t + g(x,u)\big)dx\,dt = 0, \quad \forall T > S \geq 0. \tag{4.5}$$

Note that

$$\begin{aligned}
\int_S^T E^q(t)(u,u_{tt})dt &= E^q(t)(u,u_t)|_S^T - \int_S^T \Big(qE^{q-1}(t)E'(t)(u,u_t) + E^q(t)\|u_t(t)\|_2^2\Big)dt \\
-\int_S^T E^q(t)\big(u,\Delta_p u\big)dt &= \int_S^T E^q(t)\|\nabla u(t)\|_p^p dt, \\
-\int_S^T E^q(t)(u,\Delta u_t)dt &= \int_S^T E^q(t)(\nabla u,\nabla u_t)dt.
\end{aligned} \tag{4.6}$$

Then we have from (4.5) that

$$\begin{aligned}
p\int_S^T E^{q+1}(t)dt = &-E^q(t)(u,u_t)|_S^T + q\int_S^T E^{q-1}(t)E'(t)(u,u_t)dt \\
&+ \Big(1+\frac{p}{2}\Big)\int_S^T E^q(t)\|u_t(t)\|_2^2 dt - \int_S^T E^q(t)(\nabla u,\nabla u_t)dt \\
&+ \int_S^T E^q(t)\int_\Omega (pG(u) - ug(u)dx\,dt.
\end{aligned} \tag{4.7}$$

Since $\int_\Omega G(x,u)dx \geq 0$, $E(t) \geq 0$. Further, by (4.4), we see that

$$\begin{gathered}
\|\nabla u_t(t)\|_2 \leq \big(-E'(t)\big)^{1/2}, \qquad \|\nabla u(t)\|_p \leq pE^{1/p}(t), \quad \forall t \geq 0, \\
|E^q(t)(u,u_t)| \leq E^q(t)\|u(t)\|_2\|u_t(t)\|_2 \leq C_0E^q(t)\|\nabla u(t)\|_p\|\nabla u_t(t)\|_2 \leq C_0(E(t))^{\mu_1}
\end{gathered} \tag{4.8}$$

with $\mu_1 = q + 1/2 + 1/p$.

This gives

$$E^q(t)(u,u_t)|_S^T \leq C_1E^{\mu_1}(S), \quad \forall T > S \geq 0, \tag{4.9}$$

where the fact that $E(t)$ is nonincreasing is used.

Similarly, we derive the following estimates

$$\int_S^T E^q(t)\|u_t(t)\|_2^2 dt \le C_1 \int_S^T E^q(t)\|\nabla u_t(t)\|_2^2 dt$$
$$= C_1 \int_S^T E^q(t)\left(-E'(t)\right)dt \le C_1 E^{q+1}(S), \tag{4.10}$$

$$q\int_S^T \left|E^{q-1}(t)E'(t)(u,u_t)\right|dt \le C_1 \int_S^T E^{q-1}(t)\left|E'(t)\right|\|u(t)\|_2\|u_t(t)\|_2 dt$$
$$\le C_1 \int_S^T E^{\mu_1-1}(t)\left|E'(t)\right|dt \le C_1 E^{\mu_1}(S), \tag{4.11}$$

$$\int_S^T |E^q(t)(\nabla u, \nabla u_t)|dt \le \int_S^T E^q(t)\|\nabla u(t)\|_2\|\nabla u_t(t)\|_2 dt$$
$$\le C_1 \int_S^T E^{q+1/p}(t)\left(-E'(t)\right)^{1/2}dt$$
$$\le \int_S^T E^{q+1}(t)dt + C_1\int_S^T E^{q+2/p-1}(t)\left(-E'(t)\right)dt$$
$$\le \int_S^T E^{q+1}(t)dt + C_1 E^{q+2/p}(S). \tag{4.12}$$

Then we get from (4.9)–(4.12) that

$$\int_S^T E^{q+1}(t)dt \le C_1\left(\mathrm{E}^{\mu_1}(S) + E^{q+1}(S) + E^{q+2/p}(S)\right)$$
$$\le C_1 E(S)\left(E^{\mu_1}(S) + E^q(S) + E^{q+2/p-1}(S)\right)$$
$$\le C_1 E(S)E^q(0)\left(E^{1/p-1/2}(0) + 1 + E^{2/p-1}(0)\right)$$
$$\equiv \gamma^{-1}E^q(0)E(S), \tag{4.13}$$

for any $T > S \ge 0$, letting $T \to \infty$, we find that

$$\int_S^\infty E^{q+1}(t)dt \le \gamma^{-1}E(S)E^q(0), \qquad \forall S \ge 0. \tag{4.14}$$

By Lemma 4.1, we obtain that

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \int_\Omega G(x,u)dx \le E(0)\left(\frac{1+q}{1+q\gamma t}\right)^{1/q} \le C_2 E(0)(1+t)^{-p/(p-2)}. \tag{4.15}$$

This is (2.9) and we complete the proof of Theorem 2.3.

4.2. The Proof of Theorem 2.4

By Sobolev inequality, we know that there exists $\lambda_0 > 0$ such that

$$\lambda_0\|u\|_p^p \le \|\nabla u\|_p^p, \quad \forall u \in W_0^{1,p}(\Omega). \tag{4.16}$$

Let u be a solution for (1.1)-(1.2) in Theorem 2.2. By (2.10),

$$G(u) = \frac{\lambda}{\alpha+1}|u|^{\alpha+1} - \frac{1}{\beta+1}|u|^{\beta+1}. \tag{4.17}$$

Obviously, there exists $\lambda_2 > 0$, such that $\lambda > \lambda_2$,

$$\frac{\lambda_0}{2p}|u|^p + G(u) \ge \frac{1}{2(\alpha+1)}|u|^{\alpha+1}, \quad \forall u \in \mathbf{R}. \tag{4.18}$$

This implies that

$$\begin{gathered}\frac{\lambda_0}{2p}\|u\|_p^p + \int_\Omega G(u)dx \ge \frac{1}{2(\alpha+1)}\|u\|_{\alpha+1}^{\alpha+1}, \\ E(t) \ge \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2p}\|\nabla u(t)\|_p^p + \frac{1}{2(\alpha+1)}\|u(t)\|_{\alpha+1}^{\alpha+1}.\end{gathered} \tag{4.19}$$

On the other hand, we have, from (4.18) and (4.19),

$$\begin{aligned}pG(u) - ug(u) &= \frac{\beta+1-p}{\beta+1}|u|^{\beta+1} - \frac{\lambda(\alpha+1-p)}{\alpha+1}|u|^{\alpha+1} \\ &\le \frac{\beta+1-p}{\beta+1}|u|^{\beta+1} = (\beta+1-p)\left(\frac{\lambda}{\alpha+1}|u|^{\alpha+1} - G(u)\right) \\ &\le (\beta+1-p)\left(\frac{\lambda_0}{p}|u|^p + G(u)\right).\end{aligned} \tag{4.20}$$

It shows that

$$\int_S^T E^q(t)\int_\Omega (pG(u) - gu)dxdt \le (\beta+1-p)\int_S^T E^{q+1}(t)dt. \tag{4.21}$$

Then, by (4.9) and (4.11)–(4.14), we have

$$\begin{aligned}(2p-\beta-1)\int_S^T E^{q+1}(t)dt &\le C_0\left(E^{q+1/p+2}(S) + E^{q+1}(S) + E^{q+2/p}(S)\right) \\ &\le \gamma^{-1}E(S)E^q(0).\end{aligned} \tag{4.22}$$

The applications of Lemma 4.1 and (4.19) yields that

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_{\alpha+1}^{\alpha+1} \leq C_0(1+t)^{-p/(p-2)}, \quad \forall t \geq 0. \tag{4.23}$$

This ends the proof of Theorem 2.4.

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