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Research Article

Stability of a 2-Dimensional Functional Equation in a Class of Vector Variable Functions

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We prove the Hyers-Ulam stability of a 2-dimensional quadratic functional equation in a class of vector variable functions in Banach modules over a unital $C^{\star}$-algebra.

1. Introduction

In 1940, Ulam proposed the stability problem (see [1]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [2] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability. The authors investigated various functional equations and their Hyers-Ulam stability [3–8]. This Hyers-Ulam stability is a classical type of stability, but there is another kind of stability introduced by Risteski [9] for functional equations spanned over an n-dimensional complex vector space too.

Let X and Y be real or complex vector spaces. For a mapping $g : X \to Y$, consider the quadratic functional equation

$$g(x+y) + g(x-y) = 2g(x) + 2g(y). \tag{1.1}$$

In 1989, Aczél and Dhombres [10] obtained the solution of (1.1) for the case that Y acts on X. The result also holds when X and Y are arbitrary real or complex vector spaces. For a mapping $f : X \times X \to Y$, consider the 2-dimensional quadratic functional equation:

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x,z) + 2f(y,w). \tag{1.2}$$

The quadratic form $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := ax^2 + by^2$ is a solution of (1.2). In 2008, the authors of [8] acquired the general solution and proved the stability of the 2-dimensional quadratic functional equation (1.2) for the case that X and Y are real vector spaces as follows.

The results of [8, Theorems 3 and 4] also hold for complex vector spaces X and Y. In this paper, we investigate the stability of (1.2) with two module actions in Banach modules over a unital $C^\star$-algebra.

2. Preliminaries

Let A be a unital $C^\star$-algebra with a norm $|\cdot|$, and let ${}_A\mathcal{M}$ and ${}_A\mathcal{N}$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Put $A_1 := \{a \in A \mid |a| = 1\}$, $A_{\text{in}} := \{a \in A \mid a \text{ is invertible in } A\}$, $A_{\text{sa}} := \{a \in A \mid a^\star = a\}$, $\mathcal{U}(A) := \{a \in A \mid aa^\star = a^\star a = 1\}$, $A^+ := \{a \in A_{\text{sa}} \mid Sp(a) \subset [0,\infty)\}$, and $A_1^+ := A_1 \cap A^+$.

Definition 2.1. A 2-dimensional vector variable quadratic mapping $F : {}_A\mathcal{M}\times{}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.2) is called *A-quadratic* if $F(ax, ay) = a^2F(x,y)$ for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$.

Definition 2.2. A unital $C^\star$-algebra A is said to have *real rank* 0 (see [11]) if the invertible self-adjoint elements are dense in A_{sa}.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 := (a+a^\star)/2$ and $a_2 := (a-a^\star)/2i$ are self-adjoint elements; furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+ and a_2^- are positive elements (see [12, Lemma 38.8]).

3. Results

Theorem 3.1. *Let $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$\psi(s+t, u-v) + \psi(s-t, u+v) = 2\psi(s,u) + 2\psi(t,v) \tag{3.1}$$

for all $s,t,u,v \in \mathbb{R}$. If the function ψ is a Borel function, then it also satisfies

$$\psi(s,t) = s^2\psi(1,0) + t^2\psi(0,1) \tag{3.2}$$

for all $s,t \in \mathbb{R}$.

Proof. By [8, Theorem 3], there exist two symmetric biadditive mappings $S, T : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $\psi(s,t) = S(s,s) + T(t,t)$ for all $s,t \in \mathbb{R}$. By the proof of Theorem 3 in [8], we gain

$$\psi(pu, qv) = S(pu,pu) + T(qv,qv) = p^2S(u,u) + q^2T(v,v) = p^2\psi(u,0) + q^2\psi(0,v) \tag{3.3}$$

for all $p, q \in \mathbb{Q}$ and all $u, v \in \mathbb{R}$. Letting $p = v = 1$ in the equality (3.3), we get

$$\psi(u, q) = \psi(u, 0) + q^2 \psi(0, 1) \tag{3.4}$$

for all $u \in \mathbb{R}$ and all $q \in \mathbb{Q}$. Putting $u = v = 1$ in the equality (3.3) again, we have

$$\psi(p, q) = p^2 \psi(1, 0) + q^2 \psi(0, 1) \tag{3.5}$$

for all $p, q \in \mathbb{Q}$. Since the function $v \to \psi(u, v)$ is measurable and satisfies (1.1), by [13], it is continuous. By the same reasoning, $u \to \psi(u, v)$ is also continuous. Let $s, t \in \mathbb{R}$ be fixed. Since ψ is measurable, by [14, Theorem 7.14.26], for every $m \in \mathbb{N}$ there is a closed set $F_m \subset [s, s+1]$ such that $\mu([s, s+1] \setminus F_m) < 1/m$ and $\psi|_{F_m \times \mathbb{R}}$ is continuous. Since $\mu(F_m) \to 1$, one can choose $u_m \in F_m$ satisfying $u_m \to s$. Take a sequence $\{q_n\}$ in $\mathbb{Q}$ converging to t. By the equality (3.4), we get

$$\begin{aligned}\psi(u_m, t) &= \psi\left(u_m, \lim_{n\to\infty} q_n\right) = \lim_{n\to\infty} \psi(u_m, q_n) = \lim_{n\to\infty}\left[\psi(u_m, 0) + q_n^2 \psi(0, 1)\right] \\ &= \psi(u_m, 0) + t^2 \psi(0, 1)\end{aligned} \tag{3.6}$$

for all $m \in \mathbb{N}$. For each fixed $m \in \mathbb{N}$, take a sequence $\{p_n\}$ in $\mathbb{Q}$ converging to u_m. By (3.5) and the above equality, we have

$$\begin{aligned}\psi(u_m, t) &= \psi\left(\lim_{n\to\infty} p_n, 0\right) + t^2 \psi(0, 1) = \lim_{n\to\infty} \psi(p_n, 0) + t^2 \psi(0, 1) \\ &= \lim_{n\to\infty} p_n^2 \psi(1, 0) + t^2 \psi(0, 1) = u_m^2 \psi(1, 0) + t^2 \psi(0, 1).\end{aligned} \tag{3.7}$$

Hence we see that

$$\begin{aligned}\psi(s, t) &= \psi\left(\lim_{m\to\infty} u_m, t\right) = \lim_{m\to\infty} \psi(u_m, t) = \lim_{m\to\infty}\left[u_m^2 \psi(1, 0) + t^2 \psi(0, 1)\right] \\ &= s^2 \psi(1, 0) + t^2 \psi(0, 1),\end{aligned} \tag{3.8}$$

as desired. □

Lemma 3.2. *Let X and Y be normed spaces and $r \neq 2$ a real number, and let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) - 2f(y, w)\| \le \|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r \tag{3.9}$$

for all $x, y, z, w \in X$. Suppose $f(0,0) = 0$ for $r > 2$. If Y is complete, then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leq \begin{cases} \dfrac{1}{2-2^{r-1}}\left(2\|x\|^r + 3\|y\|^r\right) + \dfrac{1}{3}\|f(0,0)\| & (r < 2), \\ \dfrac{2^{1-r}}{1-2^{2-r}}\left(2\|x\|^r + 3\|y\|^r\right) & (r > 2) \end{cases} \tag{3.10}$$

for all $x, y \in X$. The mapping F is given by

$$F(x,y) := \begin{cases} \lim\limits_{j \to \infty} \dfrac{1}{4^j} f\left(2^j x, 2^j y\right) & (r < 2), \\ \lim\limits_{m \to \infty} 4^j f\left(\dfrac{x}{2^j}, \dfrac{y}{2^j}\right) & (r > 2) \end{cases} \tag{3.11}$$

for all $x, y \in X$.

Proof. Letting $y = x$ and $w = -z$ in (3.9), we gain

$$\left\| f(x,z) + f(x,-z) - \frac{1}{2}\left[f(0,0) + f(2x,2z)\right] \right\| \leq \|x\|^r + \|z\|^r \tag{3.12}$$

for all $x, z \in X$. Putting $x = 0$ in (3.12), we get

$$\left\| f(0,z) + f(0,-z) - \frac{1}{2}\left[f(0,0) + f(0,2z)\right] \right\| \leq \|z\|^r \tag{3.13}$$

for all $z \in X$. Replacing z by $-z$ in the above inequality, we have

$$\left\| f(0,-z) + f(0,z) - \frac{1}{2}\left[f(0,0) + f(0,-2z)\right] \right\| \leq \|z\|^r \tag{3.14}$$

for all $z \in X$. By the above two inequalities, we see that

$$\|f(0,2z) - f(0,-2z)\| \leq 4\|z\|^r \tag{3.15}$$

for all $z \in X$. Setting $y = x$ and $w = z$ in (3.9), we obtain that

$$\|f(2x,0) + f(0,2z) - 4f(x,z)\| \leq 2\left(\|x\|^r + \|z\|^r\right) \tag{3.16}$$

for all $x, z \in X$. Replacing z by $-z$ in the above inequality, we see that

$$\|f(2x,0) + f(0,-2z) - 4f(x,-z)\| \leq 2\left(\|x\|^r + \|z\|^r\right) \tag{3.17}$$

for all $x, z \in X$. By the last two inequalities, we know that

$$\left\| f(x,z) - f(x,-z) - \frac{1}{4}[f(0,2z) - f(0,-2z)] \right\| \le \|x\|^r + \|z\|^r \tag{3.18}$$

for all $x, z \in X$. By (3.12) and (3.18), we obtain that

$$\left\| f(x,z) - \frac{1}{8}[f(0,2z) - f(0,-2z)] - \frac{1}{4}\ [f(0,0) + f(2x,2z)] \right\| \le \|x\|^r + \|z\|^r \tag{3.19}$$

for all $x, z \in X$. By (3.15) and the above inequality, we have

$$\left\| f(x,z) - \frac{1}{4}\ [f(0,0) + f(2x,2z)] \right\| \le \|x\|^r + \frac{3}{2}\|z\|^r \tag{3.20}$$

for all $x, z \in X$. Thus we obtain that

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j z\right) - \frac{1}{4^{j+1}}\left[f(0,0) + f\left(2^{j+1}x, 2^{j+1}z\right)\right] \right\| \le 2^{j(r-2)}\left(\|x\|^r + \frac{3}{2}\|z\|^r\right) \tag{3.21}$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j y\right) - \frac{1}{4^{j+1}}\left[f(0,0) + f\left(2^{j+1}x, 2^{j+1}y\right)\right] \right\| \le 2^{j(r-2)}\left(\|x\|^r + \frac{3}{2}\|y\|^r\right) \tag{3.22}$$

for all $x, y \in X$ and all j. For given integers l, m $(0 \le l < m)$, we obtain that

$$\left\| \frac{1}{4^m} f(2^m x, 2^m y) - \frac{1}{4^l} f\left(2^l x, 2^l y\right) + \frac{1}{3}\left(\frac{1}{4^l} - \frac{1}{4^m}\right) f(0,0) \right\| \le \frac{2^{l(r-2)} - 2^{m(r-2)}}{1 - 2^{r-2}}\left(\|x\|^r + \frac{3}{2}\|y\|^r\right) \tag{3.23}$$

for all $x, y \in X$.

Consider the case $r < 2$. By (3.23), the sequence $\{(1/4^j)f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x,y) := \lim_{j\to\infty}(1/4^j)f(2^j x, 2^j y)$ for all $x, y \in X$. By (3.9), we have

$$\begin{aligned} &\left\| \frac{1}{4^j} f\left(2^j(x+y), 2^j(z-w)\right) + \frac{1}{4^j} f\left(2^j(x-y), 2^j(z+w)\right) \right. \\ &\left. -\frac{2}{4^j} f\left(2^j x, 2^j z\right) - \frac{2}{4^j} f\left(2^j y, 2^j w\right) \right\| \le 2^{(r-2)j}(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \end{aligned} \tag{3.24}$$

for all $x, y, z, w \in X$ and all j. Letting $j \to \infty$, we see that F satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (3.23), one can obtain inequality (3.10). If $G : X \times X \to Y$ is another 2-dimensional vector variable quadratic mapping satisfying (3.10), by [8, Theorem 3], there are four symmetric biadditive mappings $S, T, U, V : X \times X \to Y$ such that $F(x,y) = S(x,x) + T(y,y)$ and $G(x,y) = U(x,x) + V(y,y)$ for all $x, y \in X$. Thus we obtain that

$$\begin{aligned}\|F(x,y) - G(x,y)\| &= \|S(x,x) + T(y,y) - U(x,x) - V(y,y)\| \\ &= \frac{1}{4^n}\|S(2^n x, 2^n x) + T(2^n y, 2^n y) - U(2^n x, 2^n x) - V(2^n y, 2^n y)\| \\ &= \frac{1}{4^n}\|F(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\le \frac{1}{4^n}\|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + \frac{1}{4^n}\|f(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\le \frac{2^{n(r-2)}}{1-2^{r-2}}(2\|x\|^r + 3\|y\|^r) + \frac{2^{1-2n}}{3}\|f(0,0)\| \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty\end{aligned} \tag{3.25}$$

for all $x, y \in X$. Hence the mapping F is the unique 2-dimensional vector variable quadratic mapping, as desired.

Next, consider the case $r > 2$. Since $f(0,0) = 0$, by inequality (3.20), we gain

$$\left\|4f\left(\frac{x}{2}, \frac{z}{2}\right) - f(x,z)\right\| \le \frac{1}{2^{r-1}}(2\|x\|^r + 3\|z\|^r) \tag{3.26}$$

for all $x, z \in X$. Thus we get

$$\left\|4^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\right) - 4^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)\right\| \le 2^{j(2-r)+1-r}(2\|x\|^r + 3\|z\|^r) \tag{3.27}$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we have

$$\left\|4^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}\right) - 4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right)\right\| \le 2^{j(2-r)+1-r}(2\|x\|^r + 3\|y\|^r) \tag{3.28}$$

for all $x, y \in X$ and all j. For given integers l, m $(0 \le l < m)$, we obtain that

$$\left\|4^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right) - 4^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right)\right\| \le \frac{2^{2-r} - 2^{(2-r)(m+1)}}{2 - 2^{3-r}}(2\|x\|^r + 3\|y\|^r) \tag{3.29}$$

for all $x, y \in X$. By (3.29), the sequence $\{4^j f(x/2^j, y/2^j)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{4^j f(x/2^j, y/2^j)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x,y) := \lim_{j\to\infty} 4^j f(x/2^j, y/2^j)$ for all $x, y \in X$. By (3.9), we have

$$\begin{aligned}
&\left\|4^j f\left(\frac{x+y}{2^j}, \frac{z-w}{2^j}\right) + 4^j f\left(\frac{x-y}{2^j}, \frac{z+w}{2^j}\right) - 2\cdot 4^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right) - 2\cdot 4^j f\left(\frac{y}{2^j}, \frac{w}{2^j}\right)\right\| \\
&\quad \le 2^{(2-r)j}\left(\|x\|^r + \|y\|^r + \|z\|^r + 3\|w\|^r\right)
\end{aligned} \tag{3.30}$$

for all $x, y, z, w \in X$ and all j. Letting $j \to \infty$, we see that F satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (3.29), one can obtain inequality (3.10). If $G : X \times X \to Y$ is another 2-dimensional vector variable quadratic mapping satisfying (3.10), by in [8, Theorem 3], there are four symmetric biadditive mappings $S, T, U, V : X \times X \to Y$ such that $F(x,y) = S(x,x) + T(y,y)$ and $G(x,y) = U(x,x) + V(y,y)$ for all $x, y \in X$. Thus we obtain that

$$\begin{aligned}
\|F(x,y) - G(x,y)\| &= \|S(x,x) + T(y,y) - U(x,x) - V(y,y)\| \\
&= 4^n \left\|S\left(\frac{x}{2^n}, \frac{x}{2^n}\right) + T\left(\frac{y}{2^n}, \frac{y}{2^n}\right) - U\left(\frac{x}{2^n}, \frac{x}{2^n}\right) - V\left(\frac{y}{2^n}, \frac{y}{2^n}\right)\right\| \\
&= 4^n \left\|F\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - G\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\
&\le 4^n \left\|F\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + 4^n \left\|f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - G\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\
&\le \frac{2^{(2-r)(n+1)}}{1 - 2^{2-r}}\left(2\|x\|^r + 3\|y\|^r\right) \\
&\longrightarrow 0, \quad \text{as } n \longrightarrow \infty
\end{aligned} \tag{3.31}$$

for all $x, y \in X$. Hence the mapping F is the unique 2-dimensional vector variable quadratic mapping, as desired. □

Theorem 3.3. *Let $r \neq 2$ be a real number, and let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\begin{aligned}
&\left\|f(ax + ay, az - aw) + f(ax - ay, az + aw) - 2a^2 f(x,z) - 2a^2 f(y,w)\right\| \\
&\quad \le \|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r
\end{aligned} \tag{3.32}$$

for all $a \in A_1$ and all $x, y, z, w \in {}_A\mathcal{M}$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique 2-dimensional vector variable A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.2) and (3.10) for all $x, y \in {}_A\mathcal{M}$.

Proof. Suppose $r < 2$. By Lemma 3.2, it follows from the inequality of the statement for $a = 1$ that there exists a unique 2-dimensional vector variable quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.2) and inequality (3.10) for all $x, y \in {}_A\mathcal{M}$.

Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let $L : {}_A\mathcal{N} \to \mathbb{R}$ be any continuous linear functional, that is, L is an arbitrary element of the dual space of ${}_A\mathcal{N}$. For $n \in \mathbb{N}$, consider two functions $\zeta_n : \mathbb{R} \to \mathbb{R}$ and $\xi_n : \mathbb{R} \to \mathbb{R}$ defined by $\zeta_n(t) := (1/4^n)L[f(2^n t x_0, 0)]$ and

$\xi_n(t) := (1/4^n)L[f(0,2^n t y_0)]$ for all $t \in \mathbb{R}$. By the assumption that $f(tx,ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x,y \in {}_A\mathcal{M}$, the functions ζ_n and ξ_n are continuous for all $n \in \mathbb{N}$. Note that $\zeta_n(t) = (1/4^n)L[f(2^n t x_0,0)] = L[(1/4^n)f(2^n t x_0,0)]$ and $\xi_n(t) = (1/4^n)L[f(0,2^n t y_0)] = L[(1/4^n)f(0,2^n t y_0)]$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. By [8], the sequences $\{\zeta_n(t)\}$ and $\{\xi_n(t)\}$ are Cauchy sequences for all $t \in \mathbb{R}$. Define two functions $\zeta : \mathbb{R} \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ by $\zeta(t) := \lim_{n\to\infty} \zeta_n(t)$ and $\xi(t) := \lim_{n\to\infty} \xi_n(t)$ for all $t \in \mathbb{R}$. Note that $\zeta(t) = L[F(tx_0,0)]$ and $\xi(t) = L[F(0,ty_0)]$ for all $t \in \mathbb{R}$. Since F satisfies (1.2), we get

$$\begin{aligned}
\zeta(s+t)+\zeta(s-t) &= L(F[(s+t)x_0,0]) + L(F[(s-t)x_0,0]) \\
&= L(F[(s+t)x_0,0] + F[(s-t)x_0,0]) = L[F(sx_0+tx_0,0)+F(sx_0-tx_0,0)] \\
&= L[2F(sx_0,0)+2F(tx_0,0)] = 2L[F(sx_0,0)] + 2L[F(tx_0,0)] = 2\zeta(s)+2\zeta(t), \\
\xi(s+t)+\xi(s-t) &= L\left(F\left[0,(s+t)y_0\right]\right) + L\left(F\left[0,(s-t)y_0\right]\right) \\
&= L\left(F\left[0,(s+t)y_0\right] + F\left[0,(s-t)y_0\right]\right) = L\left[F\left(0,sy_0+ty_0\right)+F\left(0,sy_0-ty_0\right)\right] \\
&= L\left[2F\left(0,sy_0\right)+2F\left(0,ty_0\right)\right] = 2L\left[F\left(0,sy_0\right)\right] + 2L\left[F\left(0,ty_0\right)\right] = 2\xi(s)+2\xi(t)
\end{aligned} \tag{3.33}$$

for all $s,t \in \mathbb{R}$. Since ζ and ξ are the pointwise limits of continuous functions, they are Borel functions. Thus the functions ζ and ξ as measurable quadratic functions are continuous (see [13]), so have the forms $\zeta(t) = t^2\zeta(1)$ and $\xi(t) = t^2\xi(1)$ for all $t \in \mathbb{R}$. Since F satisfies (1.2), by [8, Theorem 3], there exist two symmetric biadditive mappings $S,T : X \times X \to Y$ such as $F(x,y) = S(x,x) + T(y,y)$ for all $x,y \in X$. Hence we have

$$\begin{aligned}
L\left[F\left(tx_0,ty_0\right)\right] &= L\left[F(tx_0,0)+F\left(0,ty_0\right)\right] = L[F(tx_0,0)] + L\left[F\left(0,ty_0\right)\right] = \zeta(t)+\xi(t) \\
&= t^2\zeta(1)+t^2\xi(1) = t^2L[F(x_0,0)] + t^2L\left[F\left(0,y_0\right)\right] \\
&= t^2L\left[F(x_0,0)+F\left(0,y_0\right)\right] = t^2L\left[S(x_0,x_0)+T\left(y_0,y_0\right)\right] \\
&= t^2L\left[F\left(x_0,y_0\right)\right] = L\left[t^2F\left(x_0,y_0\right)\right]
\end{aligned} \tag{3.34}$$

for all $t \in \mathbb{R}$. Since L is any continuous linear functional, the 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(tx_0,ty_0) = t^2F(x_0,y_0)$ for all $t \in \mathbb{R}$. Therefore we obtain

$$F(tx,ty) = t^2F(x,y) \tag{3.35}$$

for all $t \in \mathbb{R}$ and all $x,y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by 2^jx and 2^jz, respectively, and letting $y = w = 0$ in inequality (3.32), we gain

$$\left\| f\left(2^jax,2^jaz\right) - a^2f\left(2^jx,2^jz\right) - a^2f(0,0)\right\| \le 2^{jr-1}\left(\|x\|^r + \|z\|^r\right) \tag{3.36}$$

for all $a \in A_1$ and all $x,z \in {}_A\mathcal{M}$. Note that there is a constant $K > 0$ such that the condition

$$\|av\| \le K|a|\|v\| \tag{3.37}$$

for each $a \in A$ and each $v \in {}_A\mathcal{N}$ (see [12, Definition 12]). For all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$, we get

$$\frac{1}{4^j}\left\|f\left(2^j ax, 2^j ay\right) - a^2 f\left(2^j x, 2^j y\right)\right\| \leq 2^{j(r-2)-1}\left(\|x\|^r + \|w\|^r\right) + \frac{K|a|^2}{4^j}\|f(0,0)\| \longrightarrow 0 \quad (3.38)$$

as $j \to \infty$. Hence we have

$$F(ax, ay) = \lim_{j\to\infty}\frac{1}{4^j}f\left(2^j ax, 2^j ay\right) = a^2\lim_{j\to\infty}\frac{1}{4^j}f\left(2^j x, 2^j y\right) = a^2 F(x,y) \quad (3.39)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$. Since $F(ax, ay) = a^2 F(x,y)$ for each $a \in A_1$, by (3.35), we obtain

$$F(ax, ay) = F\left(|a|\frac{a}{|a|}x, |a|\frac{a}{|a|}y\right) = |a|^2 F\left(\frac{a}{|a|}x, \frac{a}{|a|}y\right) = a^2 F(x,y) \quad (3.40)$$

for all nonzero $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By (3.35), we get $F(0x, 0y) = 0^2 F(x,y)$ for all $x, y \in {}_A\mathcal{M}$. Therefore the mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ is the unique 2-dimensional vector variable A-quadratic mapping satisfying (1.2) and (3.10).

The proof of the case $r > 2$ is similar to that of the case $r < 2$. □

Theorem 3.4. *Let $r \neq 2$ be a real number and A of real rank 0, and let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\begin{aligned}&\|f(ax + ay, bz - bw) + f(ax - ay, bz + bw) - 2abf(x,z) - 2ab(y,w)\|\\ &\quad < \|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\end{aligned} \quad (3.41)$$

for all $a, b \in (A_1^+ \cap A_{\text{in}}) \cup \{i\}$ and all $x, y, z, w \in {}_A\mathcal{M}$. For each fixed $x, y \in {}_A\mathcal{M}$, let the sequence $\{(1/4^j)f(2^j ax, 2^j by)\}$ converge uniformly on $A_1 \times A_1$. If $f(ax, by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique 2-dimensional vector variable mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.2) and (3.10) such that $F(ax, by) = abF(x,y)$ for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$.

Proof. Suppose $r < 2$. By [8, Theorem 4], there exists a unique 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.2) and inequality (3.10) on ${}_A\mathcal{M} \times {}_A\mathcal{M}$. Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let L be an arbitrary element of the dual space of ${}_A\mathcal{N}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\psi_n(s,t) := (1/4^n)\, L[f(2^n s x_0, 2^n t y_0)]$ for all $s, t \in \mathbb{R}$. By the assumption that $f(ax, by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(s,t) = (1/4^n)\, L[f(2^n s x_0, 2^n t y_0)] = L[(1/4^n)\, f(2^n s x_0, 2^n t y_0)]$ for all $n \in \mathbb{N}$ and all $s, t \in \mathbb{R}$. By [8], the sequence $\psi_n(s,t)$ is a Cauchy sequence for all $s, t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$\varphi(s,t) := \lim_{n\to\infty}\varphi_n(s,t)$ for all $s,t \in \mathbb{R}$. Note that $\varphi(s,t) = L[F(sx_0, ty_0)]$ for all $t \in \mathbb{R}$. Thus we have

$$\begin{aligned}
&\varphi(s_1+s_2, t_1-t_2) + \varphi(s_1-s_2, t_1+t_2)\\
&\quad = L\big(F\big[(s_1+s_2)x_0, (t_1-t_2)y_0\big]\big) + L\big(F\big[(s_1-s_2)x_0, (t_1+t_2)y_0\big]\big)\\
&\quad = L\big(F\big[(s_1+s_2)x_0, (t_1-t_2)y_0\big] + F\big[(s_1-s_2)x_0, (t_1+t_2)y_0\big]\big)\\
&\quad = L\big[F\big(s_1x_0+s_2x_0, t_1y_0-t_2y_0\big) + F\big(s_1x_0-s_2x_0, t_1y_0+t_2y_0\big)\big]\\
&\quad = L\big[2F\big(s_1x_0, t_1y_0\big) + 2F\big(s_2x_0, t_2y_0\big)\big]\\
&\quad = 2L\big[F\big(s_1x_0, t_1y_0\big)\big] + 2L\big[F\big(s_2x_0, t_2y_0\big)\big]\\
&\quad = 2\varphi(s_1, t_1) + 2\varphi(s_2, t_2)
\end{aligned} \tag{3.42}$$

for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$. Since φ is the pointwise limit of continuous functions, it is a Borel function. By Theorem 3.1, we gain $\varphi(s,t) = s^2\varphi(1,0) + t^2\varphi(0,1)$ for all $s,t \in \mathbb{R}$. Hence we get

$$\begin{aligned}
L\big[F\big(sx_0, ty_0\big)\big] &= \varphi(s,t) = s^2\varphi(1,0) + t^2\varphi(0,1) = s^2L[F(x_0,0)] + t^2L\big[F\big(0,y_0\big)\big]\\
&= L\Big[s^2F(x_0,0) + t^2F\big(0,y_0\big)\Big]
\end{aligned} \tag{3.43}$$

for all $s,t \in \mathbb{R}$. Since L is any continuous linear functional, the 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(sx_0, ty_0) = s^2F(x_0,0) + t^2F(0,y_0)$ for all $s,t \in \mathbb{R}$. Therefore we obtain

$$F\big(sx, ty\big) = s^2F(x,0) + t^2F\big(0,y\big) \tag{3.44}$$

for all $s,t \in \mathbb{R}$ and all $x,y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by 2^jx and 2^jz, respectively, and letting $y = w = 0$ in the inequality (3.41), we get

$$\Big\|f\Big(2^jax, 2^jbz\Big) - abf\Big(2^jx, 2^jz\Big) - abf(0,0)\Big\| \le 2^{jr-1}\left(\|x\|^r + \|z\|^r\right) \tag{3.45}$$

for all $a,b \in (A_1^+ \cap A_{\text{in}}) \cup \{i\}$ and all $x,z \in {}_A\mathcal{M}$. By condition (3.37), for all $a,b \in (A_1^+ \cap A_{\text{in}}) \cup \{i\}$ and all $x,y \in {}_A\mathcal{M}$, we have

$$\begin{aligned}
\frac{1}{4^j}\Big\|f\Big(2^jax, 2^jby\Big) - abf\Big(2^jx, 2^jy\Big)\Big\| &\le 2^{j(r-2)-1}\left(\|x\|^r + \|z\|^r\right) + \frac{K|a||b|}{4^j}\|f(0,0)\|\\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{3.46}$$

Hence we obtain that

$$F\big(ax, by\big) = \lim_{j\to\infty}\frac{1}{4^j}f\Big(2^jax, 2^jby\Big) = ab\lim_{j\to\infty}\frac{1}{4^j}f\Big(2^jx, 2^jy\Big) = abF\big(x,y\big) \tag{3.47}$$

for all $a,b \in (A_1^+ \cap A_{\text{in}}) \cup \{i\}$ and all $x,y \in {}_A\mathcal{M}$.

Let $c, d \in A_1^+ \setminus A_{\text{in}}$. Since $A_{in} \cap A_{\text{sa}}$ is dense in A_{sa}, there exist two sequences $\{c_j\}$ and $\{d_j\}$ in $A_{\text{in}} \cap A_{\text{sa}}$ such that $c_j \to c$ and $d_j \to d$ as $j \to \infty$. Put $p_j := (1/|c_j|)\, c_j$ and $q_j := (1/|d_j|)d_j$. Then $p_j \to c$ and $q_j \to d$ as $j \to \infty$. Set $a_j := \sqrt{p_j{}^\star p_j}$ and $b_j := \sqrt{q_j{}^\star q_j}$. Then $a_j \to c$ and $b_j \to d$ as $j \to \infty$ and $a_j, b_j \in A_1^+ \cap A_{\text{in}}$. Since $\{(1/4^j)f(2^j ax, 2^j by)\}$ is uniformly converges on $A_1 \times A_1$ and $f(ax, by)$ is continuous in $a, b \in A_1$, we see that $F(ax, by)$ is also continuous in $a, b \in A_1$ for each $x, y \in {}_A\mathcal{M}$. In fact, we gain

$$\begin{aligned}\lim_{(a,b)\to(c,d)} F(ax, by) &= \lim_{(a,b)\to(c,d)} \lim_{j\to\infty} \frac{1}{4^j} f\left(2^j ax, 2^j by\right) = \lim_{j\to\infty} \lim_{(a,b)\to(c,d)} \frac{1}{4^j} f\left(2^j ax, 2^j by\right) \\ &= \lim_{j\to\infty} \frac{1}{4^j} f\left(2^j cx, 2^j dy\right) = F(cx, dy)\end{aligned} \tag{3.48}$$

for all $x, y \in {}_A\mathcal{M}$. Thus we get

$$\lim_{j\to\infty} F(a_j\mathrm{x}, b_j y) = F\left(\lim_{j\to\infty} a_j x, \lim_{j\to\infty} b_j y\right) = F(cx, dy) \tag{3.49}$$

for all $x, y \in {}_A\mathcal{M}$. By equality (3.47), we have

$$\|F(a_j x, b_j y) - cdF(x, y)\| = \|a_j b_j F(x, y) - cdF(x, y)\| \longrightarrow \|cdF(x, y) - cdF(x, y)\| = 0 \tag{3.50}$$

as $j \to \infty$ for all $x, y \in {}_A\mathcal{M}$. By equality (3.49) and the above convergence, we see that

$$\begin{aligned}\|F(cx, dy) - cdF(x, y)\| &\le \|F(cx, dy) - F(a_j x, b_j y)\| + \|F(a_j x, b_j y) - cdF(x, y)\| \\ &\longrightarrow 0 \quad \text{as } j \longrightarrow \infty\end{aligned} \tag{3.51}$$

for all $x, y \in {}_A\mathcal{M}$. By equality (3.47) and the above convergence, we obtain $F(ax, by) = abF(x, y)$ for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$. □

The proof of the case $r > 2$ is similar to that of the case $r < 2$.

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