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*Research Article*

# Oscillatory Criteria for the Two-Dimensional Difference Systems

**Jin-Fa Cheng[1] and Yu-Ming Chu[2]**

[1] *Department of Mathematics, Xiamen University, Xiamen 361005, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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We establish some necessary and sufficient conditions for oscillation of the solutions of the following two-dimensional difference system: $\Delta x_n = f(n, y_n)$, $\Delta y_n = -g(n, x_n)$, where $f(n,u)$ and $g(n,u)$ are strongly superlinear or sublinear functions.

## 1. Introduction

We consider the following two-dimensional nonlinear difference system as follows:

$$\begin{aligned} \Delta x_n &= f(n, y_n), \\ \Delta y_n &= -g(n, x_n), \end{aligned} \tag{1.1}$$

where $\Delta x_n = x_{n+1} - x_n$, $\Delta y_n = y_{n+1} - y_n$, $f(n,u)$ and $g(n,u)$ are strongly superlinear or sublinear functions.

Now we pose some conditions on functions $f$ and $g$:

$(H1)$: $uf(n,u) > 0$ and $ug(n,u) > 0$ for $u \neq 0$;

$(H2)$: $f(n,u)$ and $g(n,u)$ are continuous real-valued functions, and nondecreasing with respect to $u$;

$(H3)$: it is

$$\sum_{n=n_0}^{\infty} f(n, \pm c) = \pm\infty \tag{1.2}$$

for each $c > 0$.

*Definition 1.1.* Suppose that $f, g : N \times R \to R$ are real-valued functions. $\alpha$ and $\beta$ are the quotients of positive odd numbers.

(1) $f$ and $g$ are said to be strongly superlinear if there exist constants $\alpha > 0$ and $\beta > 0$ with $\alpha\beta > 1$, such that $f(n,u)/|u|^{\alpha} \operatorname{sgn} u$ and $g(n,u)/|u|^{\beta} \operatorname{sgn} u$ are nondecreasing with respect to $|u|$ for each fixed $n \in N$.

(2) $f$ and $g$ are said to be strongly sublinear if there exist constants $\alpha > 0$ and $\beta > 0$ with $\alpha\beta < 1$, such that $f(n,u)/|u|^{\alpha} \operatorname{sgn} u$ and $g(n,u)/|u|^{\beta} \operatorname{sgn} u$ are nonincreasing with respect to $|u|$ for each fixed $n \in N$.

The solutions of (1.1) are said to be nonoscillatory if $\{x_n\}$ or $\{y_n\}$ is eventually positive or negative. Otherwise the solutions are called oscillatory.

Some oscillation results for the difference system (1.1) in the case of $g(n, x_n) = a_n x_n^{\beta}$ with $a_n > 0$ have been established by many authors. In particular, if $f(n, y_n) = b_n y_n$ and $b_n > 0$, then the difference system (1.1) is reduced to the well-known second-order nonlinear difference equation:

$$\Delta\left(\frac{1}{b_n}\Delta x_n\right) + a_n x_n^{\beta} = 0. \tag{1.3}$$

Also, if $b_n = 1$, then (1.3) becomes

$$\Delta^2 x_n + a_n x_n^{\beta} = 0. \tag{1.4}$$

Furthermore, if $f(n, y_n) = b_n y_n^{\alpha}$ and $\alpha$ is a ratio of odd positive integers, then (1.1) reduces to the well-known quasilinear difference equation:

$$\Delta\left(\frac{1}{b_n^{1/\alpha}}(\Delta x_n)^{1/\alpha}\right) + a_n x_n^{\beta} = 0. \tag{1.5}$$

For (1.4), the following well-known Theorem A was established by Hooker and Patula [1, 2].

**Theorem A.** *For* (1.4), *the following statements are true.*

*(1) If* $0 < \beta < 1$, *then every solution of* (1.4) *oscillates if and only if*

$$\sum_{n=1}^{\infty} n^{\beta} a_n = \infty. \tag{1.6}$$

*(2) If* $\beta > 1$, *then every solution of* (1.4) *oscillates if and only if*

$$\sum_{n=1}^{\infty} n a_n = \infty. \tag{1.7}$$

*For* (1.3), *if one denotes*

$$B_n = \sum_{s=0}^{n-1} b_s \tag{1.8}$$

*and assumes that*

$$\lim_{n\to\infty} B_n = \sum_{s=0}^{\infty} b_s = \infty, \tag{1.9}$$

*then the following theorem is proved in [3].*

**Theorem B.** *If* (1.9) *holds, then the following statements are true.*

*(1) If* $0 < \beta < 1$, *then every solution of* (1.3) *oscillates if and only if*

$$\sum_{n=1}^{\infty} B_n^{\beta} a_n = \infty. \tag{1.10}$$

*(2) If* $\beta > 1$, *then every solution of* (1.3) *oscillates if and only if*

$$\sum_{n=1}^{\infty} B_n a_n = \infty. \tag{1.11}$$

The problem of oscillation of second-order nonlinear difference equations has attracted the attention of many mathematicians because of its physical applications [2, 4]. For some results regarding the growth of solutions of some equations related to the above mentioned see book [5], as well as the following papers [6–8]. It is an interesting problem to extend oscillation criteria for second-order nonlinear difference equations to the case of nonlinear two-dimensional difference systems since such systems include, in particular, the second-order nonlinear, half-linear, and quasilinear difference equations that are the special cases of the nonlinear two-dimensional difference systems [5, 9, 10].

The main purpose of this paper is to establish some necessary and sufficient conditions for oscillation of the nonlinear two-dimensional difference systems.

## 2. Main Results

In order to establish our main results, we need the following lemma.

**Lemma 2.1.** *Suppose that conditions* $(H1)$–$(H3)$ *are satisfied. If* $\{x_n\}$ *and* $\{y_n\}$ *are nonoscillatory solutions of* (1.1) *for* $n > n_0$*, then*

$$\operatorname{sgn} x_n = \operatorname{sgn} y_n. \tag{2.1}$$

*Proof.* Without loss of generality, we assume that $x_n$ is eventually positive; that is, $x_n > 0$ for $n > n_0 > 0$. From (1.1), we clearly see that $\Delta y_n < 0$, then we know that either $y_n > 0$ or $y_n < 0$ eventually holds.

If $y_n < y_{N_1} < 0$ for $n > N_1 > n_0$, then we have

$$\Delta x_n = f(n, y_n) \leq f(n, y_N) < 0, \tag{2.2}$$

summing up from $N_1$ to $n$, and by (1.2) of $(H3)$, we get

$$x_n - x_{N_1} \leq \sum_{s=N_1}^{n-1} f(s, y_N) \longrightarrow -\infty \quad (n \longrightarrow \infty). \tag{2.3}$$

This contradiction completes the proof of the lemma. □

**Theorem 2.2.** *If* $f$ *and* $g$ *are strongly sublinear (i.e.,* $0 < \alpha\beta < 1$*), then a necessary and sufficient condition for* (1.1) *to oscillate is that*

$$\sum_{n=n_1}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right) = +\infty \tag{2.4}$$

*for every* $c > 0$*, where* $n_1 > n_0$*.*

*Proof. Sufficiency*. If (1.1) has a nonoscillatory solution $x_n$, then without loss of generality, we assume that $x_n$ is eventually positive. Then, by Lemma 2.1, for $n_0$ sufficiently large,

$$y_n > 0, \quad \Delta x_n > 0, \quad \Delta y_n < 0, \quad \text{for } n \geq n_0. \tag{2.5}$$

Since $\{y_n\}$ is decreasing, hence there exists $c > 0$ such that

$$y_n \leq y_N \leq c, \quad \text{for } N \geq n_0. \tag{2.6}$$

Summing up

$$\Delta x_n = f(n, y_n) \tag{2.7}$$

from $s = n_0$ to $n-1$, we obtain

$$x_n - x_{n_0} = \sum_{s=n_0}^{n-1} f(s, y_s) \geq y_n^{\alpha} \sum_{s=n_0}^{n-1} \frac{f(s, y_s)}{y_s^{\alpha}} \geq y_n^{\alpha} \sum_{s=n_0}^{n-1} \frac{f(s, c)}{c^{\alpha}},$$
$$x_n \geq c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c), \tag{2.8}$$

and so

$$g(n, x_n) \geq g\left(n, c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right). \tag{2.9}$$

Therefore

$$-\Delta y_n = g(n, x_n) \geq g\left(n, c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right). \tag{2.10}$$

Since

$$c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c) \leq \sum_{s=n_0}^{n-1} f(s, c), \tag{2.11}$$

we have

$$\begin{aligned} -\Delta y_n \geq g\left(n, c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right) &= \frac{g\left(n, c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right)}{\left(c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right)^{\beta}} \left(c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right)^{\beta} \\ &\geq \frac{g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right)}{\left(\sum_{s=n_0}^{n-1} f(s, c)\right)^{\beta}} \left(c^{-\alpha} y_n^{\alpha} \sum_{s=n_0}^{n-1} f(s, c)\right)^{\beta} \\ &= (y_n)^{\alpha\beta} c^{-\alpha\beta} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right), \\ -\frac{\Delta y_n}{(y_n)^{\alpha\beta}} \geq c^{-\alpha\beta} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right), \end{aligned} \tag{2.12}$$

and let $n_1 > n_0$, we have

$$\sum_{i=n_1}^{n-1} \left(-\frac{\Delta y_i}{(y_i)^{\alpha\beta}}\right) \geq c^{-\alpha\beta} \sum_{i=n_1}^{n-1} g\left(i, \sum_{s=n_0}^{i-1} f(s, c)\right). \tag{2.13}$$

From

$$\int_{y_{n+1}}^{y_n} \frac{1}{u^{\alpha\beta}} du = \frac{1}{\xi^{\alpha\beta}} (y_n - y_{n+1}) \geq -\frac{\Delta y_n}{(y_n)^{\alpha\beta}}, \quad y_{n+1} < \xi < y_n, \tag{2.14}$$

we get

$$\begin{gathered} \sum_{i=n_1}^{\infty} \int_{y_{i+1}}^{y_i} \frac{1}{u^{\alpha\beta}} du \geq \sum_{i=n_1}^{\infty} -\frac{\Delta y_i}{(y_i)^{\alpha\beta}} \geq c^{-\alpha\beta} \sum_{i=n_1}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,c)\right), \\ \sum_{i=n_1}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,c)\right) \leq c^{\alpha\beta} \int_{c}^{y_{n_1}} \frac{1}{u^{\alpha\beta}} du < +\infty, \end{gathered} \tag{2.15}$$

which leads to a contradiction.
*Necessity*. If

$$\sum_{n=n_1}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s,c)\right) < +\infty \tag{2.16}$$

for some $c > 0$, then there exist $M > n_0 > 0$ and $c/2 > d > 0$, such that

$$\sum_{n=M}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s,c)\right) < d. \tag{2.17}$$

Let $X$ be the Banach space of all the real-valued sequences $\{x_n\}$ with the norm

$$\|x\| = \sup_{n \geq M} \frac{|x_n|}{\sum_{s=n_0}^{n-1} f(s,c)}, \tag{2.18}$$

let $\Psi$ be the subset of $X$ defined by

$$\Psi = \left\{ \{x_n\} \in X : \sum_{s=n_0}^{n-1} f(s,d) \leq x_n \leq \sum_{s=n_0}^{n-1} f(s,2d) \right\} \tag{2.19}$$

and let $F : \Psi \to X$ be the operator defined by

$$(Fx)_n = \sum_{s=n_0}^{n-1} f\left(s, d + \sum_{i=s}^{\infty} g(i, x_i)\right). \tag{2.20}$$

Then the mapping $F$ satisfies the assumptions of Knaster's fixed-point theorem (see [11, page 8]): $F$ maps $\Psi$ into itself and $F$ is increasing. The latter statement is easy to see, and the former statement follows from

$$\begin{aligned}
(Fx)_n &\geq \sum_{n=n_0}^{n-1} f(s,d),\\
(Fx)_n &= \sum_{s=n_0}^{n-1} f\left(s, d+\sum_{i=s}^{\infty} g(i,x_i)\right) \leq \sum_{s=n_0}^{n-1} f\left(s, d+\sum_{i=s}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,2d)\right)\right)\\
&\leq \sum_{s=n_0}^{n-1} f\left(s, d+\sum_{i=s}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,c)\right)\right) \leq \sum_{s=n_0}^{n-1} f(s,2d)
\end{aligned} \tag{2.21}$$

for any $\{x_n\} \in \Psi$. From Knaster's fixed-point theorem, we know that there exists $\{x_n\} \in \Psi$ such that $x_n = (Fx)_n$. Let

$$y_n = d + \sum_{s=n}^{\infty} g(s, x_s), \tag{2.22}$$

then $\lim_{n\to\infty} y_n = d$ and $\Delta y_n = -g(n, x_n)$. On the other hand, we have

$$x_n = (Fx)_n = \sum_{i=n_0}^{n-1} f(i, y_i). \tag{2.23}$$

Then by (1.2) and the continuity of function $f$, we have that $\lim_{n\to\infty} x_n = \infty$ and $\Delta x_n = f(n, x_n)$, which leads to a contradiction and the proof of Theorem 2.2 is completed. □

*Example 2.3.* Considering the difference system,

$$\begin{aligned}
\Delta x_n &= 2(n+1)^{1/3} y_n^{1/3},\\
\Delta y_n &= -\frac{2n+3}{(n+1)(n+2)} x_n^{5/3}, \quad n \geq n_0.
\end{aligned} \tag{2.24}$$

Here $\alpha = 1/3$, $\beta = 5/3$, and $f$ and $g$ are strongly sublinear. It is easy to verify that the conditions of Theorem 2.2 are satisfied and hence all solutions are oscillatory. In fact, we clearly see that the sequence $\{(x_n, y_n)\} = \{((-1)^n, (-1)^{n+1}/(n+1))\}$ is such a solution for the difference system.

*Example 2.4.* Considering the difference system,

$$\begin{aligned}
\Delta x_n &= \left(\frac{n}{n+1}\right)^{1/3} y_n^{1/3},\\
\Delta y_n &= -\frac{1}{n^{5/3}} \frac{1}{n(n+1)} x_n^{5/3}, \quad n \geq n_0.
\end{aligned} \tag{2.25}$$

Here $\alpha = 1/3$, $\beta = 5/3$, and $f$ and $g$ are strong sublinear. We clearly see that the conditions of Theorem 2.2 are not satisfied and hence there exists a nonoscillatory solution. In fact, the sequence $\{(x_n, y_n)\} = \{(n, (n+1)/n)\}$ is such a solution.

**Theorem 2.5.** *If $f$ and $g$ are strongly superlinear (i.e., $\alpha\beta > 1$), then a necessary and sufficient condition for* (1.1) *to oscillate is that*

$$\sum_{n=1}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s, c)\right) = +\infty \tag{2.26}$$

*for every $c > 0$.*

*Proof. Sufficiency*. If (1.1) has a nonoscillatory solution $x_n$, then without loss of generality, we may assume that $x_n$ is eventually positive. Then by Lemma 2.1, we have for $N_1$ sufficiently large,

$$x_n > 0, \quad \Delta x_n > 0, \quad \Delta y_n < 0, \quad \text{for } n > N_1. \tag{2.27}$$

Since $\{x_n\}$ is increasing, hence there exists $c > 0$ such that

$$x_n \geq y_N \geq c, \quad \text{for } N > n_0. \tag{2.28}$$

Summing up

$$\Delta y_n = -g(n, x_n) \tag{2.29}$$

from $s = n$ to $\infty$, we have

$$\begin{gathered} -y_n \leq y_\infty - y_n = -\sum_{s=n}^{\infty} g(s, x_s), \\ y_n \geq \sum_{s=n}^{\infty} g(s, x_s) \geq \sum_{s=n+1}^{\infty} \frac{g(s, x_s)}{x_s^{\beta}} x_s^{\beta} \geq x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} \frac{g(s, c)}{c^{\beta}} = c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s, c). \end{gathered} \tag{2.30}$$

Therefore

$$\Delta x_n = f(n, y_n) \geq f\left(n, c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s, c)\right). \tag{2.31}$$

From $x_n \ge c$ and $c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c) \ge \sum_{s=n+1}^{\infty} g(s,c)$, we get

$$\begin{aligned}
\Delta x_n &\ge \frac{f\left(n, c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)}{\left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha}} \left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha} \\
&\ge \frac{f\left(n, \sum_{s=n+1}^{\infty} g(s,c)\right)}{\left(\sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha}} \left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha} \\
&= x_{n+1}^{\alpha\beta} c^{-\alpha\beta} f\left(n, \sum_{s=n+1}^{\infty} g(s,c)\right), \\
\frac{\Delta x_n}{x_{n+1}^{\alpha\beta}} &\ge c^{-\alpha\beta} f\left(n, \sum_{s=n+1}^{\infty} g(s,c)\right), \\
\sum_{i=n}^{\infty} \frac{\Delta x_i}{x_{i+1}^{\alpha\beta}} &\ge c^{-\alpha\beta} \sum_{i=n}^{\infty} f\left(i, \sum_{s=i+1}^{\infty} g(s,c)\right).
\end{aligned} \tag{2.32}$$

But

$$\begin{gathered}
\int_{x_n}^{x_{n+1}} \frac{1}{u^{\alpha\beta}} du = \frac{1}{\xi^{\alpha\beta}} (x_{n+1} - x_n) \ge \frac{\Delta x_n}{(x_{n+1})^{\alpha\beta}}, \quad x_n < \xi < x_{n+1}, \\
\sum_{s=n}^{\infty} \int_{x_s}^{x_{s+1}} \frac{1}{u^{\alpha\beta}} du \ge \sum_{s=n}^{\infty} \frac{\Delta x_s}{(x_{s+1})^{\alpha\beta}} \ge c^{-\alpha\beta} \sum_{s=n}^{\infty} f\left(s, \sum_{t=s+1}^{\infty} g(t,c)\right).
\end{gathered} \tag{2.33}$$

Therefore

$$\sum_{s=N_1}^{\infty} f\left(s, \sum_{t=s+1}^{\infty} g(t,c)\right) \le c^{\alpha\beta} \int_{x_{N_1}}^{\infty} \frac{1}{u^{\alpha\beta}} du < +\infty, \tag{2.34}$$

which leads to a contradiction.

*Necessity*. If

$$\sum_{n=1}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s,c)\right) < +\infty \tag{2.35}$$

for some $c > 0$, then there exists $M > 0$ large enough, such that

$$\sum_{n=M}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s,c)\right) < \frac{c}{2}. \tag{2.36}$$

Let $X$ be the set of all bounded and real-valued sequences $\{x_n\}$ with the norm

$$\|x\| = \sup_{n\geq M}|x_n| \tag{2.37}$$

and $\Psi$ be the subset of $X$ defined by

$$\Psi = \left\{\{x_n\} \in X : \frac{c}{2} \leq x_n \leq c\right\}, \tag{2.38}$$

then $\Psi$ is a bounded, convex, and closed subset of $X$. Let $F : \Psi \to X$ be the operator defined by

$$(Fx)_n = c - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right). \tag{2.39}$$

Then $F$ maps $\Psi$ into $\Psi$. In fact, if $\{x_n\} \in \Psi$, then

$$\begin{aligned} c \geq (Fx)_n &\geq c - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, c)\right) \\ &\geq c - \sum_{i=M}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, c)\right) \geq \frac{c}{2}. \end{aligned} \tag{2.40}$$

Next, we show that $F$ is continuous. Let $\{x_n^{(j)}\}$ be a convergent sequence in $\Psi$ such that $\lim_{i\to\infty}\|x_n^{(j)} - x_n\| = 0$, then from that $\Psi$ is closed ($\{x_n\} \in \Psi$) and the definition of $F$, we have

$$\begin{aligned} \left|\left(Fx^j\right)_n - (Fx)_n\right| &= \left|\sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g\left(s, x_s^j\right)\right) - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right)\right| \\ &\leq \sum_{i=n}^{\infty}\left| f\left(i, \sum_{s=i}^{\infty} g\left(s, x_s^j\right)\right) - f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right)\right|. \end{aligned} \tag{2.41}$$

Since $\sum_{i=n}^{\infty} f(i, \sum_{s=i}^{\infty} g(s, x_s)) < \sum_{i=n}^{\infty} f(i, \sum_{s=i}^{\infty} g(s, c)) < \infty$, now from the continuity of $f$ and $g$ together with the well-known Lebesgue's dominated convergence theorem (see [11, page 263]), we know that $\lim_{j\to\infty}\|(Fx^j)_n - (Fx)_n\| = 0$ for $\|x^{(j)} - x\| \to 0$.

Finally, we show that $F\Psi$ is precompact. Let $\{x_m\} \in \Psi$, $\{x_n\} \in \Psi$, then for large enough $m > n$ we have

$$\begin{aligned} |(Fx)_m - (Fx)_n| &= \left| \sum_{i=m}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right) - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right) \right| \\ &= \left| \sum_{i=n}^{m} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right) \right| \\ &\le \sum_{i=n}^{m} f\left(i, \sum_{s=i}^{\infty} g(s, c)\right) < \varepsilon \end{aligned} \tag{2.42}$$

for any $\varepsilon > 0$. From Schauder's fixed-point theorem (see [11]), we know that there exists $\{x_n\} \in \Psi$ such that $x_n = (Fx)_n$.

Let

$$y_n = \sum_{s=n}^{\infty} g(s, x_s), \tag{2.43}$$

then $\lim_{n\to\infty} y_n = 0$ and $\Delta y_n = -g(n, x_n)$. On the other hand, we have

$$x_n = (Fx)_n = c - \sum_{s=n}^{\infty} f(s, y_s). \tag{2.44}$$

Therefore, $\lim_{n\to\infty} x_n = c$ and $\Delta x_n = f(n, y_n)$, which leads to a contradiction. The proof of Theorem 2.5 is completed. □

*Example 2.6.* Considering the difference system,

$$\begin{aligned} \Delta x_n &= 2^{3n+1} y_n^3, \\ \Delta y_n &= -\frac{3}{2^n} x_n^3, \quad n \ge n_0. \end{aligned} \tag{2.45}$$

Here $\alpha = 3$, $\beta = 3$, and $f$ and $g$ are strongly suplinear. We clearly see that the conditions of Theorem 2.5 are satisfied and hence all solutions are oscillatory. In fact, the sequence $\{(x_n, y_n)\} = \{((-1)^n, (-1)^{n+1}/2^n)\}$ is such a solution.

*Example 2.7.* Considering the difference system,

$$\begin{aligned} \Delta x_n &= \left(\frac{n}{n+1}\right)^{2/3} y_n^{2/3}, \\ \Delta y_n &= -\frac{1}{n^{5/3}} \frac{1}{n(n+1)} x_n^{5/3}, \quad n \ge n_0. \end{aligned} \tag{2.46}$$

Here $\alpha = 2/3$, $\beta = 5/3$, and $f$ and $g$ are strong sublinear. However, it is easy to see that the conditions of Theorem 2.5 are not satisfied and hence there exists a nonoscillatory solution. In fact, the sequence $\{(x_n, y_n)\} = \{(n, (n+1)/n)\}$ is such a solution.

If we set $f(n, y_n) = b_n y_n^{\alpha}, g(n, x_n) = a_n x_n^{\beta}$, then the difference system (1.1) is reduced to (1.5). From Theorems 2.2 and 2.5, we get the following results for (1.5).

**Corollary 2.8.** *If* $0 < \alpha\beta < 1$, *then every solution of* (1.5) *oscillation if and only if*

$$\sum_{n=n_1}^{\infty} a_n \left( \sum_{s=n_0}^{n-1} b_s \right)^{\beta} = \infty, \tag{2.47}$$

*where* $n_1 > n_0$.

**Corollary 2.9.** *If* $\alpha\beta > 1$, *then every solution of* (1.5) *oscillation if and only if*

$$\sum_{n=1}^{\infty} b_n \left( \sum_{s=n}^{\infty} a_s \right)^{\alpha} = \infty. \tag{2.48}$$

*Remark 2.10.* It is easy to see that Theorems A and B are the special cases of our Corollaries 2.8 and 2.9, respectively.

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