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Research Article

Some Properties of Multiple Parameters Linear Programming

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We consider a linear programming problem in which the right-hand side vector depends on multiple parameters. We study the characters of the optimal value function and the critical regions based on the concept of the optimal partition. We show that the domain of the optimal value function f can be decomposed into finitely many subsets with disjoint relative interiors, which is different from the result based on the concept of the optimal basis. And any directional derivative of f at any point can be computed by solving a linear programming problem when only an optimal solution is available at the point.

1. Introduction

Parametric and sensitivity analyses are classic subject in linear programming problems. They are of great importance in the analysis of practical linear models. Almost any textbook includes a section about them and many commercial optimization package offer an option to perform postoptimal analysis. Over the years we have learned to use an optimal basic solution to perform parametric and sensitivity analyses. However, this approach has led to the existing misuse of parametric optimization in commercial packages [1]. This misuse is of course a shortcoming of the packages and by no means a shortcoming in the model existing theoretical literature. In [2–4], an alternative optimal partition approach to one-parameter linear programming and sensitivity analysis was proposed, which is based on the concept of an optimal partition. The optimal partition corresponding to a pair of primal-dual strictly complementary optimal solutions is uniquely determined (unlike the optimal basis). The approach has the advantage that contains the information needed to defined the local behavior of the optimal solutions and the optimal objective function value of a parametric linear programming, and avoids any misunderstanding. Goldfarb and Scheinberg [5] extend the optimal partition approach to one-parameter semidefinite programming and Yildirim [6] to one-parameter conic optimization. They investigate mainly the range of perturbations for which the optimal partition remains constant. In this paper, we extend this approach to multiple parameters linear programming. Our special attention is paid to investigate some properties of the whole range of perturbations for which the given problem has a finite optimal solution and the optimal value function on it.

The paper is organized as follows. In the next section we introduce the related concepts. In Section 3 the property of optimal value function is discussed. In Section 4 the character of the critical region is described. In the last section our conclusions are summarized.

2. Preliminaries

In this paper we deal with a problem (P) in standard format:

$$\min\left\{c^T x : Ax = b,\ x \geq 0\right\}, \tag{P}$$

and the dual problem (D) is written as

$$\max\left\{b^T y : A^T y + s = c,\ s \geq 0\right\}, \tag{D}$$

where matrix $A \in R^{m\times n}$ with rank m, vector $x, c, s \in R^n$ and $y, b \in R^m$. $x \geq 0$ means that each coordinate of x is greater than or equal to zero. We assume that (P) and (D) are both feasible hereafter. The feasible regions of (P) and (D) are denoted, respectively, by

$$\begin{aligned} P &:= \{x : Ax = b,\ x \geq 0\}, \\ D &:= \left\{(y, s) : A^T y + s = c,\ s \geq 0\right\}. \end{aligned} \tag{2.1}$$

The optimal solutions set of (P) and (D) are denoted by P^* and D^*, respectively. We define the index sets B and N by

$$\begin{aligned} B &:= \{i : x_i > 0 \text{ for some } x \in P^*\}, \\ N &=: \left\{i : s_i > 0 \text{ for some } (y, s) \in D^*\right\}. \end{aligned} \tag{2.2}$$

Then from [4], we have $B \cap N = \emptyset$ and $B \cup N = \{1, 2, \ldots, n\}$. Thus B and N form a partition of the full index set. This partition, denoted by $\pi = (B, N)$, is called the optimal partition of (P) and\or (D).

Given the optimal partition $\pi = (B, N)$ of (P) and\or (D), the optimal solutions x and (y, s) such that $x_i > 0$, $s_i = 0$, for $i \in B$ and $x_i = 0$, $s_i > 0$, for $i \in N$ are called strictly complementary optimal solutions of (P) and (D), respectively. The unique strictly complementary optimal solutions x and (y, s) generated from the interior point method are called the central solutions of (P) and (D), respectively.

It is well known that the optimal partition is uniquely determined by the central solution and the converse is true. We have a one-to-one correspondence between the optimal partition and the central solution. The following lemmas come from [4] and are stated without proof.

Lemma 2.1. *Let $x^* \in P^*$ and $(y^*, s^*) \in D^*$. Then*

$$\begin{aligned} P^* &= \left\{x : x \in P,\ x^T s^* = 0\right\}, \\ D^* &= \left\{(y, s) : (y, s) \in D,\ s^T x^* = 0\right\}. \end{aligned} \tag{2.3}$$

Lemma 2.2. *Let $\pi = (B, N)$ be the optimal partition of (P) and (D). Then*

$$\begin{aligned} P^* &= \{x : x \in P,\ x_N = 0\}, \\ D^* &= \{(y, s) : (y, s) \in D,\ s_B = 0\}, \end{aligned} \tag{2.4}$$

where x_N and s_B refer to the restriction of vectors x and s to the coordinate sets N and B, respectively.

3. The Optimal Value Function

In this section we consider multiple parameters perturbation of b and investigate the effect of change in b on the optimal value function.

Suppose that $b(t) = b + Ht$, where matrix $H \in R^{m\times s}$ with rank $s, t \in R^s$. Parametric linear programming problems are defined as follows:

$$\min\left\{c^T x : Ax = b(t),\ x \geq 0\right\}, \tag{P_t}$$

$$\max\left\{b(t)^T y : A^T y + s = c,\ s \geq 0\right\}. \tag{D_t}$$

The feasible regions of (P_t) and (D_t) are denoted by P_t and D_t, and the optimal solutions set by P_t^* and D_t^*, respectively. The optimal value of (P_t) and (D_t) is denoted by $f(t)$ which is a function of the parameter t, with $f(t) = -\infty$ if (P_t) is unbounded and (D_t) infeasible; and $f(t) = +\infty$ if (D_t) is unbounded and (P_t) infeasible. If (P_t) and (D_t) are both infeasible then $f(t)$ is undefined. The region, in which $f(t)$ is finite, is called the domain of f, denoted by K. By the Linear Programming theory, we have that $f(t)$ is finite if and only if (P_t) and (D_t) are both feasible. Thus

$$\begin{gathered} K = \{t \in R^s : P_t \neq \emptyset, D_t \neq \emptyset\}, \\ f(t) = \min\left\{c^T x_t : t \in K, x_t \in P_t\right\} = \max\left\{b(t)^T y_t : t \in K, (y_t, s_t) \in D_t\right\} = c^T x_t^* = b(t)^T y_t^*, \end{gathered} \tag{3.1}$$

where $t \in K,\ x_t^* \in P_t^*$, and $(y_t^*, s_t^*) \in D_t^*$.

To characterize K, we use the following sets, The polyhedral convex set in R^n is defined as the intersection of finitely many closed half-spaces of R^n, that is, as the set of the form

$$\{x \in R^n : Bx \leq b\}, \tag{3.2}$$

where $B \in R^{p\times n}$, $b \in R^p$. The polyhedral convex cone in R^n is defined as the set which is a polyhedral convex set and a cone. It is clear that a set is a polyhedral convex cone if and only if it can be expressed as the set of the form

$$\{x \in R^n : Bx \leq 0\}, \tag{3.3}$$

where $B \in R^{p\times n}$.

Here is the assumption that (P_0) and (D_0) are both feasible. It follows that (D_t) is feasible for all values of t. Therefore, the domain K of f consists of all t for which (P_t) is feasible.

Theorem 3.1. *The domain K of f is a polyhedral convex set in R^s.*

Proof. Note that the domain K of f consists of all t for which (P_t) is feasible. (P_t) is feasible if and only if $b + Ht \in \{Ax : x \geq 0\}$. Since the set $\{Ax : x \geq 0\}$ is a polyhedral convex cone, it may be represented in the form of $\{x \in R^m : Bx \leq 0\}$, where $B \in R^{p\times m}$. Thus $t \in K$ if and only if $B(b + Ht) = Bb + (BH)t \leq 0$. This means that

$$K = \{t : (BH)t \leq -Bb\}, \tag{3.4}$$

where $BH \in R^{p\times s}$ and -$Bb \in R^p$. The result now follows by the definition of the polyhedral convex set. □

It is convenient to introduce another notation. Let K be a convex subset of R^s. We define a function f to be piecewise linear on K if there exist finitely many convex subsets R_i, $i = 1, 2, \ldots, p$ of K such that $K = \bigcup_{i=1}^{p} R_i$ and f is an affine function on every R_i.

Theorem 3.2. *The optimal value function f is continuous, convex, and piecewise linear on K.*

Proof. By definition,

$$f(t) = \max\left\{b(t)^T y(t) : \left(y(t), s(t)\right) \in D_t\right\}. \tag{3.5}$$

For each $t \in K$, due to the feasibilities of (P_t) and (D_t), we have $P_t^* \neq \emptyset$ and $D_t^* \neq \emptyset$. Further, there is a unique optimal partition of (P_t) and\or (D_t), and then a unique central solution $(y^*(t), s^*(t))$ of (D_t). We may assume now that the maximum value is attained at the central solution and write

$$f(t) = b(t)^T y^*(t). \tag{3.6}$$

Noting that the number of partition of the full index $\{1,2,\ldots,n\}$ is finite and that there is a unique optimal partition of (D_t) for any $t\in K$, we may define the index set $\overline{\Gamma}$ by

$$\overline{\Gamma}:=\left\{i:\pi_i=(B_i,N_i)\text{ is an optimal partition for some }(D_t)\text{ with }t\in K\right\}. \tag{3.7}$$

It is obvious that $\overline{\Gamma}$ is a finite set. Due to the definition of (D_t), the feasible region of (D_t) is constant when t varies. We have that $D_t=D_0$ and $(y^*(t),s^*(t))\in D_0$ for all t. By Lemma 2.2, we have that if central solutions $(y^*(t_1),s^*(t_1))$ and $(y^*(t_2),s^*(t_2))$ associate with same π_i, then these two central solutions are central solutions of (D_{t_1}) and (D_{t_2}) each other. Thus, we may take a representative, say (y_i,s_i), among all the central solutions associated with same π_i. Further, the optimal solution of (D_t) must be attained at some (y_i,s_i) for any $t\in K$. The set $\{(y_i,s_i):i\in\overline{\Gamma}\}$ is a finite subset of D_0 clearly. We may write

$$f(t)=\max\left\{b(t)^Ty_i:i\in\overline{\Gamma}\right\}. \tag{3.8}$$

For each $i\in\overline{\Gamma}$, we have

$$b(t)^Ty_i=b^Ty_i+(Ht)^Ty_i, \tag{3.9}$$

which is an affine function of t. Thus $f(t)$ is the maximum of a finite set of affine functions.

Let $t^1,t^2\in K$ and $\bar{t}=\lambda t^1+(1-\lambda)t^2$, where $\lambda\in[0,1]$. Recalling that K is convex, we have $\bar{t}\in K$. By (3.8), there exist $i_1,i_2,j\in\overline{\Gamma}$ such that $f(t^1)=b(t^1)^Ty_{i_1}$, $f(t^2)=b(t^2)^Ty_{i_2}$, $f(\bar{t})=b(\bar{t})^Ty_j$, and

$$\begin{aligned}f\left(t^1\right)&\geq b\left(t^1\right)^Ty_i,\quad\forall i\in\overline{\Gamma},\\ f\left(t^2\right)&\geq b\left(t^2\right)^Ty_i,\quad\forall i\in\overline{\Gamma}.\end{aligned} \tag{3.10}$$

Since $b(t)^Ty_i$ is an affine function of t for each $i\in\overline{\Gamma}$, we have

$$\begin{aligned}f\left(\bar{t}\right)&=b\left(\bar{t}\right)^Ty_j\\ &=b\left(\lambda t^1+(1-\lambda)t^2\right)^Ty_j\\ &=\lambda b\left(t^1\right)^Ty_j+(1-\lambda)b\left(t^2\right)^Ty_j\\ &\leq\lambda b\left(t^1\right)^Ty_{i_1}+(1-\lambda)b\left(t^2\right)^Ty_{i_2}\\ &=\lambda f\left(t^1\right)+(1-\lambda)f\left(t^2\right).\end{aligned} \tag{3.11}$$

We conclude that f is a convex function on K.

For each $i \in \overline{\Gamma}$, let

$$R_i = \left\{ t : f(t) = b(t)^T y_i,\ t \in K \right\}. \tag{3.12}$$

Since, for any $t \in K$, there exists an $i \in \overline{\Gamma}$ such that $f(t) = b(t)^T y_i$, we have $K = \bigcup_{i\in\overline{\Gamma}} R_i$. For any $t \in R_i$, by (3.8), we obtain

$$b(t)^T y_i - b(t)^T y_j \geq 0, \quad j \in \overline{\Gamma},\ j \neq i. \tag{3.13}$$

This means that $t \in E_i := \{t : b(t)^T (y_i - y_j) \geq 0$, for all $j \in \overline{\Gamma}, j \neq i\}$. In turn, if $t \in K$ and $t \in E_i$, then $f(t)$ is finite and $f(t) = b(t)^T y_i$ follows from the definition of $f(t)$. Thus, we conclude that $R_i = E_i \cap K$. Note that $b(t)^T (y_i - y_j)$ is an affine function of t for every $j \in \overline{\Gamma}$ $(j \neq i)$. So E_i is a polyhedral convex set. By Theorem 3.1, R_i is a polyhedral convex set, of course a convex set. Thus, f is an affine function on R_i. It follows that f is piecewise linear on K.

Let $\varepsilon > 0$. For each $t^0 \in K$, since $b(t)^T y_i$ is continuous at the point t^0 for every $i \in \overline{\Gamma}$ and $\overline{\Gamma}$ is a finite set, there exists a positive number δ such that

$$b\left(t^0\right)^T y_i - \varepsilon < b(t)^T y_i < b\left(t^0\right)^T y_i + \varepsilon \tag{3.14}$$

for all $i \in \overline{\Gamma}$ and all points t in K with $\|t - t^0\| < \delta$. So from the inequalities above, we have

$$f\left(t^0\right) - \varepsilon < f(t) < f\left(t^0\right) + \varepsilon \tag{3.15}$$

for all points t in K with $\|t - t^0\| < \delta$. Thus, f is continuous at t^0. It follows that f is continuous on K.

Summarizing the above results, the proof of the theorem is completed. □

From the arguments above, we have known that the domain K of f is a polyhedral convex set and the union of finite polyhedral convex sets. To explore further the construction properties of the domain K, we do the following. It is possible of course that the dimension of R_i is less than the dimension of K. For this case, we have the result below.

Lemma 3.3. *If* $\dim R_i < \dim K$, *then* $K = \bigcup_{j\in\overline{\Gamma}, j\neq i} R_j$.

Proof. For any $t^0 \in R_i$ and natural number n, since $\dim\{(t^0 + (1/n)B) \cap K\} = \dim K$, (where B is the Euclidean unit ball in R^s) thus there exists $t_n \in K$ with $t_n \notin R_i$ such that $t_n \in t^0 + (1/n)B$. This means that t^0 is the limit point of the sequence $\{t_n\}$ in $\bigcup_{j\in\overline{\Gamma}, j\neq i} R_j$. As the set $\bigcup_{j\in\overline{\Gamma}, j\neq i} R_j$ is closed, we have $t^0 \in \bigcup_{j\in\overline{\Gamma}, j\neq i} R_j$, as required.

We now define the new index set

$$\Gamma = \left\{ i : i \in \overline{\Gamma},\ \dim R_i = \dim K \right\} \tag{3.16}$$

and call $R_i = \{t : f(t) = b(t)^T y_i, t \in K\}$ with $\dim R_i = \dim K$ as a critical region of f. By Lemma 3.3, we have $K = \bigcup_{j\in\Gamma} R_j$, and Γ is a finite set clearly. The following result describes a construction property of K. □

Lemma 3.4. *If R_i and R_j are two different critical regions of f, then ri $R_i \cap$ ri $R_j = \emptyset$.*

Proof. To see this, we argue by contradiction. Suppose that there exists $t^0 \in K$ such that $t^0 \in \operatorname{ri} R_i \cap \operatorname{ri} R_j$. By $\operatorname{aff} K = \operatorname{aff} R_i = \operatorname{aff} R_j$, we may choose a positive number ε such that $(t^0 + \varepsilon B) \cap (\operatorname{aff} K) \subset R_i$ and $(t^0 + \varepsilon B) \cap (\operatorname{aff} K) \subset R_j$. For any $t \in K$, as $(1-\lambda)t^0 + \lambda t \in K \subset \operatorname{aff} K$ for any number $0 \le \lambda \le 1$, we may choose a number $0 \le \lambda_0 \le 1$ such that $(1-\lambda_0)t^0 + \lambda_0 t \in R_i$ and $(1-\lambda_0)t^0 + \lambda_0 t \in R_j$. Due to definitions of R_i and R_j, we have

$$\begin{aligned} f\Big((1-\lambda_0)t^0 + \lambda_0 t\Big) &= \Big(b + H\Big((1-\lambda_0)t^0 + \lambda_0 t\Big)\Big)^T y_i, \\ f\Big((1-\lambda_0)t^0 + \lambda_0 t\Big) &= \Big(b + H\Big((1-\lambda_0)t^0 + \lambda_0 t\Big)\Big)^T y_j, \end{aligned} \tag{3.17}$$

that is,

$$\Big(b + H\Big((1-\lambda_0)t^0 + \lambda_0 t\Big)\Big)^T y_i = \Big(b + H\Big((1-\lambda_0)t^0 + \lambda_0 t\Big)\Big)^T y_j. \tag{3.18}$$

Using $(b + Ht^0)^T y_i = (b + Ht^0)^T y_j$, we have $(b + Ht)^T y_i = (b + Ht)^T y_j$. Thus $R_i = R_j$ follows from definitions of them. This contradicts the supposition of R_i and R_j being different. □

Summarizing the above results, we have the following consequence.

Theorem 3.5. *Every critical region of f is a polyhedral convex set in R^s and the number of critical regions is finite. The domain K of f can be expressed as the union of all critical regions. Different critical regions have disjoint relative interiors.*

4. The Optimal Solution Sets on Critical Regions

We established in the previous section that optimal value function $f(t)$ is continuous, convex, and piecewise linear and that the domain K and every critical region R_i are polyhedral convex sets. In this section we will see some characters of optimal solution set at the points in some critical region R_i. Before proceeding, we introduce several notations. Let K be a nonempty convex subset of R^s and $d \in R^s$ with $d \neq 0$. We call d as an admissible direction of K at point $\bar{t}$ in K, if $K \cap \{\bar{t} + \lambda d : \lambda > 0\} \neq \emptyset$. Let f be a convex function from R^s to $[-\infty, +\infty]$, and let t be a point where f is finite. The directional derivative of f at t with respect to a direction d $(d \neq 0)$ is defined to be the limit

$$f'(t; d) = \lim_{\lambda \downarrow 0} \frac{f(t + \lambda d) - f(t)}{\lambda}. \tag{4.1}$$

If d is not an admissible direction of K at t, the directional derivative $f'(t; d)$ may be taken as $+\infty$.

Theorem 4.1. *Let R_i be a critical region of f. Then the dual optimal set D_t^* is constant (i.e., invariant) over relative interior region ri R_i of R_i.*

Proof. Let $t_1, t_2 \in \operatorname{ri} R_i$ and $t_1 \neq t_2$. Since R_i is convex, there are two points $\bar{t}_1$ and $\bar{t}_2$ of R_i such that t_1, t_2 are relative interiors of the line segment $[\bar{t}_1, \bar{t}_2]$ included in R_i. The fact that f is linear on R_i implies that f is linear on $[\bar{t}_1, \bar{t}_2]$. Supposing that $g(\lambda) = f(\lambda\bar{t}_2 + (1-\lambda)\bar{t}_1)$, $g(\lambda)$ is a linear function on $[0,1]$. Let $\overline{\lambda} \in (0,1)$ be arbitrary and let $(\overline{y}, \overline{s}) \in D_{\bar{t}}^*$ be arbitrary as well, where $\bar{t} = \overline{\lambda}\bar{t}_2 + (1-\overline{\lambda})\bar{t}_1$. Since $(\overline{y}, \overline{s})$ is optimal for $(D_{\bar{t}})$, we have

$$g\left(\overline{\lambda}\right) = f\left(\bar{t}\right) = \left(b + H\bar{t}\right)^T \overline{y} = b^T\overline{y} + \left(\overline{\lambda}H\bar{t}_2 + \left(1-\overline{\lambda}\right)H\bar{t}_1\right)^T \overline{y}, \tag{4.2}$$

and, since $(\overline{y}, \overline{s})$ is dual feasible for all t,

$$\begin{gathered} \left(b + H\bar{t}_1\right)^T \overline{y} \le f\left(\bar{t}_1\right) = g(0), \\ \left(b + H\bar{t}_2\right)^T \overline{y} \le f\left(\bar{t}_2\right) = g(1). \end{gathered} \tag{4.3}$$

Hence we find that

$$\begin{gathered} g(1) - g\left(\overline{\lambda}\right) \ge \left(1-\overline{\lambda}\right)\left(H\left(\bar{t}_2 - \bar{t}_1\right)\right)^T \overline{y}, \\ g\left(\overline{\lambda}\right) - g(0) \le \overline{\lambda}\left(H\left(\bar{t}_2 - \bar{t}_1\right)\right)^T \overline{y}. \end{gathered} \tag{4.4}$$

The linearity of g on $[0,1]$ implies that

$$\frac{g(1) - g\left(\overline{\lambda}\right)}{1-\overline{\lambda}} = \frac{g\left(\overline{\lambda}\right) - g(0)}{\overline{\lambda}}. \tag{4.5}$$

Hence, the last two inequalities are equalities. This means that the derivative of g with respect to λ on the interval $(0,1)$ satisfies

$$g'(\lambda) = \left(H\left(\bar{t}_2 - \bar{t}_1\right)\right)^T \overline{y}, \quad \forall \lambda \in (0,1), \tag{4.6}$$

or equivalently

$$\begin{aligned} g(\lambda) = g(0) + \lambda g'(\lambda) &= b^T\overline{y} + \left(H\bar{t}_1\right)^T \overline{y} + \lambda\left(H\left(\bar{t}_2 - \bar{t}_1\right)\right)^T \overline{y} \\ &= b^T\overline{y} + \left(H\left(\lambda\bar{t}_2 + (1-\lambda)\bar{t}_1\right)\right)^T \overline{y} = b(t)^T\overline{y}, \quad \forall \lambda \in (0,1), \end{aligned} \tag{4.7}$$

where $t = \lambda\bar{t}_2 + (1-\lambda)\bar{t}_1$. We conclude that $\overline{y}$ is optimal for any (D_t) with t being an interior of the line segment $[\bar{t}_1, \bar{t}_2]$. Since $\overline{\lambda}$ and $\overline{y}$ are arbitrary, it follows that $D_{t_1}^* = D_{t_2}^*$. The theorem is proved. □

Corollary 4.2. *One has*

$$f(t) = b^T y + (Ht)^T y, \quad f'(t;d) = \left(H^T y\right)^T d, \quad \forall t \in R_i, \tag{4.8}$$

where d is an admissible direction of R_i at t, (y,s) is in $D^*_{\bar{t}}$, and $\bar{t}$ is in ri R_i.

Proof. The above theorem reveals that $D^*_{t_1} = D^*_{t_2}$ for all $t_1, t_2 \in \operatorname{ri} R_i$. This implies that

$$f(t) = b(t)^T y = b^T y + (Ht)^T y, \quad \forall t \in \operatorname{ri} R_i,\ \forall (y,s) \in D^*_t. \tag{4.9}$$

By continuity of f, we conclude that

$$f(t) = b(t)^T y = b^T y + (Ht)^T y, \quad \forall t \in R_i,\ \forall (y,s) \in D^*_{\bar{t}} \text{ for some } \bar{t} \in \operatorname{ri} R_i. \tag{4.10}$$

Moreover, if d is an admissible direction of R_i at t, we have

$$f'(t;d) = \left(H^T y\right)^T d, \quad \forall (y,s) \in D^*_{\bar{t}} \text{ for some } \bar{t} \in \operatorname{ri} R_i, \tag{4.11}$$

as required. □

Corollary 4.3. *Let $\bar{t}$ be an arbitrary relative interior of R_i, and let t be an arbitrary boundary point of R_i. Then $D^*_{\bar{t}} \subseteq D^*_t$.*

Proof. Let $(\bar{y}, \bar{s}) \in D^*_{\bar{t}}$. Since $(\bar{y}, \bar{s})$ is dual feasible for all t, $(\bar{y}, \bar{s}) \in D_t$. Using $\bar{t} \in \operatorname{ri} R_i$ and (4.10), we have $f(t) = b(t)^T \bar{y}$. That is, $(\bar{y}, \bar{s}) \in D^*_t$. The proof is completed. □

From the argument of the theorem above, we have the following consequences.

Corollary 4.4. *If $f(t)$ is linear on the line segment $[t_1, t_2]$, where $t_1 \neq t_2$, then the dual optimal set D^*_t is constant for $t \in (t_1, t_2)$ and the slope of $f(t)$ on (t_1, t_2) is equal to $(H(t_2 - t_1))^T y^*$ for any $t \in (t_1, t_2)$ and any $(y^*, s^*) \in D^*_t$.*

Theorem 4.5. *Let t_1 and t_2 be any two different points of the domain K of f such that $D^*_{t_1} \cap D^*_{t_2} \neq \emptyset$. Then D^*_t is constant for all $t \in (t_1, t_2)$ and $f(t)$ is linear on the line segment $[t_1, t_2]$.*

Proof. Let $(y,s) \in D^*_{t_1} \cap D^*_{t_2}$. Then

$$f(t_1) = b(t_1)^T y, \qquad f(t_2) = b(t_2)^T y. \tag{4.12}$$

Consider the following linear function h:

$$h(t) = b(t)^T y = (b + Ht)^T y, \quad \forall t \in [t_1, t_2]. \tag{4.13}$$

h coincides with f at t_1 and t_2. Since $f(t)$ is convex, this implies that

$$f(t) \leq h(t), \quad \forall t \in [t_1, t_2]. \tag{4.14}$$

Now (y, s) is dual feasible for all $t \in [t_1, t_2]$. Since $f(t)$ is the optimal value of (D_t), it follows that

$$f(t) \geq b(t)^T y = (b + Ht)^T y = h(t). \tag{4.15}$$

Therefore, f coincides with h on $[t_1, t_2]$. As a consequence, f is linear on $[t_1, t_2]$. By Corollary 4.4, we have that D_t^* is constant on (t_1, t_2), and we complete the proof. □

Theorem 4.6. *If R_i and R_j are any two different critical regions, then*

$$D_{t_i}^* \cap D_{t_j}^* = \emptyset, \quad \forall t_i \in ri\, R_i, \ \forall t_j \in ri\, R_j. \tag{4.16}$$

Proof. Let $\pi_i = (B_i, N_i)$ and $\pi_j = (B_j, N_j)$ be the optimal partitions of (D_{t_i}) and (D_{t_j}), (y_i, s_i) and (y_j, s_j) be the central solutions, respectively. By Lemma 2.2, we have

$$\begin{aligned} D_{t_i}^* &= \{(y, s) : (y, s) \in D_{t_i},\ s_{B_i} = 0\}, \\ D_{t_j}^* &= \left\{(y, s) : (y, s) \in D_{t_j},\ s_{B_j} = 0\right\}. \end{aligned} \tag{4.17}$$

$B_i \neq B_j$ and $t_i \neq t_j$ follow from R_i and R_j being different. Further, either $(s_i)_{B_j} \neq 0$ or $(s_j)_{B_i} \neq 0$ holds, where $(s_i)_{B_j}$ and $(s_j)_{B_i}$ are the restrictions of s_i and s_j to the coordinate sets B_j and B_i, respectively. Otherwise, by the definition of the central solution, $(s_i)_{B_i} = 0$, $(s_i)_{N_i} > 0$, $(s_j)_{B_j} = 0$, $(s_j)_{B_j} > 0$, $(s_i)_{B_j} = 0$, and $(s_j)_{B_i} = 0$ hold simultaneously. This implies that $B_j \subseteq B_i$ and $B_i \subseteq B_j$, which contradicts $B_i \neq B_j$.

Since $\dim R_i = \dim R_j = \dim K$, we have $\operatorname{aff} R_i = \operatorname{aff} R_j = \operatorname{aff} K$. The inclusive relation $\{t : t = \lambda t_i + (1 - \lambda) t_j,\ \lambda \in R\} \subset \operatorname{aff} R_i = \operatorname{aff} R_j$ follows from $t_i, t_j \in K$, $t_i \neq t_j$. Due to t_i and t_j being the relative interiors of R_i and R_j separately, there exists a number $\lambda_0 > 1$ such that $\bar{t}_i = \lambda_0 t_i + (1 - \lambda_0) t_j$ and $\bar{t}_j = \lambda_0 t_j + (1 - \lambda_0) t_i$ are relative interiors of R_i and R_j separately. By Theorem 4.1, it holds that $D_{t_i}^* = D_{\bar{t}_i}^*$ and $D_{t_j}^* = D_{\bar{t}_j}^*$. In order to prove the theorem, we now argue by contradiction. If $D_{t_i}^* \cap D_{t_j}^* \neq \emptyset$, then $D_{\bar{t}_i}^* \cap D_{\bar{t}_j}^* \neq \emptyset$. Using Theorem 4.5 and $t_i, t_j \in (\bar{t}_i, \bar{t}_j)$, we conclude that $D_{t_i}^* = D_{t_j}^*$. Hence we have $(y_i, s_i) \in D_{t_j}^*$ and $(y_j, s_j) \in D_{t_i}^*$. This contradicts the definition above of $D_{t_i}^*$ if $(s_j)_{B_i} \neq 0$ or the definition above of $D_{t_j}^*$ if $(s_i)_{B_j} \neq 0$. The theorem is proved. □

Theorem 4.7. *Let $\bar{t}$ be an arbitrary point of K and x^* an arbitrary optimal solution of $(P_{\bar{t}})$. Then for any direction d $(d \neq 0)$,*

$$\begin{aligned} f'\left(\bar{t}; d\right) &= \max_{y,s}\left\{(Hd)^T y : (y, s) \in D_{\bar{t}}^*\right\} \\ &= \max_{y,s}\left\{(Hd)^T y : A^T y + s = c,\ s \geq 0,\ s^T x^* = 0\right\}. \end{aligned} \tag{4.18}$$

Proof. The second equality obviously holds owing to

$$D^*_{\bar{t}} = \left\{(y,s) : A^T y + s = c,\ s \geq 0,\ s^T x^* = 0\right\}. \tag{4.19}$$

Below we proceed by considering two cases separately.

Case 1. One has $K \cap \{\bar{t} + \lambda d : \lambda > 0\} \neq \emptyset$.

Since K is the union of finitely many critical regions and each of them is polyhedral, there is certainly a critical region R_i such that $[\bar{t}, \bar{t} + \overline{\lambda} d] \subseteq R_i$, where $\overline{\lambda} > 0$. Let $(\overline{y}, \overline{s}) \in D^*_{\tilde{t}}$, where $\tilde{t} \in \operatorname{ri} R_i$. From Corollary 4.2, we have

$$f(t) = (b + Ht)^T \overline{y}, \quad \forall t \in R_i. \tag{4.20}$$

By the definition of $f'(\bar{t}; d)$, we easily obtain that

$$f'\left(\bar{t}; d\right) = (Hd)^T \overline{y}. \tag{4.21}$$

Since $(\overline{y}, \overline{s})$ is optimal for $(D_{\bar{t}+\lambda d})$ and any $(y,s) \in D^*_{\bar{t}}$ is feasible for $(D_{\bar{t}+\lambda d})$ with respect to $\lambda \in [0, \overline{\lambda}]$, so we have

$$\left(b + H\left(\bar{t} + \overline{\lambda} d\right)\right)^T \overline{y} \geq \left(b + H\left(\bar{t} + \overline{\lambda} d\right)\right)^T y, \quad \forall (y,s) \in D^*_{\bar{t}}. \tag{4.22}$$

We also have $(\overline{y}, \overline{s}) \in D^*_{\bar{t}}$. Therefore

$$\left(b + H\bar{t}\right)^T \overline{y} = \left(b + H\bar{t}\right)^T y, \quad \forall (y,s) \in D^*_{\bar{t}}. \tag{4.23}$$

Subtracting both sides of this equality from the corresponding sides in the last inequality, we get

$$\overline{\lambda}(Hd)^T \overline{y} \geq \overline{\lambda}(Hd)^T y, \quad \forall (y,s) \in D^*_{\bar{t}}. \tag{4.24}$$

Dividing both sides by the positive number $\overline{\lambda}$, we obtain

$$(Hd)^T \overline{y} \geq (Hd)^T y, \quad \forall (y,s) \in D^*_{\bar{t}}, \tag{4.25}$$

thus proving that

$$f'\left(\bar{t}; d\right) = \max_{y,s}\left\{(Hd)^T y : (y,s) \in D^*_{\bar{t}}\right\} = (Hd)^T \overline{y}. \tag{4.26}$$

The theorem follows in this case.

Case 2. One has $K \cap \{\bar{t} + \lambda d : \lambda > 0\} = \emptyset$.

In this case, we point out first that $(P_{\bar{t}+\lambda d})$ is infeasible for any positive number λ and $f'(\bar{t}; d) = +\infty$. Since $(P_{\bar{t}})$ has an optimal solution x^*, $(D_{\bar{t}})$ has an optimal solution as well. This implies that the problem

$$\max_{y,s}\left\{(Hd)^T y : A^T y + s = c,\ s \geq 0,\ s^T x^* = 0\right\} \tag{4.27}$$

is feasible. Hence, if the problem is not unbounded, the problem and its dual have optimal solutions. The dual problem is given by

$$\min_{\xi,\lambda}\left\{c^T \xi : A\xi = Hd,\ \xi + \lambda x^* \geq 0\right\}. \tag{4.28}$$

We conclude that there are a vector $\xi \in R^n$ and a real number λ such that $A\xi = Hd$, $\xi + \lambda x^* \geq 0$. This implies that we cannot have $\xi_i < 0$ and $x_i^* = 0$ for $1 \leq i \leq n$. In other words,

$$x_i^* = 0 \Longrightarrow \xi_i \geq 0, \quad \forall 1 \leq i \leq n. \tag{4.29}$$

Therefore, there is a positive number ε such that $\overline{x} := x^* + \varepsilon\xi \geq 0$. Now we have

$$A\overline{x} = A(x^* + \varepsilon\xi) = Ax^* + \varepsilon A\xi = b + H\bar{t} + \varepsilon Hd = b + H\left(\bar{t} + \varepsilon d\right). \tag{4.30}$$

Thus we find that $(P_{\bar{t}+\varepsilon d})$ admits $\overline{x}$ as a feasible point. This contradicts the fact that $(P_{\bar{t}+\lambda d})$ is infeasible for any positive number λ. We conclude that the problem is unbounded, proving the theorem. □

5. Conclusions

Using the properties of the optimal partition, we give some description to a multiple parameters linear programming problem. The results in Section 3 show the geometric structures of the optimal value function and its domain. In Section 4, we point out that the character of the domain K of f is completely decided by the structure of dual optimal solutions and the directional derivative of f at any point can be obtained by solving a linear programming problem. Similarly, we may study multiple parameters perturbation of the cost coefficient vector problem or other parameter values problem. Our results maybe become as a theoretical foundation of summarizing critical regions.

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