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Research Article

Convergence Theorems for Partial Sums of Arbitrary Stochastic Sequences

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By using Doob's martingale convergence theorem, this paper presents a class of strong limit theorems for arbitrary stochastic sequence. Chow's two strong limit theorems for martingale-difference sequence and Loève's and Petrov's strong limit theorems for independent random variables are the particular cases of the main results.

1. Introduction

Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a stochastic sequence on the probability space $(\Omega, \mathcal{F}, P)$ that is, the sequence of σ-fields $\{\mathcal{F}_n, n \geq 1\}$ in $\mathcal{F}$ is increasing in n (that is $\mathcal{F}_n \uparrow$), and $\{\mathcal{F}_n\}$ are adapted to random variables $\{X_n\}$.

Almost sure behavior of partial sums of random variables has enjoyed both a rich classical period and a resurgence of research activity. Some famous researchers, such as Borel, Kolmogorov, Khintchine, Loève, Chung, and so on, were interested in convergence theorem of partial sums of random variables and obtained lots of classical results for sequences of independent random variables and martingale differences. For a detailed survey of strong limit theorems of sequences for random variables, interested readers can refer to the books [1, 2].

In recent years, some work has been done on the strong limit theorems for arbitrary stochastic sequences. Liu and Yang [3] established two strong limit theorems for arbitrary stochastic sequences, which generalized Chung's [4] strong law of large numbers for sequence of independent random variables as well as Chow's [5] strong law of large numbers for sequence of martingale differences. Then, Yang [6] established two more general strong limit theorems in 2007, which generalized a result by Jardas et al. [7] for sequences of independent random variables and the results by Liu and Yang [3] for arbitrary stochastic sequences in 2003. In 2008, W. Yang and X. Yang [8] proved two strong limit theorems for stochastic sequences, which generalized results by Freedman [9], Isaac, [10] and Petrov [2]. Qiu and Yang [11] established another type strong limit theorem for stochastic sequence in 1999. Then, Wang and Guo [12] extended the main result of Qiu and Yang in 2009. In addition, Wang and Yang [13] established a strong limit theorem for arbitrary stochastic sequences in 2005, which generalized Chow's [5] series convergence theorem for sequence of martingale differences. Then, Qiu [14] extended the result of Wang and Yang in 2008.

The purpose of this paper is to discuss further the strong limit theorems for arbitrary stochastic sequences. By using Doob's [1] convergence theorem for martingale-difference sequence, we establish a class of new strong limit theorems for stochastic sequences. Chow's two strong limit theorems for martingale-difference sequence, Loève's series convergence theorem, and Petrov's strong law of large numbers for sequences of independent random variables are the particular cases of this paper. In addition, the main theorems of this paper extend the main results by Wang and Guo in 2009, Qiu and Yang in 1999, and the result by Wang and Yang in 2005, respectively. The remainder of this paper is organized as follows. In Section 2, we present the main theorems of this paper. In Section 3, the proofs of the main theorems in this paper are presented.

2. Main Theorems

In this section, we will introduce the main results of this paper.

Let $\{c_n, n \geq 1\}$ be a positive real numbers sequence and $a(x)$ and $b(x)$ two positive real-valued functions on $[0,+\infty)$ satisfying $a(x) \geq a > 0$ when $x \in [0, c_n)$ and $b(x) \geq b > 0$ when $x \in (c_n, +\infty)$.

Theorem 2.1. *Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a stochastic sequence defined as in Section 1 and $\{\phi_n(x), n \geq 1\}$ a sequence of nondecreasing and nonnegative Borel functions on $[0,+\infty)$. For some $1 \leq p \leq 2$, suppose that*

$$h_n(x) = a(x)x^p I_{[0,c_n]}(x) + b(x)I_{(c_n,+\infty)}(x), \quad n \geq 1, \tag{2.1}$$

where $a(x), b(x)$ and c_n defined as above. Assume that

$$\phi_n(x) \geq h_n(x), \quad x \in [0,+\infty). \tag{2.2}$$

Set

$$A = \left\{\omega : \sum_{n=1}^{\infty} E\left[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}\right] < \infty\right\}. \tag{2.3}$$

(i) *If there exists some $c > 0$ such that*

$$\phi_n(x) \geq cx \tag{2.4}$$

holds when $x \in [0, c_n)$, *then*

$$\sum_{n=1}^{\infty} X_n \quad \text{converges a.e. on } A. \tag{2.5}$$

(ii) *If there exists some* $c > 0$ *such that* (2.4) *holds when* $x \in (c_n, +\infty)$, *then*

$$\sum_{n=1}^{\infty} (X_n - E[X_n \mid \mathcal{F}_{n-1}]) \quad \text{converges a.e. on } A. \tag{2.6}$$

Corollary 2.2 (Chow). *Let* $\{X_n, \mathcal{F}_n, n \geq 1\}$ *be a* L_P *martingale-difference sequence and* $\{a_n, n \geq 1\}$ *be an increasing sequence of positive numbers. For* $1 \leq p \leq 2$, *let*

$$A = \left\{ \omega : \sum_{n=1}^{\infty} a_n^{-p} E\left[|X_n|^p \mid \mathcal{F}_{n-1} \right] < \infty \right\}. \tag{2.7}$$

If $a_n \uparrow \infty$, *then*

$$\lim_{n \to \infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0 \quad \text{a.e. on } A. \tag{2.8}$$

Proof. By using Kroncker's lemma, it is a special case of Theorem 2.1 when the random variables X_n are replaced by X_n / a_n and $\phi_n(x) = |x|^p$. □

Theorem 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of arbitrary random variables. Let* $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ *and* $\mathcal{F}_0 = \{\Omega, \Phi\}, n \geq 1$. *Let* ϕ_n *and* h_n *be defined as Theorem 2.1. If*

$$\sum_{n=1}^{\infty} E\left[\phi_n(|X_n|) \right] < \infty, \tag{2.9}$$

then $\sum_{n=1}^{\infty} X_n$ *and* $\sum_{n=1}^{\infty} (X_n - E[X_n \mid \mathcal{F}_{n-1}])$ *converge a.e. under the same conditions (i) and (ii) as in Theorem 2.1, respectively.*

Corollary 2.4 (Loève). *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables, and* $0 < r_n \leq 2$. *Suppose that*

$$\sum_{n=1}^{\infty} E|X_n|^{r_n} < \infty. \tag{2.10}$$

If $0 < r_n \leq 1$, *then* $\sum_{n=1}^{\infty} X_n$ *converges a.e. If* $1 < r_n \leq 2$, *then* $\sum_{n=1}^{\infty} (X_n - E[X_n])$ *converges a.e.*

Corollary 2.5 (Petrov). *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. If $0 < a_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{E|X_n|^{r_n}}{a_n^{r_n}} < \infty, \tag{2.11}$$

then $\lim_{n\to\infty}(1/a_n)\sum_{i=1}^{n} X_i = 0$ a.e. when $0 < r_n < 1$, and $\lim_{n\to\infty}(1/a_n)\sum_{i=1}^{n}(X_i - E[X_i]) = 0$ a.e. when $1 \leq r_n \leq 2$.

Theorem 2.6. *Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a stochastic sequence defined as in Section 1 and $\{\phi_n(x), n \geq 1\}$ a sequence of nondecreasing and nonnegative Borel functions with $\phi_n(x/y) \leq \phi_n(x)/\phi_n(y)$ on $[0, +\infty)$. Let $h_n(x)$ be defined as Theorem 2.1 and*

$$\frac{\phi_n(x)}{\phi_n(d_n)} \geq h_n(x), \quad x \in [0, +\infty), \tag{2.12}$$

where $\{d_n, n \geq 1\}$ is a sequence of positive real numbers. Set

$$B = \left\{\omega : \sum_{n=1}^{\infty} \frac{E[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}]}{\phi_n(d_n)} < \infty\right\}. \tag{2.13}$$

Under the same conditions (i) and (ii) as in Theorem 2.1, $\sum_{n=1}^{\infty} d_n^{-1} X_n$ and $\sum_{n=1}^{\infty} d_n^{-1}(X_n - E[X_n \mid \mathcal{F}_{n-1}])$ converge a.e. on B, respectively.

Remark 2.7. By using Kronecker's lemma, if $d_n \uparrow \infty$, we have

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{k=1}^{n} X_k = 0 \quad \text{a.e. on } B, \tag{2.14}$$

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{k=1}^{n} (X_k - E[X_k \mid \mathcal{F}_{k-1}]) = 0 \quad \text{a.e. on } B, \tag{2.15}$$

respectively.

Theorem 2.8. *Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a stochastic sequence defined as in Section 1, $\{\xi_n, n \geq 1\}$ a sequence of nonzero random variables such that ξ_n is $\mathcal{F}_{n-1}$-measurable, and $c_n \geq 1$, $(n \geq 1)$ a sequence of real numbers. Let $\phi_n(x), \varphi_n(x)$ be two sequences of nonnegative Borel functions on $\mathfrak{R}$. Suppose that for $p \geq 2$, $\phi_n(x)/x^p$ does not decrease as $x > 0$, and for $0 < x_1 < x_2$,*

$$\frac{\varphi_n(x_1)}{x_1^p} \leq \frac{\phi_n(x_2)}{x_2^p} \tag{2.16}$$

holds. Let

$$A=\left\{\omega:\sum_{n=1}^{\infty}\left[\frac{\xi_n^2}{\varphi_n(|\xi_n|)}\right]E\left[\phi_n(|X_n|)\mid \mathcal{F}_{n-1}\right]<\infty\right\},$$
$$B=\left\{\omega:\sum_{n=1}^{\infty}\frac{\xi_n^2}{c_n^2}<\infty\right\}. \tag{2.17}$$

Then,

$$\sum_{n=1}^{\infty}c_n^{-1}(X_n-E[X_n\mid \mathcal{F}_{n-1}])\quad \text{converges a.e. on } AB. \tag{2.18}$$

Furthermore, if $c_n\uparrow\infty$, one has

$$\lim_{n\to\infty}\frac{1}{c_n}\sum_{k=1}^{n}(X_k-E[X_k\mid \mathcal{F}_{k-1}])=0\quad \text{a.e. on } AB. \tag{2.19}$$

Corollary 2.9 (Chow). *Let $\{X_n,\mathcal{F}_n,n\geq 1\}$ be a sequence of martingale differences, and let $\{a_n,n\geq 1\}$ be a sequence of positive real numbers with $\sum_{n=1}^{\infty}a_n<\infty$. For $p\geq 2$, let*

$$\sum_{n=1}^{\infty}a_n^{1-p/2}E\left[|X_n|^p\mid \mathcal{F}_{n-1}\right]<\infty. \tag{2.20}$$

Then,

$$\sum_{n=1}^{\infty}X_n\quad \text{converges a.e.} \tag{2.21}$$

Corollary 2.10. *Let $\{X_n,\mathcal{F}_n,n\geq 1\}$ be an arbitrary stochastic sequence. For $p\geq 2$, let*

$$A=\left\{\omega:\sum_{n=1}^{\infty}(n\log n)^{p/2-1}E\left[|X_n|^p\mid \mathcal{F}_{n-1}\right]<\infty\right\}. \tag{2.22}$$

Then,

$$\sum_{n=1}^{\infty}\frac{1}{n}(X_n-E[X_n\mid \mathcal{F}_{n-1}])\ \text{converges}\quad \text{a.e. on } A,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(X_k-E[X_k\mid \mathcal{F}_{k-1}])=0\quad \text{a.e. on } A, \tag{2.23}$$

where the log is to the base 2.

Proof. It is a special case of Theorem 2.8 when $\xi_n = \sqrt{\log n}, c_n = n$, $\phi_n(x) = |x|^p$, and $\varphi_n(x) = |x|^p / n^{p/2-1} (\log n)^{p-2}$ (here, we set $\varphi_1(x) = |x|^p$). □

3. Proofs of Theorems

We first give a lemma.

Lemma 3.1 (see [1]). *Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1\}$ be a martingale. Then, for some $1 \leq p \leq 2$, S_n converges a.e. on the set $\{\sum_{i=1}^\infty E[X_i^p \mid \mathcal{F}_{i-1}] < \infty\}$.*

Proof of Theorem 2.1. Let $X_n^* = X_n I_{(|X_n| \leq c_n)}$ and k a positive integer number. Let $Z_n = \phi_n(|X_n|)$,

$$\begin{aligned} A_k &= \left\{\omega : \sum_{n=1}^{\infty} E[Z_n \mid \mathcal{F}_{n-1}] \leq k\right\}, \\ \tau_k &= \min\left\{n : n \geq 1, \ \sum_{i=1}^{n+1} E[Z_i \mid \mathcal{F}_{i-1}] > k\right\}, \end{aligned} \tag{3.1}$$

where $\tau_k = +\infty$, if the right-hand side of (18) is empty. Then, $\sum_{n=1}^{\tau_k} Z_n = \sum_{n=1}^{\infty} I_{(\tau_k \geq n)} Z_n$. Since $I_{(\tau_k \geq n)}$ is measurable $\mathcal{F}_{n-1}$, and Z_n is nonnegative, we have

$$\begin{aligned} E\left[\sum_{n=1}^{\tau_k} Z_n\right] &= E\left[\sum_{n=1}^{\infty} I_{(\tau_k \geq n)} Z_n\right] \\ &= E\left\{\sum_{n=1}^{\infty} E\left[I_{(\tau_k \geq n)} Z_n \mid \mathcal{F}_{n-1}\right]\right\} \\ &\leq E\left\{\sum_{n=1}^{\infty} E[Z_n \mid \mathcal{F}_{n-1}]\right\} \leq k. \end{aligned} \tag{3.2}$$

Since $A_k = \{\tau_k = +\infty\}$, we have by (3.2)

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{A_k} Z_n dP &= \sum_{n=1}^{\infty} E\left[I_{(A_k)} Z_n\right] \\ &\leq E\left[\sum_{n=1}^{\tau_k} Z_n\right] \\ &\leq k. \end{aligned} \tag{3.3}$$

By (2.1), (2.2), and (3.3), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} P[A_k(X_n^* \neq X_n)] &= \sum_{n=1}^{\infty} \int_{A_k(X_n^* \neq X_n)} dP \\
&\leq \sum_{n=1}^{\infty} \frac{1}{b} \int_{A_k(|X_n|>c_n)} b(|X_n|)dP \\
&\leq \frac{1}{b} \sum_{n=1}^{\infty} \int_{A_k(|X_n|>c_n)} Z_n dP \\
&\leq \frac{1}{b} \sum_{n=1}^{\infty} \int_{A_k} Z_n dP \\
&\leq \frac{k}{b}.
\end{aligned} \tag{3.4}$$

It follows from Borel-Cantelli lemma and (3.4) that $P(A_k(X_n^* \neq X_n)\ i.o.) = 0$ holds. Hence, we have

$$\sum_{n=1}^{\infty} (X_n - X_n^*) \quad \text{converges a.e. on } A_\mathrm{k}. \tag{3.5}$$

Since $A = \bigcup_k A_k$, it follows from (3.5) that

$$\sum_{n=1}^{\infty} (X_n - X_n^*) \quad \text{converges a.e. on } A. \tag{3.6}$$

Let

$$Y_n = X_n^* - E[X_n^* \mid \mathcal{F}_{n-1}]. \tag{3.7}$$

It is clear that $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is a sequence for martingale difference. By using Cr inequality, we have

$$E\left[Y_n^p \mid \mathcal{F}_{n-1}\right] \leq 2^p E\left[(X_n^*)^p \mid \mathcal{F}_{n-1}\right] \leq 2^p E\left[|X_n^*|^p \mid \mathcal{F}_{n-1}\right] \quad \text{a.e.} \tag{3.8}$$

By using (2.1) and (2.2), we have

$$|X_n^*|^p \leq \frac{1}{a(|X^*|)} \phi_n(|X_n^*|) \leq \frac{1}{a} \phi_n(|X_n^*|). \tag{3.9}$$

Thus, the following inequality holds from (2.3), (3.8), and (3.9)

$$\sum_{n=1}^{\infty} E\left[Y_n^p \mid \mathcal{F}_{n-1}\right] < \infty \quad \text{a.e. on } A. \tag{3.10}$$

By using Lemma 3.1, we obtain

$$\sum_{n=1}^{\infty} Y_n \quad \text{converges a.e. on } A. \tag{3.11}$$

Hence, it follows from (3.6), (3.7), and (3.11) that

$$\sum_{n=1}^{\infty} (X_n - E[X_n^* \mid \mathcal{F}_{n-1}]) \quad \text{converges a.e. on } A. \tag{3.12}$$

The following argument breaks down into two cases.

Case 1. If there exists some $c > 0$ such that (2.4) holds when $0 \le x \le c_n$, then

$$\begin{aligned} \sum_{n=1}^{\infty} E[X_n^* \mid \mathcal{F}_{n-1}] &\le \frac{1}{c} \sum_{n=1}^{\infty} E\left[\phi_n(|X_n^*|) \mid \mathcal{F}_{n-1}\right] \\ &\le \frac{1}{c} \sum_{n=1}^{\infty} E\left[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}\right] \quad \text{a.e.} \end{aligned} \tag{3.13}$$

By using (2.3) and (3.13), we obtain

$$\sum_{n=1}^{\infty} E[X_n^* \mid \mathcal{F}_{n-1}] \quad \text{converges a.e. on A.} \tag{3.14}$$

It follows from (3.12) and (3.14) that (2.5) holds.

Case 2. If there exists some $c > 0$ such that the inequality (2.4) holds when $x > c_n$, then

$$\begin{aligned} &|E[X_n \mid \mathcal{F}_{n-1}] - E[X_n^* \mid \mathcal{F}_{n-1}]| \\ &\quad \le E[|X_n - X_n^*| \mid \mathcal{F}_{n-1}] \\ &\quad \le E[|X_n| \mid \mathcal{F}_{n-1}] \\ &\quad \le \frac{1}{c} E\left[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}\right] \quad \text{a.e.} \end{aligned} \tag{3.15}$$

By using (2.3) and (3.15), we obtain that

$$\sum_{n=1}^{\infty} (E[X_n \mid \mathcal{F}_{n-1}] - E[X_n^* \mid \mathcal{F}_{n-1}]) \quad \text{converges a.e. on A.} \tag{3.16}$$

It follows from (3.12) and (3.16) that (2.6) holds. The theorem is proved. □

Proof of Theorem 2.3. Since $\sum_{n=1}^{\infty} E[\phi_n(|X_n|)] = \sum_{n=1}^{\infty} E\{E[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}]\}$, we have by (2.9)

$$\sum_{n=1}^{\infty} E\{E[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}]\} < \infty. \tag{3.17}$$

It follows from the nonnegative property of $\phi_n(x)$ that

$$\sum_{n=1}^{\infty} E[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}] \quad \text{converges a.e.} \tag{3.18}$$

That is $P(A) = 1$. By Theorem 2.1, the conclusion of Theorem 2.3 holds. The theorem is proved. □

Proof of Theorem 2.6. It is a similar way with Theorem 2.1 except $Z_n = \phi_n(|X_n|)/\phi_n(d_n)$. □

Proof of Theorem 2.8. For $n \geq 1$, let $Z_n = X_n/c_n, Y_n = Z_n - E[Z_n \mid \mathcal{F}_{n-1}]$. Then, $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is a martingale-difference sequence. It follows from $p \geq 2$ and Jensen's inequality that

$$\begin{aligned} E\left[Y_n^2 \mid \mathcal{F}_{n-1}\right] &= E\left[Z_n^2 \mid \mathcal{F}_{n-1}\right] - E^2[Z_n \mid \mathcal{F}_{n-1}] \\ &\leq E\left[Z_n^2 \mid \mathcal{F}_{n-1}\right] \\ &= E\left[|Z_n|^{p \cdot 2/p} \mid \mathcal{F}_{n-1}\right] \\ &\leq E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}] \quad \text{a.e.} \end{aligned} \tag{3.19}$$

Furthermore,

$$\begin{aligned} E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}] &= E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}]\left[I_{(E^{2/p}[|Z_n|^p|\mathcal{F}_{n-1}]\leq \xi_n^2/c_n^2)} + I_{(E^{2/p}[|Z_n|^p|\mathcal{F}_{n-1}]> \xi_n^2/c_n^2)}\right] \\ &\leq \frac{\xi_n^2}{c_n^2} + E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}]\, I_{((\frac{E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}]}{\xi_n^2/c_n^2})>1)} \\ &\leq \frac{\xi_n^2}{c_n^2} + \frac{\xi_n^2}{c_n^2}\left(\frac{E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}]}{\xi_n^2/c_n^2}\right)^{p/2} I_{((\frac{E^{2/p}[|Z_n|^p \mid \mathcal{F}_{n-1}]}{\xi_n^2/c_n^2})>1)} \\ &\leq \frac{\xi_n^2}{c_n^2} + \frac{\xi_n^2}{c_n^2} E\left[\frac{|X_n|^p}{|\xi_n|^p \mid \mathcal{F}_{n-1}}\right] \quad \text{a.e.} \end{aligned} \tag{3.20}$$

It follows from (2.16) that

$$\frac{|X_n|^p}{|\xi_n|^p} \leq \frac{\phi_n(|X_n|)}{\varphi_n(|\xi_n|)} \tag{3.21}$$

holds when $|\xi_n| < |X_n|$. By (3.21), we have

$$E\left[\frac{|X_n|^p}{|\xi_n|^p} \mid \mathcal{F}_{n-1}\right] = E\left[\left(\frac{|X_n|^p}{|\xi_n|^p}\right) I_{(|X_n|\le|\xi_n|)} \mid \mathcal{F}_{n-1}\right] + E\left[\left(\frac{|X_n|^p}{|\xi_n|^p}\right) I_{(|X_n|>|\xi_n|)} \mid \mathcal{F}_{n-1}\right] \le 1 + E\left[\frac{\phi_n(|X_n|)}{\varphi_n(|\xi_n|)} \mid \mathcal{F}_{n-1}\right] \quad \text{a.e.} \tag{3.22}$$

Note that $c_n \ge 1$, it follows from (3.19), (3.21), and (3.22) that

$$E\left[Y_n^2 \mid \mathcal{F}_{n-1}\right] \le 2\frac{\xi_n^2}{c_n^2} + \frac{\xi_n^2}{c_n^2} E\left[\frac{\phi_n(|X_n|)}{\varphi_n(|\xi_n|)} \mid \mathcal{F}_{n-1}\right] \le 2\frac{\xi_n^2}{c_n^2} + \left[\frac{\xi_n^2}{\varphi_n(|\xi_n|)}\right] E\left[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}\right] \quad \text{a.e.} \tag{3.23}$$

And it follows from (2.17) and (3.23) that

$$\sum_{n=1}^{\infty} E\left[Y_n^2 \mid \mathcal{F}_{n-1}\right] \le 2\sum_{n=1}^{\infty} \frac{\xi_n^2}{c_n^2} + \sum_{n=1}^{\infty} \left[\frac{\xi_n^2}{\varphi_n(|\xi_n|)}\right] E\left[\phi_n(|X_n|) \mid \mathcal{F}_{n-1}\right] < \infty \quad \text{a.e. on } AB. \tag{3.24}$$

It follows from Lemma 3.1 that (2.18) holds. Furthermore, when $c_n \uparrow \infty$, it follows from Kronecker's lemma that (2.19) holds. The theorem is proved. □

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