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Research Article

Applications of Wirtinger Inequalities on the Distribution of Zeros of the Riemann Zeta-Function

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On the hypothesis that the $(2k)$th moments of the Hardy Z-function are correctly predicted by random matrix theory and the moments of the derivative of Z are correctly predicted by the derivative of the characteristic polynomials of unitary matrices, we establish new large spaces between the zeros of the Riemann zeta-function by employing some Wirtinger-type inequalities. In particular, it is obtained that $\Lambda(15) \geq 6.1392$ which means that consecutive nontrivial zeros often differ by at least 6.1392 times the average spacing.

1. Introduction

The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{for } \operatorname{Re}(s) > 1, \tag{1.1}$$

and by analytic continuation elsewhere except for a simple pole at $s = 1$. The identity between the Dirichlet series and the Euler product (taken over all prime numbers p) is an analytic version of the unique prime factorization in the ring of integers and reflects the importance of the zeta-function for number theory. The functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.2}$$

implies the existence of so-called trivial zeros of $\zeta(s)$ at $s = -2n$ for any positive integer n; all other zeros are said to be nontrivial and lie inside the so-called critical strip $0 < \mathrm{Re}(s) < 1$. The number $N(T)$ of nontrivial zeros of $\zeta(s)$ with ordinates in the interval $(0,T]$ is asymptotically given by the Riemann-von Mangoldt formula (see [1])

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T). \tag{1.3}$$

Consequently, the frequency of their appearance is increasing and the distances between their ordinates is tending to zero as $T \to \infty$.

The Riemann zeta-function is one of the most studied transcendental functions, having in view its many applications in number theory, algebra, complex analysis, and statistics as well as in physics. Another reason why this function has drawn so much attention is the celebrated Riemann conjecture regarding nontrivial zeros which states that all nontrivial zeros of the Riemann zeta-function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. The distribution of zeros of $\zeta(s)$ is of great importance in number theory. In fact any progress in the study of the distribution of zeros of this function helps to investigate the magnitude of the largest gap between consecutive primes below a given bound. Clearly, there are no zeros in the half plane of convergence $\mathrm{Re}(s) > 1$, and it is also known that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$. In the negative half plane, $\zeta(s)$ and its derivative are oscillatory and from the functional equation there exist so-called trivial (real) zeros at $s = -2n$ for any positive integer n (corresponding to the poles of the appearing Gamma-factors), and all nontrivial (nonreal) zeros are distributed symmetrically with respect to the critical line $\mathrm{Re}(s) = 1/2$ and the real axis.

There are three directions regarding the studies of the zeros of the Riemann zeta-function. The first direction is concerned with the existence of simple zeros. It is conjectured that all or at least almost all zeros of the zeta-function are simple. For this direction, we refer to the papers by Conrey [2] and Cheer and Goldston [3].

The second direction is the most important goal of number theorists which is the determination of the moments of the Riemann zeta-function on the critical line. It is important because it can be used to estimate the maximal order of the zeta-function on the critical line, and because of its applicability in studying the distribution of prime numbers and divisor problems. For more details of the second direction, we refer the reader to the papers in [4–6] and the references cited therein. For further classical results from zeta-function theory, we refer to the monograph [7] of Ivić and the papers by Kim [8–11].

For completeness in the following we give a brief summary of some of these results in this direction that we will use in the proof of the main results. It is known that the behavior of $\zeta(s)$ on the critical line is reflected by the Hardy Z-function $Z(t)$ as a function of a real variable, defined by

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right), \quad \text{where } \theta(t) := \pi^{-it/2}\frac{\Gamma(1/4 + (1/2)it)}{|\Gamma(1/4 + (1/2)it)|}. \tag{1.4}$$

It follows from the functional equation (1.2) that $Z(t)$ is an infinitely often differentiable function which is real for real t and moreover $|Z(t)| = |\zeta(1/2 + it)|$. Consequently, the zeros of $Z(t)$ correspond to the zeros of the Riemann zeta-function on the critical line. An important problem in analytic number theory is to gain an understanding of the moments of the Hardy

Z-function $Z(t)$ function $I_k(T)$ and the moments of its derivative $M_k(T)$ which are defined by

$$I_k(T) := \int_0^T |Z(t)|^{2k} dt, \qquad M_k(T) := \int_0^T \left|Z'(t)\right|^{2k} dt. \tag{1.5}$$

For positive real numbers k, it is believed that

$$\begin{aligned} I_k(T) &\sim C_k T (\log T)^{k^2}, \\ M_k(T) &\sim L_k T (\log T)^{k^2+2k}, \end{aligned} \tag{1.6}$$

for positive constants C_k and L_k will be defined later.

Keating and Snaith [12] based on considerations from random matrix theory conjectured that

$$I_k(T) \sim a(k) b_k T (\log T)^{k^2}, \tag{1.7}$$

where

$$a_k := \prod_p \left(1 - \frac{1}{p^2}\right) \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}, \qquad b_k := \frac{G^2(k+1)}{G(2k+1)} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}, \tag{1.8}$$

where G is the Barnes G-function (for the definition of the Barnes G-function and its properties, we refer to [5]).

Hughes [5] used the Random Matrix Theory (RMT) and stated an interesting conjecture on the moments of the Hardy Z-function and its derivatives at its zeros subject to the truth of Riemann's hypothesis when the zeros are simple. This conjecture includes for fixed $k > -3/2$ the asymptotic formula of the moments of the form

$$\int_0^T Z^{2k-2h}(t) \left(Z'(t)\right)^{2h} dt \sim a(k) b(h,k) T (\log T)^{k^2+2h}, \tag{1.9}$$

where $a(k)$ is defined as in (1.8) and the product is over the primes. Hughes [5] was able to establish the explicit formula

$$b(h,k) = b(0,k) \left(\frac{(2h)!}{8^h h!}\right) H(h,k), \tag{1.10}$$

in the range $\min(h, k-h) > -1/2$, where $H(h,k)$ is an explicit rational function of k for each fixed h. The functions $H(h,k)$ as introduced by Hughes [5] are given in the following:

$$
\begin{aligned}
H(0,k) &= 1, \\
H(1,k) &= \frac{1}{K^2-1}, \\
H(2,k) &= \frac{1}{(K^2-1)(K^2-9)}, \\
H(3,k) &= \frac{1}{(K^2-1)^2(K^2-25)}, \\
H(4,k) &= \frac{K^2-33}{(K^2-1)^2(K^2-9)(K^2-25)(K^2-49)}, \\
H(5,k) &= \frac{K^4-90K^2+1497}{(K^2-1)^2(K^2-9)^2(K^2-25)(K^2-49)(K^2-81)} \\
H(6,k) &= \frac{K^6-171K^4+6867K^2-27177}{(K^2-1)^3(K^2-9)^2(K^2-25)(K^2-49)(K^2-81)(K^2-121)}, \\
H(7,k) &= \frac{K^8-316K^6+30702K^4-982572K^2+6973305}{(K^2-1)^3(K^2-9)^2(K^2-25)^2(K^2-49)(K^2-81)(K^2-121)(K^2-169)},
\end{aligned}
\tag{1.11}
$$

where $K = 2k$. This sequence is continuous, and it is believed that both the nominator and denominator are monic polynomials in k^2. Using (1.10) and the definitions of the functions $H(h,k)$, we can obtain the values of $b(0,k)/b(k,k)$ for $k = 1, 2, \ldots, 7$. As indicated in [13] Hughes [5] evaluated the first four functions and then writes a numerical experiment suggesting the next three. The values of $b(0,k)/b(k,k)$ for $k = 1, 2, \ldots, 7$ have been collected in [6]. To the best of my knowledge there is no explicit formula to find the values of the function $H(h,k)$ for $k, h \geq 8$. This limitation of the values of $H(h,k)$ leads to the limitation of the values of the lower bound between the zeros of the Riemann zeta-function by applying the moments (1.9). To overcame this restriction, we will use a different explicit formula of the moments to establish new values of the distance between zeros.

Conrey et al. [4] established the moments of the derivative, on the unit circle, of characteristic polynomials of random unitary matrices and used this to formulate a conjecture for the moments of the derivative of the Riemann zeta-function on the critical line. Their method depends on the fact that the distribution of the eigenvalues of unitary matrices gives insight into the distribution of zeros of the Riemann zeta-function and the values of the characteristic polynomials of the unitary matrices give a model for the value distribution of the Riemann zeta-function. Their formulae are expressed in terms of a determinant of a matrix whose entries involve the I-Bessel function and, alternately, by a combinatorial sum. They conjectured that

$$M_k(T) \sim a(k) c_k T (\log T)^{k^2+2k}, \tag{1.12}$$

where $a(k)$ is the arithmetic factor and defined as in (1.8) and

$$c_k := (-1)^{k(k+1)/2} \sum_{m \in P_o^{k+1}(2k)}^{k} \binom{2k}{m} \left(\frac{-1}{2}\right)^{m_0} \left(\prod_{i=1}^{k} \frac{1}{(2k-i+m_i)!}\right) M_{i,j}, \tag{1.13}$$

where

$$M_{i,j} := \left(\prod_{1 \le i,j \le k}^{k} (m_j - m_i + i - j)\right), \tag{1.14}$$

and $P_o^{k+1}(2k)$ denotes the set of partitions $m = (m_0, \ldots, m_k)$ of $2k$ into nonnegative parts. They also gave some explicit values of c_k for $k = 1, 2, \ldots, 15$. These values will be presented in Section 2 and will be used to establish the main results in this paper.

The third direction in the studies of the zeros of the Riemann zeta-function is the gaps between the zeros (finding small gaps and large gaps between the zeros) on the critical line when the Riemann hypothesis is satisfied. In the present paper we are concerned with the largest gaps between the zeros on the critical line assuming that the Riemann hypothesis is true.

Assuming the truth of the Riemann hypothesis Montgomery [14] studied the distribution of pairs of nontrivial zeros $1/2 + i\gamma$ and $1/2 + i\gamma'$ and conjectured, for fixed α, β satisfying $0 < \alpha < \beta$, that

$$\lim_{T \to \infty} \frac{1}{N(T)} \# \left\{ 0 < \gamma, \gamma' < T : \alpha \le \frac{\gamma' - \gamma'}{(2\pi / \log T)} \le \beta \right\} = \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx. \tag{1.15}$$

This so-called pair correlation conjecture plays a complementary role to the Riemann hypothesis. This conjecture implies the essential simplicity hypothesis that almost all zeros of the zeta-function are simple. On the other hand, the integral on the right hand side is the same as the one observed in the two-point correlation of the eigenvalues which are the energy levels of the corresponding Hamiltonian which are usually not known with uncertainty. This observation is due to Dyson and it restored some hope in an old idea of Hilbert and Polya that the Riemann hypothesis follows from the existence of a self-adjoint Hermitian operator whose spectrum of eigenvalues correspond to the set of nontrivial zeros of the zeta-function.

Now, we assume that $(\beta_n + i\gamma_n)$ are the zeros of $\zeta(s)$ in the upper half-plane (arranged in nondecreasing order and counted according multiplicity) and $\gamma_n \le \gamma_{n+1}$ are consecutive ordinates of all zeros and define

$$r_n := \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi / \log \gamma_n)}, \tag{1.16}$$

and set

$$\lambda := \limsup_{n \to \infty} r_n, \qquad \mu := \liminf_{n \to \infty} r_n. \tag{1.17}$$

These numbers have received a great deal of attention. In fact, important results concerning the values of them have been obtained by some authors. It is generally believed that $\mu = 0$ and $\lambda = \infty$. Selberg [15] proved that

$$0 < \mu < 1 < \lambda, \tag{1.18}$$

and the average of r_n is 1. Note that $2\pi / \log \gamma_n$ is the average spacing between zeros. Fujii [16] also showed that there exist constants $\lambda > 1$ and $\mu < 1$ such that

$$\frac{(\gamma_{n+1} - \gamma_n)}{(2\pi / \log \gamma_n)} \geq \lambda, \qquad \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi / \log \gamma_n)} \leq \mu, \tag{1.19}$$

for a positive proportion of n. Mueller [17] obtained

$$\lambda > 1.9, \tag{1.20}$$

assuming the Riemann hypothesis. Montgomery and Odlyzko [18] showed, assuming the Riemann hypothesis, that

$$\lambda > 1.9799, \qquad \mu < 0.5179. \tag{1.21}$$

Conrey et al. [19] improved the bounds in (1.21) and showed that, if the Riemann hypothesis is true, then

$$\lambda > 2.337, \qquad \mu < 0.5172. \tag{1.22}$$

Conrey et al. [20] obtained a new lower bound and proved that

$$\lambda > 2.68, \tag{1.23}$$

assuming the generalized Riemann hypothesis for the zeros of the Dirichlet L-functions. Bui et al. [21] improved (1.23) and obtained

$$\lambda > 2.69, \qquad \mu < 0.5155, \tag{1.24}$$

assuming the Riemann hypothesis. Ng in [22] improved (1.24) and proved that

$$\lambda > 3, \tag{1.25}$$

assuming the generalized Riemann hypothesis for the zeros of the Dirichlet L-functions.

Hall in [23] (see also Hall [24]) assumed that $\{t_n\}$ is the sequence of distinct positive zeros of the Riemann zeta-function $\zeta(1/2+it)$ arranged in nondecreasing order and counted according multiplicity and defined the quantity

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1}-t_n}{(2\pi/\log t_n)}, \tag{1.26}$$

and showed that $\Lambda \geq \lambda$, and the lower bound for Λ bear direct comparison with such bounds for λ dependent on the Riemann hypothesis, since if this were true the distinction between Λ and λ would be nugatory. Of course $\Lambda \geq \lambda$ and the equality holds if the Riemann hypothesis is true. Hall [23] used a Wirtinger-type inequality of Beesack and proved that

$$\Lambda \geq \left(\frac{105}{4}\right)^{1/4} = 2.2635. \tag{1.27}$$

In [25] Hall proved a Wirtinger inequality and used the moment

$$\int_0^T Z^4(t)dt = \frac{1}{2\pi^2} T\log^4(t) + O\left(T\log^3\right), \tag{1.28}$$

due to Ingham [26], and the moments

$$\begin{gathered}\int_0^T (Z'(t))^4 dt = \frac{1}{1120\pi^2} T\log^8(t) + O\left(T\log^7\right),\\ \int_0^T Z^2(t)(Z'(t))^2 dt = \frac{1}{120\pi^2} T\log^6(t) + O\left(T\log^5\right),\end{gathered} \tag{1.29}$$

due to Conrey [27], and obtained

$$\Lambda \geq \sqrt{\frac{11}{2}} = 2.3452. \tag{1.30}$$

Hall [24] proved a new generalized Wirtinger-type inequality by using the calculus of variation and obtained a new value of Λ which is given by

$$\Lambda \geq \sqrt{\frac{7533}{901}} = 2.8915. \tag{1.31}$$

Hall [28] employed the generalized Wirtinger inequality obtained in [24], simplified the calculus used in [24] and converted the problem into one of the classical theory of equations involving Jacobi-Schur functions. Assuming that the moments in (1.9) are correctly predicted by RMT, Hall [28] proved that

$$\Lambda(4) \geq 3.392272\ldots, \quad \Lambda(5) \geq 3.858851\ldots, \quad \Lambda(5) \geq 4.2981467\ldots. \tag{1.32}$$

In [29] the authors applied a technique involving the comparison of the continuous global average with local average obtained from the discrete average to a problem of gaps between the zeros of zeta-function assuming the Riemann hypothesis. Using this approach, which takes only zeros on the critical line into account, the authors computed similar bounds under assumption of the Riemann hypothesis when (1.9) holds. They then showed that for fixed positive integer r

$$(\gamma_{n+r} - \gamma_n) \geq \theta\left(\frac{2\pi r}{\log \gamma_n}\right), \tag{1.33}$$

holds for any $\theta \leq 4k/\pi r e$ for more than $c(\log T)^{-4k^2}$ proportion of the zeros $\gamma_n \in [0, T]$ with a computable constant $c = c(k, \theta, r)$.

Hall [13] developed the technique used in [28] and proved that

$$\Lambda(7) \geq 4.215007. \tag{1.34}$$

The improvement of this value as obtained in [13] is given by

$$\Lambda(7) \geq 4.71474396\ldots. \tag{1.35}$$

In this paper, first we apply some well-known Wirtinger-type inequalities and the moments of the Hardy Z-function and the moments of its derivative to establish some explicit formulas for $\Lambda(k)$. Using the values of b_k and c_k, we establish some lower bounds for $\Lambda(15)$ which improves the last value of $\Lambda(7)$. In particular it is obtained that $\Lambda(15) \geq 6.1392$ which means that consecutive nontrivial zeros often differ by at least 6.1392 times the average spacing. To the best of the author knowledge the last value obtained for Λ in the literature is the value obtained by Hall in (1.35) and nothing is known regarding $\Lambda(k)$ for $k \geq 8$.

2. Main Results

In this section, we establish some explicit formulas for $\Lambda(k)$ and by using the same explicit values of b_k and c_k we establish new lower bounds for $\Lambda(15)$. The explicit values of b_k using the formula

$$b_k := \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}, \tag{2.1}$$

are calculated in the following for $k = 1, 2, \ldots, 15$:

$$\begin{gathered}
b_1 = 1, \quad b_2 = \frac{1}{2^2 3}, \quad b_3 = \frac{1}{2^6 3^3 5}, \\
b_4 = \frac{1}{2^{12} 3^5 5^3 7}, \quad b_5 = \frac{1}{2^{20} 3^9 5^5 7^3}, \\
b_6 = \frac{1}{2^{30} 3^{15} 5^7 7^5 11}, \quad b_7 = \frac{1}{2^{42} 3^{21} 5^9 7^7 11^3 13}, \\
b_8 = \frac{1}{2^{56} 3^{28} 5^{12} 7^9 11^5 13^3}, \quad b_9 = \frac{1}{2^{72} 3^{36} 5^{16} 7^{11} 11^7 13^5 17}, \\
b_{10} = \frac{1}{2^{90} 3^{44} 5^{20} 7^{13} 11^9 13^7 17^3 19}, \\
b_{11} = \frac{1}{2^{110} 3^{53} 5^{24} 7^{16} 11^{11} 13^9 17^5 19^3}, \\
b_{12} = \frac{1}{2^{132} 3^{63} 5^{28} 7^{20} 11^{13} 13^{11} 17^7 19^5 23}, \\
b_{13} = \frac{1}{2^{156} 3^{73} 5^{34} 7^{24} 11^{15} 13^{13} 17^9 19^7 23^3}, \\
b_{14} = \frac{1}{2^{182} 3^{86} 5^{42} 7^{28} 11^{17} 13^{15} 17^{11} 19^9 23^5}, \\
b_{15} = \frac{1}{2^{210} 3^{102} 5^{50} 7^{32} 11^{19} 13^{17} 17^{13} 19^{11} 23^7 29}.
\end{gathered} \tag{2.2}$$

The explicit values of the parameter c_k that has been determined by Conrey et al. [4] for $k = 1, 2, \ldots, 15$ are given in the following:

$$\begin{gathered}
c_1 = \frac{1}{2^2 \cdot 3}, \quad c_2 = \frac{1}{2^6 \cdot 3 \cdot 5 \cdot 7}, \quad c_3 = \frac{1}{2^{12} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11}, \quad c_4 = \frac{31}{2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13}, \\
c_5 = \frac{227}{2^{30} \cdot 3^{12} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19}, \quad c_6 = \frac{67 \cdot 1999}{2^{42} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}, \\
c_7 = \frac{43 \cdot 46663}{2^{56} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23}, \\
c_8 = \frac{46743947}{2^{72} \cdot 3^{34} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}, \\
c_9 = \frac{19583 \cdot 16249}{2^{90} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}, \\
c_{10} = \frac{3156627824489}{2^{110} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 31 \cdot 37}, \\
c_{11} = \frac{59 \cdot 11332613 \cdot 33391}{2^{132} \cdot 3^{63} \cdot 5^{31} \cdot 7^{18} \cdot 11^{12} \cdot 13^{10} \cdot 17^5 \cdot 19^5 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43},
\end{gathered}$$

$$c_{12} = \frac{241 \cdot 251799899121593}{2^{156} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^{8} \cdot 19^{7} \cdot 23^{4} \cdot 29^{3} \cdot 31^{2} \cdot 41 \cdot 43 \cdot 47},$$

$$c_{13} = \frac{285533 \cdot 37408704134429}{2^{182} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^{8} \cdot 23^{5} \cdot 29^{3} \cdot 31^{3} \cdot 37^{2} \cdot 41 \cdot 43 \cdot 47},$$

$$c_{14} = \frac{197 \cdot 1462253323 \cdot 6616773091}{2^{210} \cdot 3^{100} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^{7} \cdot 29^{4} \cdot 31^{4} \cdot 37^{2} \cdot 41 \cdot 43 \cdot 47 \cdot 53},$$

$$c_{15} = \frac{1625537582517468726519545837}{2^{240} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{11} \cdot 23^{9} \cdot 29^{5} \cdot 31^{5} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53 \cdot 59}. \tag{2.3}$$

Now, we are in a position to prove our first results in this section which gives an explicit formula of the gaps between the zeros of the Riemann zeta-function. This will be proved by applying an inequality due to Agarwal and Pang [30].

Theorem 2.1. *Assuming the Riemann hypothesis, one has*

$$\Lambda(k) \geq \frac{1}{2\pi}\left(\frac{b_k}{c_k}\frac{2\Gamma(2k+1)}{\Gamma^2((2k+1)/2)}\right)^{1/2k}. \tag{2.4}$$

Proof. To prove this theorem, we employ the inequality

$$\int_0^{\pi} \left(x'(t)\right)^{2k} dt \geq \frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)} \int_0^{\pi} x^{2k}(t)dt, \quad \text{for } k \geq 1, \tag{2.5}$$

with $x(t) \in C^1[0,\pi]$ and $x(0) = x(\pi) = 0$, that has been proved by Agarwal and Pang [30]. As in [25] by a suitable linear transformation, we can deduce from (2.5) that if $x(t) \in C^1[a,b]$ and $x(a) = x(b) = 0$, then

$$\int_a^b \left(\frac{b-a}{\pi}\right)^{2k} \left(x'(t)\right)^{2k} dt \geq \frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)} \int_a^b x^{2k}(t)dt, \quad \text{for } k \geq 1. \tag{2.6}$$

Now, we follow the proof of [24] and supposing that t_l is the first zero of $Z(t)$ not less than T and t_m the last zero not greater than $2T$. Suppose further that for $l \leq n < m$, we have

$$L_n = t_{n+1} - t_n \leq \frac{2\pi\kappa}{\log T}, \tag{2.7}$$

and apply the inequality (2.6), to obtain

$$\int_{t_n}^{t_{n+1}} \left[\left(\frac{L_n}{\pi}\right)^{2k} \left(Z'(t)\right)^{2k} - \frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)} \left(Z(t)\right)^{2k}\right] dt \geq 0. \tag{2.8}$$

Since the inequality remains true if we replace L_n/π by $2\kappa/\log T$, we have

$$\int_{t_n}^{t_{n+1}}\left[\left(\frac{2\kappa}{\log T}\right)^{2k}\left(Z'(t)\right)^{2k}-\frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}\left(Z(t)\right)^{2k}\right]dt\geq 0. \tag{2.9}$$

Summing (2.9) over n, applying (1.7), (1.12) and as in [24], we obtain

$$\begin{aligned} a(k)c_k\left(\frac{2\kappa}{\log T}\right)^{2k}T(\log T)^{k^2+2k}-\frac{2a(k)b_k\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}T(\log T)^{k^2} \\ =\left(a(k)c_k\kappa^{2k}\left(2^{2k}\right)-\frac{2a(k)b_k\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}\right)T(\log T)^{k^2} \\ \geq O\left(T\log^{k^2}T\right), \end{aligned} \tag{2.10}$$

whence

$$\kappa^{2k}\geq\frac{a(k)b_k}{2^{2k}a(k)c_k}\frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}=\frac{b_k}{2^{2k}c_k}\frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}\quad(\text{as } T\longrightarrow\infty). \tag{2.11}$$

This implies that

$$\Lambda^{2k}(k)\geq\frac{b_k}{2^{2k}c_k}\frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}, \tag{2.12}$$

and then we obtain the desired inequality (2.1). The proof is complete. □

Using the values of b_k and c_k and (2.1) we have the new lower values for $\Lambda(k)$ for $k=1,2,\ldots,15$ in Table 1.

One can easily see that the value of $\Lambda(7)$ in Table 1 does not improve the lower bound in (1.35) due to Hall, but the the approach that we used is simple and depends only on a well-known Wirtinger-type inequality and the asymptotic formulas of the moments. In the following, we employ a different inequality due to Brnetić and Pečarić [31] and establish a new explicit formula for $\Lambda(k)$ and then use it to find new lower bounds.

Theorem 2.2. *Assuming the Riemann hypothesis, one has*

$$\Lambda(k)\geq\frac{1}{2\pi}\left(\frac{b_k}{c_k}\frac{1}{I(k)}\right)^{1/2k}, \tag{2.13}$$

where I_k is defined by

$$I(k):=\int_0^1\frac{1}{\left(t^{1-2k}+(1-t)^{1-2k}\right)}dt. \tag{2.14}$$

Table 1

$\Lambda(1)$	$\Lambda(2)$	$\Lambda(3)$	$\Lambda(4)$	$\Lambda(5)$
1.2442	1.7675	2.2265	2.6544	3.0545
$\Lambda(6)$	$\Lambda(7)$	$\Lambda(8)$	$\Lambda(9)$	$\Lambda(10)$
3.4259	3.7676	4.0806	4.3681	4.6342
$\Lambda(11)$	$\Lambda(12)$	$\Lambda(13)$	$\Lambda(14)$	$\Lambda(15)$
4.8827	5.1169	5.3393	5.5515	5.7550

Table 2

$I(1)$	$I(2)$	$I(3)$	$I(4)$	$I(5)$
$\frac{16\,667}{100\,000}$	$\frac{2863}{125000}$	$\frac{19\,581}{5000\,000}$	$\frac{743}{1000\,000}$	$\frac{14\,961}{100\,000\,000}$
$I(6)$	$I(7)$	$I(8)$	$I(9)$	$I(10)$
$\frac{15\,653}{500\,000\,000}$	$\frac{16\,823}{2500\,000\,000}$	$\frac{7377}{5000\,000\,000}$	$\frac{8211}{25000\,000\,000}$	$\frac{37\,001}{500\,000\,000\,000}$
$I(11)$	$I(12)$	$I(13)$	$I(14)$	$I(15)$
$\frac{8419}{500\,000\,000\,000}$	$\frac{19\,311}{5000\,000\,000\,000}$	$\frac{89\,199}{100\,000\,000\,000\,000}$	$\frac{20\,721}{100\,000\,000\,000\,000}$	$\frac{48\,377}{1000\,000\,000\,000\,000}$

Proof. To prove this theorem, we apply the inequality

$$\int_0^{\pi} \left(x'(t)\right)^{2k} dt \geq \frac{1}{\pi^{2k} I(k)} \int_0^{\pi} x^{2k}(t)dt, \quad \text{for } k \geq 1, \tag{2.15}$$

that has been proved by Brnetić and Pečarić [31], where $x(t)$ is continuous function on $[0, \pi]$ with $x(0) = x(\pi) = 0$. Proceeding as in the proof of Theorem 2.1 and employing (2.15), we may have

$$\kappa^{2k} \geq \frac{a(k)b_k}{2^{2k}a(k)c_k} \frac{1}{\pi^{2k}I(k)} = \frac{b_k}{2^{2k}c_k} \frac{1}{\pi^{2k}I(k)} \quad (\text{as } T \longrightarrow \infty). \tag{2.16}$$

This implies that

$$\Lambda^{2k}(k) \geq \frac{b_k}{2^{2k}c_k} \frac{1}{\pi^{2k}I(k)}. \tag{2.17}$$

which is the desired inequality (2.13). The proof is complete. □

To find the new lower bounds for $\Lambda(k)$ we need the values of $I(k)$ for $k = 1, \ldots, 15$. These values are calculated numerically in Table 2.

Using these values and the values of b_k,c_k, and the explicit formula (2.13) we have the new lower bounds for $\Lambda(k)$ in Table 3.

Table 3

$\Lambda(1)$	$\Lambda(2)$	$\Lambda(3)$	$\Lambda(4)$	$\Lambda(5)$
1.3505	1.9902	2.4905	2.9389	3.3508
$\Lambda(6)$	$\Lambda(7)$	$\Lambda(8)$	$\Lambda(9)$	$\Lambda(10)$
3.7287	4.0736	4.3875	4.6742	4.9384
$\Lambda(11)$	$\Lambda(12)$	$\Lambda(13)$	$\Lambda(14)$	$\Lambda(15)$
5.1845	5.4159	5.6353	5.8444	6.0449

We note from Table 3 that the value of $\Lambda(15)$ improves the value $\Lambda(7)$ that has been obtained by Hall.

Finally, in the following we will employ an inequality to Beesack [32, page 59] and establish a new explicit formula for $\Lambda(k)$ and use it to find new values of its lower bounds.

Theorem 2.3. *Assuming the Riemann hypothesis, one has*

$$\Lambda(k) \geq \frac{1}{2k\sin(\pi/2k)}\left((2k-1)\frac{b_k}{c_k}\right)^{1/2k}. \tag{2.18}$$

Proof. To prove this theorem, we apply the inequality

$$\int_0^\pi \left(x'(t)\right)^{2k} dt \geq \frac{2k-1}{(k\sin(\pi/2k))^{2k}}\int_0^\pi x^{2k}(t)dt, \quad \text{for } k \geq 1, \tag{2.19}$$

that has been proved by Beesack [32, page 59], where $x(t)$ is continuous function on $[0,\pi]$ with $x(0) = x(\pi) = 0$. Proceeding as in Theorem 2.1 by using (2.19), we may have

$$\kappa^{2k} \geq \frac{a(k)b_k}{2^{2k}a(k)c_k}\frac{2k-1}{(k\sin(\pi/2k))^{2k}} = \frac{b_k}{2^{2k}c_k}\frac{2k-1}{(k\sin(\pi/2k))^{2k}} \quad (\text{as } T \longrightarrow \infty). \tag{2.20}$$

This implies that

$$\Lambda^{2k}(k) \geq \frac{b_k}{2^{2k}c_k}\frac{2k-1}{(k\sin(\pi/2k))^{2k}}, \tag{2.21}$$

which is the desired inequity (2.18). The proof is complete. □

Using these values and the values of b_k,c_k, and the explicit formula in (2.18) we have the new lower bounds for $\Lambda(k)$ in Table 4.

We note from Table 4, that the values of $\Lambda(k)$ for $k = 1,\ldots,7$ are compatible with the values of $\Lambda(k)$ for $k = 1,\ldots,7$ that has been obtained by Hall [13, Table 1(i)] and since there is no explicit value of $H(h,k)$ for $h,k \geq 8$, to obtain the values of $\Lambda(k)$ for $k \geq 8$ the author in [13] stopped the estimation for $\Lambda(k)$ for $k \geq 8$.

Table 4

$\Lambda(1)$	$\Lambda(2)$	$\Lambda(3)$	$\Lambda(4)$	$\Lambda(5)$
1.7321	2.2635	2.7080	3.1257	3.5177
$\Lambda(6)$	$\Lambda(7)$	$\Lambda(8)$	$\Lambda(9)$	$\Lambda(10)$
3.8814	4.215	4.5196	4.7985	5.0560
$\Lambda(11)$	$\Lambda(12)$	$\Lambda(13)$	$\Lambda(14)$	$\Lambda(15)$
5.2962	5.5225	5.7373	5.9424	6.1392

We notice that the calculations can be continued as above just if one knows the explicit values of c_k for $k \geq 16$ where the values

$$b_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \tag{2.22}$$

are easy to calculate. Note that the values of c_k that we have used in this paper are adapted from the paper by Conrey et al. [4]. It is clear that the values of $\Lambda(k)$ are increasing with the increase of k and this may help in proving the conjecture of the distance between of the zeros of the Riemann zeta-function.

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