

Research Article

A System of Nonlinear Operator Equations for a Mixed Family of Fuzzy and Crisp Operators in Probabilistic Normed Spaces

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By using a random version of the theory of contractor introduced by Altman, we introduce and study a system of nonlinear operator equations for a mixed family of fuzzy and crisp operators in probabilistic normed spaces. We construct some new iterative algorithms for solving this kind of nonlinear operator equations. We also prove some new existence theorems of solutions of a new system of nonlinear operator equations for a mixed family of fuzzy and crisp operators and some new convergence results of sequences generated by iterative algorithms under joint orbitally complete conditions.

1. Introduction

Altman [1, 2] introduced the theory of contractor and contractor direction, which has a very strong significance for the study of existence and uniqueness for solving nonlinear operator equations in Banach spaces. The theory of contractor offers a unified approach to a very large class of iterative methods including the most important ones. Chang [3] introduced the concept of probabilistic contractor and studied the existence and uniqueness of solution for nonlinear operator equations with probabilistic contractor in Menger PN-spaces. By using the theory of countable extension of t -norms [4–6] and the results from [7, 8], many results for the more general classes of t -norms have been proved (see [9] and the references therein).

On the other hand, since then, several kinds of variational inequalities, variational inclusions, complementarity problems, and nonlinear equations with fuzzy mappings were

introduced and studied by many authors (see, e.g., [8–15]). Sharma et al. [15] considered two nonfuzzy mappings and a sequence of fuzzy mappings to define a hybrid D -compatible condition. They also showed the existence of common fixed points under such condition, where the range of the one of the two nonfuzzy mappings is joint orbitally complete. Furthermore, Cho et al. [10] introduced the concept of probabilistic contractor couple in probabilistic normed spaces and discuss the solution for nonlinear equations of fuzzy mappings and the convergence of sequences generated by the algorithms in Menger probabilistic normed spaces. Very recently, Hadžić and Pap [16] introduced some new classes of probabilistic contractions in probabilistic metric spaces. They also obtained a new fixed point theorem for the ψ -probabilistic contraction and gave some applications to random operators.

Motivated and inspired by the above works, in this paper, by using a random version of the theory of contractor introduced by Altman, we introduce and study a system of nonlinear operator equations for a mixed family of fuzzy and crisp operators in probabilistic normed spaces. We construct some new iterative algorithms for solving this kind of nonlinear operator equations. We also prove some new existence theorems of solution for the system of nonlinear operator equations for a mixed family of fuzzy and crisp operators and new convergence results of sequences generated by the iterative algorithms under joint orbitally complete condition. The results presented in this paper improve and generalize corresponding results of [9, 15–17].

2. Preliminaries

Let Δ^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from \mathbb{R} into $[0, 1]$ such that $\sup_{x \in \mathbb{R}} F(x) = 1$). The special distribution function H is defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (2.1)$$

The ordered pair (S, \mathcal{F}) is said to be a probabilistic metric space if S is a nonempty set and $\mathcal{F} : S \times S \rightarrow \Delta^+$ ($\mathcal{F}(p, q)$ is written by $F_{p,q}$ for all $(p, q) \in S \times S$) satisfying the following conditions:

- (i) $F_{u,v}(x) = 1$ for all $x > 0 \Leftrightarrow u = v$ for all $u, v \in S$;
- (ii) $F_{u,v} = F_{v,u}$ for all $u, v \in S$;
- (iii) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for all $u, v, w \in S$ and $x, y \in \mathbb{R}^+$.

A Menger space (see [18]) is a triple (S, \mathcal{F}, T) , where (S, \mathcal{F}) is a probabilistic metric space, T is a triangular norm (abbreviated t -norm), and the following inequality holds:

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y)), \quad \forall u, v, w \in S, x, y > 0. \quad (2.2)$$

Recall that a mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a triangular norm (shortly, a t -norm) if the following conditions are satisfied:

- (i) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (ii) $T(a, b) = T(b, a)$ for all $a, b \in [0, 1]$;
- (iii) if $a \geq b$ and $c \geq d$, then $T(a, c) \geq T(b, d)$ for all $a, b, c, d \in [0, 1]$;
- (iv) $T(a, T(b, c)) = T(T(a, b), c)$ for all $a, b, c \in [0, 1]$.

Example 2.1. The following are the four basic examples.

- (1) The minimum t -norm T_M is defined by

$$T_M(x, y) = \min(x, y). \quad (2.3)$$

- (2) The product t -norm T_P is defined by

$$T_P(x, y) = x \cdot y. \quad (2.4)$$

- (3) The Lukasiewicz t -norm T_L is defined by

$$T_L(x, y) = \max(x + y - 1, 0). \quad (2.5)$$

- (4) The weakest t -norm, the drastic product T_D , is defined by

$$T_D(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

The (ϵ, λ) -topology in S is introduced by the family of neighbourhoods $\mathcal{U} = \{U_v(\epsilon, \lambda)\}_{(v, \epsilon, \lambda) \in S \times \mathbb{R}_+ \times (0, 1)}$, where

$$U_v(\epsilon, \lambda) = \{u; F_{u,v}(\epsilon) > 1 - \lambda\}. \quad (2.7)$$

If a t -norm T is such that $\sup_{x < 1} T(x, x) = 1$, then S is a metrizable topological space under the (ϵ, λ) -topology.

Let X be a vector space over the real or complex number field \mathcal{K} , $\mathcal{F} : X \rightarrow \Delta^+$, and T a t -norm. The ordered triple (X, \mathcal{F}, T) is a Menger probabilistic normed space (briefly, a Menger PN-space) if and only if the following conditions are satisfied, where $\mathcal{F}(x) = F_x$ for all $x \in S$:

- (i) $F_x(0) = 0$ for all $x \in S$ and $F_x = H \Leftrightarrow x = \theta$ (θ is a neutral element for $+$ in X);
- (ii) $F_{\lambda x}(t) = F_x(t/|\lambda|)$ for all $x \in X$ and $\lambda \in \mathcal{K}$ ($\lambda \neq 0$);
- (iii) $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2))$ for all $x, y \in X$ and $t_1, t_2 > 0$.

If (X, \mathcal{F}, T) is a Menger PN-space and, for all $x, y \in X$, $\overline{\mathcal{F}} : X \times X \rightarrow \Delta^+$ is defined by

$$\overline{\mathcal{F}}(x, y) = F_{x-y}, \quad (2.8)$$

then $(X, \overline{\mathcal{F}}, T)$ is a Menger space.

The following definition can be found in [7].

Definition 2.2 (see [7]). (X, \mathcal{F}, T) is called a non-Archimedean Menger PN-space (briefly, an N.A. Menger PN-space) if (X, \mathcal{F}, T) is a Menger PN-space satisfying the following condition:

$$F_{x+y}(\max\{t_1, t_2\}) \geq T(F_x(t_1), F_y(t_2)), \quad \forall x, y \in X, t_1, t_2 \geq 0. \quad (2.9)$$

If (X, \mathcal{F}, T) is a Menger PN-space and T is a t -norm which satisfies the condition:

$$\sup_{a < 1} T(a, a) = 1, \quad (2.10)$$

then (X, \mathcal{F}, T) is a Hausdorff topological vector space in the topology τ induced by the base of neighborhoods of θ

$$\{U(\varepsilon, \lambda) \mid \varepsilon > 0, \lambda \in (0, 1]\}, \quad (2.11)$$

where $U(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}$.

Example 2.3 (see [9]). It is easy to see that an ultra-metric space (M, d) belongs to the class of N.A. Menger PN-spaces, where $M \neq \emptyset$ and $d : M \times M \rightarrow [0, \infty)$ satisfying the following conditions:

- (i) $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in M$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in M$;
- (iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in M$.

A fuzzy set A in X is a function from X into $[0, 1]$. If $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A . The α -level set of A , denoted by A_α , is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\}, \quad \forall \alpha \in (0, 1]. \quad (2.12)$$

Let $W(X)$ denote the collection of all fuzzy sets A in X such that A_α is compact and convex for all $\alpha \in (0, 1]$ and $\sup_{x \in X} A(x) = 1$. For any $A, B \in W(X)$, $A \subset B$ means $A(x) \leq B(x)$ for all $x \in X$.

Let M be an arbitrary set and X any linear metric space. A function $\tilde{S} : M \rightarrow W(X)$ is called fuzzy operator. Now, we define an orbit for mixed operators (\tilde{S}_n, f, g) and a joint orbitally complete space as follows.

Definition 2.4 (see [15]). Let f, g be two operators from X into itself and $\{\tilde{S}_n\}_{n=1}^\infty$ a sequence of fuzzy operators from X into $W(X)$. If, for some $x_0 \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{cases} \{y_{2n+1}\} = \{f(x_{2n+1})\} \subset \tilde{S}_{2n+1}(x_{2n}), \\ \{y_{2n+2}\} = \{g(x_{2n+2})\} \subset \tilde{S}_{2n+2}(x_{2n+1}), \quad \forall n \geq 0, \end{cases} \quad (2.13)$$

then $\mathfrak{O}(\tilde{S}_n, f, g, x_0) = \{y_n : n \geq 1\}$ is called an orbit for the mixed operators (\tilde{S}_n, f, g) .

Definition 2.5 (see [15]). X is called x_0 -joint orbitally complete if every Cauchy sequence of each orbit at x_0 is convergent in X .

Remark 2.6 (see [15]). Clearly, if X is an any complete space and $x_0 \in X$, then X is x_0 -joint orbitally complete, while the converse is not necessarily true.

3. Some Countable t -Norms

Let T be a t -norm and, for each $n \geq 1$, and a mapping $T_n : [0, 1] \rightarrow [0, 1]$ let defined in the following way:

$$T_1(x) = T(x, x), \quad T_{n+1}(x) = T(T_n(x), x), \quad \forall n \geq 1, x \in [0, 1]. \quad (3.1)$$

A t -norm T is of H -type if the family $\{T_n(x)\}_{n \geq 1}$ is equicontinuous at $x = 1$ (see [19]).

Each t -norm T can be extended (by the associativity) in a unique way to the n -ary operation taking the values $T(x_1, x_2, \dots, x_n) = \mathbf{T}_{i=1}^n x_i$ for any $(x_1, \dots, x_n) \in [0, 1]^n$, which is defined by

$$\mathbf{T}_{i=1}^0 x_i = 1, \quad \mathbf{T}_{i=1}^n x_i = T\left(\mathbf{T}_{i=1}^{n-1} x_i, x_n\right). \quad (3.2)$$

A t -norm T can be extended to countable infinitely operations taking the value

$$\mathbf{T}_{i=1}^\infty x_i = \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^n x_i \quad (3.3)$$

for any sequence $\{x_n\}_{n \geq 1}$ in $[0, 1]$. Also, the sequence $\{\mathbf{T}_{i=1}^n x_i\}_{n \geq 1}$ is nonincreasing and bounded from below and hence the limit $\mathbf{T}_{i=1}^\infty x_i$ exists.

By (3.3) and fixed point theory in the book by Hadžić and Pap [4], it is interested to investigate the classes of t -norms T and sequences $\{x_n\}$ in the interval $[0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^\infty x_i &\triangleq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbf{T}_{i=1}^m x_i \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbf{T}_{i=1}^{m-n+1} x_{i+n-1} \\ &= \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^\infty x_{i+n-1} \\ &= \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^\infty x_{n+i-1} = 1. \end{aligned} \quad (3.4)$$

It is obvious that

$$\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^{\infty} x_i = 1 \iff \sum_{i=1}^{\infty} (1 - x_i) < \infty \quad (3.5)$$

for $T = T_L$ and $T = T_P$.

The important classes of t -norms are given in the following example.

Example 3.1 (see [16]). (1) The Dombi family of t -norms $(T_{\lambda}^D)_{\lambda \in [0, \infty]}$ is defined by

$$T_{\lambda}^D(x, y) = \begin{cases} T_D(x, y), & \lambda = 0, \\ T_M(x, y), & \lambda = \infty, \\ \frac{1}{1 + \left(((1-x)/x)^{\lambda} + ((1-y)/y)^{\lambda} \right)^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases} \quad (3.6)$$

(2) The Aczél-Alsina family of t -norms $(T_{\lambda}^{AA})_{\lambda \in [0, \infty]}$ is defined by

$$T_{\lambda}^{AA}(x, y) = \begin{cases} T_D(x, y), & \lambda = 0, \\ T_M(x, y), & \lambda = \infty, \\ e^{-((-\log x)^{\lambda} + (-\log y)^{\lambda})^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases} \quad (3.7)$$

(3) The family $(T_{\lambda}^{SW})_{\lambda \in [-1, \infty]}$ of Sugeno-Weber t -norms is given by

$$T_{\lambda}^{SW}(x, y) = \begin{cases} T_D(x, y), & \lambda = -1, \\ T_P(x, y), & \lambda = \infty, \\ \max\left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right), & \lambda \in (-1, \infty). \end{cases} \quad (3.8)$$

(4) The Schweizer-Sklar family of t -norms $(T_{\lambda}^{SS})_{\lambda \in [-\infty, \infty]}$ is defined by

$$T_{\lambda}^{SS}(x, y) = \begin{cases} T_M(x, y), & \lambda = -\infty, \\ T_P(x, y), & \lambda = 0, \\ (\max(x^{\lambda} + y^{\lambda} - 1, 0))^{1/\lambda}, & \lambda \in (-\infty, 0) \cup (0, \infty), \\ T_D(x, y), & \lambda = \infty. \end{cases} \quad (3.9)$$

The condition $T \geq T_L$ is fulfilled by the families $(T_{\lambda}^{SS})_{\lambda \in (-\infty, 1)}$, $(T_{\lambda}^{SW})_{\lambda \in [0, \infty]}$.

There exists a member of the family $(T_{\lambda}^D)_{\lambda \in (0, \infty)}$ which is incomparable with T_L and there exists a member of the family $(T_{\lambda}^{AA})_{\lambda \in (0, \infty)}$ which is incomparable with T_L .

In [4], the following results and proposition are obtained.

- (1) If $(T_\lambda^D)_{\lambda \in (0, \infty)}$ is the Dombi family of t -norms and $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$, then we have the following equivalence:

$$\sum_{n=1}^{\infty} (1 - x_n)^\lambda < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^D \right)_{i=n}^{\infty} x_i = 1. \quad (3.10)$$

- (2) If $(T_\lambda^{\text{SW}})_{\lambda \in (-1, \infty]}$ is the Sugeno-Weber family of t -norms and $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$, then we have the following equivalence:

$$\sum_{n=1}^{\infty} (1 - x_n) < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^{\text{SW}} \right)_{i=n}^{\infty} x_i = 1. \quad (3.11)$$

- (3) The equivalence (3.10) holds also for the family $(T_\lambda^{\text{AA}})_{\lambda \in (0, \infty)}$, that is,

$$\sum_{n=1}^{\infty} (1 - x_n)^\lambda < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^{\text{AA}} \right)_{i=n}^{\infty} x_i = 1. \quad (3.12)$$

Proposition 3.2. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$ and T a t -norm of H -type. Then $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} x_i = \lim_{n \rightarrow \infty} T_{i=1}^{\infty} x_{n+i} = 1$.

4. The Main Results

Let (X, \mathcal{F}, T) be a Menger PN-space with the t -norm T satisfying condition $\sup_{a < 1} T(a, a) = 1$ and A a nonempty subset of X . If $\sup_{t > 0} D_A(t) = 1$, where

$$D_A(t) = \sup_{s < t} \inf_{p, q \in A} F_{p-q}(s), \quad \forall s, t \in \mathbb{R}, \quad (4.1)$$

then A is called a probabilistically bounded set. Let Ω_X be the collection of all nonempty closed probabilistically bounded subsets of X . For any $A, B \in \Omega_X$, define the distribution functions $F_{A,B}$ and F_A by

$$F_{A,B}(t) = \sup_{s < t} T \left(\inf_{a \in A} \sup_{b \in B} F_{a-b}(s), \inf_{b \in B} \sup_{a \in A} F_{a-b}(s) \right), \quad \forall s, t \in \mathbb{R}, \quad (4.2)$$

$$F_A(t) = \sup_{s < t} \sup_{a \in A} F_a(s), \quad \forall s, t \in \mathbb{R}, \quad (4.3)$$

respectively.

Let $\tilde{S}_i, \tilde{S}_j : X \rightarrow W(Y)$ be two fuzzy operators satisfying the following condition (I).

(I) There exist two mappings $a, b : X \rightarrow (0, 1]$ such that, for all $x \in X$, the set $(\tilde{S}_{ix})_{a(x)} \in \Omega_Y$ and $(\tilde{S}_{jx})_{b(x)} \in \Omega_Y$.

We note that

$$(\tilde{S}_{ix})_{a(x)} = \{y \mid \tilde{S}_{ix}(y) \geq a(x)\} \in \Omega_Y, \quad (4.4)$$

where $a(x) \in (0, 1)$ is a real number and $\tilde{S}_{ix} \in W(Y)$ is a fuzzy set in Y decided by the fuzzy operator \tilde{S}_i at $x \in X$. By using each pair of fuzzy operators \tilde{S}_i and \tilde{S}_j , we can define two set-valued mappings S_i and S_j as follows:

$$\begin{cases} S_i : X \rightarrow \Omega_Y, & x \mapsto (\tilde{S}_{ix})_{a(x)}, & \forall x \in X, \\ S_j : X \rightarrow \Omega_Y, & x \mapsto (\tilde{S}_{jx})_{b(x)}, & \forall x \in X. \end{cases} \quad (4.5)$$

In the sequel, for some $i, j \in \mathbb{N}$, S_i and S_j are called the set-valued mappings induced by the fuzzy mappings \tilde{P} and \tilde{S}_j , respectively.

We need the following lemma and definitions.

Lemma 4.1 (see [7]). *Let (X, \mathcal{F}, T) be a Menger PN-space with a t -norm T satisfying $\sup_{a < 1} T(a, a) = 1$ and let $A, B \in \Omega_X$. Then we have the following.*

- (1) $F_A(0) = 0$.
- (2) $F_A(t) = 1$ for all $t > 0$ if and only if $\theta \in A$.
- (3) $F_{\alpha A}(t) = F_A(t/|\alpha|)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
- (4) If $\theta \in B$, then we have $F_A(t) \geq F_{A,B}(t)$ for all $t \in \mathbb{R}$.

Definition 4.2. Let $(X, \hat{\mathcal{F}}, T)$ and (Y, \mathcal{F}, T) be two Menger PN-spaces. A set-valued mapping $P : D(P) \subset X \rightarrow \Omega_Y$ is said to be τ -closed if, for any $x_n \in D(P)$ and $y_n \in P(x_n)$, whenever $x_n \xrightarrow{\tau_X} x$ and $y_n \xrightarrow{\tau_Y} y$, we have $x \in D(P)$ and $y \in P(x)$.

Definition 4.3. A function $\Psi : [0, 1] \rightarrow [0, 1]$ is said to satisfy the condition (Ψ) if it is nondecreasing and $\lim_{n \rightarrow \infty} \Psi^n(t) = 1$ for all $t \in [0, 1]$.

It is easy to prove that if $\Psi : [0, 1] \rightarrow [0, 1]$ satisfies the condition (Ψ) , then $\Psi(t) > t$ for all $t \in [0, 1)$.

Definition 4.4 (see [7]). Let $(X, \hat{\mathcal{F}}, T)$ and (Y, \mathcal{F}, T) be two Menger PN-spaces and $P : D(P) \subset X \rightarrow 2^Y$, $Q : D(Q) \subset X \rightarrow 2^Y$ be two set-valued mappings. Let $\Gamma_1, \Gamma_2 : X \rightarrow L(Y, X)$ be two mappings, where $L(Y, X)$ denotes the space of all linear operators from Y to X . (Γ_1, Γ_2) is called a probabilistic Ψ -contractor couple of P and Q if there exists a function $\Psi : [0, 1] \rightarrow [0, 1]$ satisfying the condition (Ψ) such that

$$F_{P(x+\Gamma_1(x)y), Q(x)+y}(t) \geq \Psi(\min\{F_Y(t), F_{Q(x)}(t), F_{P(x+\Gamma_1(x)y)}(t)\}) \quad (4.6)$$

for all $x \in D(Q)$, $y \in \{y \in Y : x + \Gamma_1(x)y \in D(P)\}$, $t \geq 0$, and

$$F_{Q(x+\Gamma_2(x)y), P(x)+y}(t) \geq \Psi(\min\{F_y(t), F_{P(x)}(t), F_{Q(x+\Gamma_2(x)y)}(t)\}) \tag{4.7}$$

for all $x \in D(P)$, $y \in \{y \in Y : x + \Gamma_2(x)y \in D(Q)\}$, and $t \geq 0$.

Now, we introduce two algorithms for our main results as follows.

Algorithm 1. Let $(X, \widehat{\mathcal{F}}, T)$ be an N.A. Menger PN-space with a t -norm T and (Y, \mathcal{F}, T_Y) be a Menger PN-space with a t -norm T_Y . Let f, g be two operators from X into itself, $\{\tilde{S}_n\}_{n=1}^\infty$ a sequence of fuzzy operators from X into $W(X)$ satisfying the condition (I), and S_n the τ -closed set-valued operators induced by the fuzzy operators \tilde{S}_n for all $n \in \mathbb{N}$. Let $\Gamma_1, \Gamma_2 : X \rightarrow L(Y, X)$ and $\Psi : [0, 1] \rightarrow [0, 1]$ satisfy the condition (Ψ) . Suppose that

- (i) $S_i(x) \subset f(X)$ and $S_j(x) \subset g(X)$ for all $x \in X$;
- (ii) $x + \Gamma_1(x)y \in D(S_i)$ for all $x \in D(S_j)$ and $y \in Y$, $x + \Gamma_2(x)y \in D(S_j)$ for all $x \in D(S_i)$ and $y \in Y$;
- (iii) (Γ_1, Γ_2) is a probabilistic Ψ -contractor couple of S_i and S_j ;
- (iv) for all $x \in D(S_j)$ and $y \in S_j(x)$, there exists $v \in S_i(x + \Gamma_1(x)y)$ such that

$$F_v(t) \geq F_{S_i(x+\Gamma_1(x)y), S_j(x)-y}(t), \quad \forall t \geq 0, \tag{4.8}$$

and, for all $x \in D(S_i)$ and $y \in S_i(x)$, there exists $w \in S_j(x + \Gamma_2(x)y)$ such that

$$F_w(t) \geq F_{S_j(x+\Gamma_2(x)y), S_i(x)-y}(t), \quad \forall t \geq 0. \tag{4.9}$$

For any $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$, put $x_1 = (1 - \alpha_0)x_0 + \alpha_0(x_0 - \Gamma_1(x_0)y_0)$, where $\alpha_0 \in (0, 1]$ is a real number. It follows from the assumption (ii) that $x_1 \in D(S_i)$. Replacing x and y by x_0 and $-y_0$ in (4.6), respectively, from (3.11) of Lemma 4.1, the assumption (iii), and $\theta \in S_j(x_0) - y_0$, it follows that

$$\begin{aligned} F_{S_i(x_1)}(t) &= F_{S_i((1-\alpha_0)x_0+\alpha_0(x_0-\Gamma_1(x_0)y_0))}(t) \\ &\geq F_{S_i((1-\alpha_0)x_0+\alpha_0(x_0-\Gamma_1(x_0)y_0)), S_j(x_0)-y_0}(t) \\ &\geq \Psi\left(\min\{F_{y_0}(t), F_{S_j(x_0)}(t), F_{S_i(x_1)}(t)\}\right), \quad \forall t \geq 0. \end{aligned} \tag{4.10}$$

Since $y_0 \in S_j(x_0)$ and $F_{S_j(x_0)}(t) = \sup_{s < t} \sup_{a \in S_j(x_0)} F_a(s)$, it follows that $\sup_{a \in S_j(x_0)} F_a(s) \geq F_{y_0}(s)$ and so

$$F_{S_j(x_0)}(t) \geq \sup_{s < t} F_{y_0}(s) = F_{y_0}(t), \quad \forall t \geq 0. \tag{4.11}$$

Since F_{y_0} is left continuous, now we have

$$F_{S_i(x_1)}(t) \geq \Psi(\min\{F_{y_0}(t), F_{S_i(x_1)}(t)\}), \quad \forall t \geq 0, \tag{4.12}$$

and so $F_{S_i(x_1)}(t) \geq F_{y_0}(t)$ for all $t > 0$. In fact, if there exists $t_0 > 0$ such that $F_{S_i(x_1)}(t_0) < F_{y_0}(t_0)$, then it follows from (4.12) that

$$F_{S_i(x_1)}(t_0) \geq \Psi(F_{S_i(x_1)}(t_0)) > F_{S_i(x_1)}(t_0), \quad (4.13)$$

which is a contradiction. Therefore, $F_{S_i(x_1)}(t) \geq F_{y_0}(t)$ for all $t > 0$. Thus, from (4.12), we have

$$F_{S_i(x_1)}(t) \geq F_{S_i(x_1), S_j(x_0)-y_0}(t) \geq \Psi(F_{y_0}(t)), \quad \forall t \geq 0. \quad (4.14)$$

By the assumption (iv) and (4.14), for any $\theta \in S_j(x_0) - y_0$, there exists $y_1 \in S_i(x_1)$ such that

$$F_{y_1}(t) \geq F_{S_i(x_1), S_j(x_0)-y_0}(t) \geq \Psi(F_{y_0}(t)), \quad \forall t > 0. \quad (4.15)$$

Let $x_2 = (1 - \alpha_1)x_1 + \alpha_1(x_1 - \Gamma_2(x_1)y_1)$, where α_1 is a real number which satisfies inequality $0 < \alpha_1 \leq \alpha_0 \leq 1$. By the assumption (ii), we know that $x_2 \in D(S_j)$. Similarly, since $\theta \in S_i(x_1) - y_1$, it follows from (4.6) that

$$\begin{aligned} F_{S_j(x_2)}(t) &= F_{S_j((1-\alpha_1)x_1 + \alpha_1(x_1 - \Gamma_2(x_1)y_1))}(t) \\ &\geq F_{S_j((1-\alpha_1)x_1 + \alpha_1(x_1 - \Gamma_2(x_1)y_1)), S_i(x_1) - y_1}(t) \\ &\geq \Psi\left(\min\{F_{y_1}(t), F_{S_i(x_1)}(t), F_{S_j(x_2)}(t)\}\right) \\ &\geq \Psi\left(\min\{F_{y_1}(t), F_{S_j(x_2)}(t)\}\right), \quad \forall t \geq 0. \end{aligned} \quad (4.16)$$

It is easy to check that $F_{y_1}(t) \leq F_{S_j(x_2)}(t)$ for all $t > 0$ and so it follows from (4.15) that

$$F_{S_j(x_2)}(t) \geq F_{S_j(x_2), S_i(x_1) - y_1}(t) \geq \Psi(F_{y_1}(t)) \geq \Psi^2(F_{y_0}(t)), \quad \forall t > 0. \quad (4.17)$$

Now, for any $\theta \in S_i(x_1) - y_1$, the assumption (iv) implies that there exists $y_2 \in S_j(x_2)$ such that

$$F_{y_2}(t) \geq F_{S_j(x_2), S_i(x_1) - y_1}(t) \geq \Psi^2(F_{y_0}(t)), \quad \forall t > 0. \quad (4.18)$$

Inductively, we can get two sequences $\{x_n\}$ in X and $\{y_n\}$ in Y , respectively, as follows:

$$\begin{cases} x_{2n+1} = (1 - \alpha_{2n})x_{2n} + \alpha_{2n}(x_{2n} - \Gamma_1(x_{2n})y_{2n}), \\ x_{2n+2} = (1 - \alpha_{2n+1})x_{2n+1} + \alpha_{2n+1}(x_{2n+1} - \Gamma_2(x_{2n+1})y_{2n+1}), \end{cases} \quad \forall n \geq 0, \quad (4.19)$$

where $\{\alpha_n\}$ is a real monotone decreasing sequence in $(0, 1]$ and $\alpha_n \rightarrow \alpha \in (0, 1]$ as $n \rightarrow \infty$; the sequence $\{y_n\}$ in Y is defined by (2.13) and satisfies the following:

$$F_{y_n}(t) \geq \Psi^n(F_{y_0}(t)), \quad \forall t > 0. \quad (4.20)$$

Algorithm 2. Let $(X, \widehat{\mathcal{F}}, T)$ be a N.A. Menger PN-space with a t -norm T , (Y, \mathcal{F}, T_Y) a Menger PN-space with a t -norm T_Y , and $A, V : X \rightarrow X$ be two operators. Let $\{\tilde{S}_n\}_{n=1}^\infty$ a sequence of fuzzy operators from X into $W(X)$ satisfying the condition (I) and S_n the set-valued operators induced by the fuzzy operators \tilde{S}_n for all $n \geq 1$. Let $\Gamma_1, \Gamma_2 : X \rightarrow L(Y, X)$ and $\Psi : [0, 1] \rightarrow [0, 1]$ satisfy the condition (Ψ) . Suppose that the conditions (ii)–(iv) in Algorithm 1 are satisfied. If

- (i)' $S_i(x) \subset X - A(X)$ and $S_j(x) \subset X - V(X)$ for all $x \in X$, then, for any $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$, we have two sequences $\{x_n\}$ in X and $\{y_n\}$ in Y , respectively, defined as follows:

$$\begin{cases} x_{2n+1} = x_{2n} - \Gamma_1(x_{2n})y_{2n}, \\ x_{2n+2} = x_{2n+1} - \Gamma_2(x_{2n+1})y_{2n+1}, \quad \forall n \geq 0, \end{cases} \tag{4.21}$$

where the sequence $\{y_n\}$ in Y is defined by (2.13).

Now, we state our main results by using the similar ideas as in [9].

Theorem 4.5. *Let $(X, \widehat{\mathcal{F}}, T)$ be an N.A. Menger PN-space with a t -norm T and (Y, \mathcal{F}, T_Y) be a Menger PN-space with a t -norm T_Y . Let $f, g, \{\tilde{S}_n\}_{n=1}^\infty, S_n, \Gamma_1, \Gamma_2,$ and Ψ be the same as in Algorithm 1. Suppose that the conditions (i)–(iv) in Algorithm 1 hold and the following conditions are satisfied:*

- (v) $g(X)$ is x_0 -joint orbitally complete for some $x_0 \in X$;
- (vi) there exists a constant $M > 0$ such that, for any constant $\lambda_1 > \lambda_2 > 0$,

$$\begin{cases} \widehat{F}_{\lambda_1 \Gamma_1(x)y}(t) \geq F_y\left(\frac{t}{\lambda_2 M}\right), \quad \forall x \in D(S_j), y \in Y, t \geq 0, \\ \widehat{F}_{\lambda_1 \Gamma_2(x)y}(t) \geq F_y\left(\frac{t}{\lambda_2 M}\right), \quad \forall x \in D(S_i), y \in Y, t \geq 0; \end{cases} \tag{4.22}$$

- (vii) there exist $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$ such that the t -norm T satisfies the following condition:

$$\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^\infty \Psi^i \left(F_{y_0} \left(\frac{t}{\lambda M} \right) \right) = 1, \quad \forall t \geq 0, \tag{4.23}$$

where λ is a real nonnegative number.

Then the following system of nonlinear operator equations:

$$\begin{cases} \theta = f(x), \\ \theta = g(x) \end{cases} \tag{4.24}$$

has a solution z such that $\{f(z)\} = \{g(z)\} \subset \bigcap_{i=1}^\infty S_i(z)$. Further, $\{x_n\}$ τ -converges to a solution of (4.24) and $\{y_n\}$ τ -converges to θ , where $\{x_n\}$ in X and $\{y_n\}$ in Y are two sequences generated by Algorithm 1.

Proof. By (4.19), (4.20), and the assumption (vi), since $\{\alpha_n\} \subset (0, 1]$ is a monotone decreasing sequence with $\alpha_n \rightarrow \alpha \in (0, 1]$, we have

$$\begin{cases} \widehat{F}_{x_{2n+1}-x_{2n}}(t) = \widehat{F}_{\alpha_{2n}\Gamma_1(x_{2n})y_{2n}}(t) \geq F_{y_{2n}}\left(\frac{t}{\alpha M}\right) \geq \Psi^{2n}\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \\ \widehat{F}_{x_{2n+2}-x_{2n+1}}(t) = \widehat{F}_{\alpha_{2n+1}\Gamma_2(x_{2n+1})y_{2n+1}}(t) \geq F_{y_{2n+1}}\left(\frac{t}{\alpha M}\right) \geq \Psi^{2n+1}\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \quad \forall t \geq 0, \end{cases} \quad (4.25)$$

which imply that

$$\widehat{F}_{x_{n+1}-x_n}(t) \geq \Psi^n\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \quad \forall t > 0, \quad n \geq 1. \quad (4.26)$$

Since $(X, \widehat{\mathcal{F}}, T)$ is N.A. Menger PN-space, it follows from (4.26) that, for any positive integers $m, n (m > n)$,

$$\begin{aligned} & \widehat{F}_{x_n-x_m}(t) \\ & \geq T\left(\widehat{F}_{x_n-x_{n+1}}(t), \widehat{F}_{x_{n+1}-x_m}(t)\right) \\ & \geq T\left(\widehat{F}_{x_n-x_{n+1}}(t), T\left(\widehat{F}_{x_{n+1}-x_{n+2}}(t), \widehat{F}_{x_{n+2}-x_m}(t)\right)\right) \\ & \geq T\left(\widehat{F}_{x_n-x_{n+1}}(t), T\left(\widehat{F}_{x_{n+1}-x_{n+2}}(t), \underbrace{T\left(\dots, T\left(\widehat{F}_{x_{m-2}-x_{m-1}}(t), \widehat{F}_{x_{m-1}-x_m}(t)\right)\dots\right)}_{(m-n)-3}\right)\right) \\ & \geq T\left(\Psi^n\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), T\left(\Psi^{n+1}\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \right. \right. \\ & \quad \left. \left. T\left(\dots, T\left(\Psi^{m-2}\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \Psi^{m-1}\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right)\right)\dots\right)\right)\right) \\ & = \mathbf{T}_{i=n}^{m-1}\Psi^i\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right) \\ & \geq \mathbf{T}_{i=n}^\infty\Psi^i\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right), \quad \forall t \geq 0. \end{aligned} \quad (4.27)$$

Since $\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^\infty \Psi^i(F_{y_0}(t/\alpha M)) = 1$, it follows that, for all $\lambda \in (0, 1)$ and $t > 0$, there exists a positive integer $n(t, \lambda)$ such that, for all $n \geq n(t, \lambda)$ and $m > n$,

$$\mathbf{T}_{i=n}^\infty \Psi^i\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right) > 1 - \lambda \tag{4.28}$$

and so

$$\widehat{F}_{x_n-x_m}(t) \geq \mathbf{T}_{i=n}^{m-1} \Psi^i\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right) \geq \mathbf{T}_{i=n}^\infty \Psi^i\left(F_{y_0}\left(\frac{t}{\alpha M}\right)\right) > 1 - \lambda. \tag{4.29}$$

Hence $\{x_n\}$ is a τ -Cauchy sequence in X . Since $g(X)$ is x_0 -joint orbitally complete, we can assume that $x_n \xrightarrow{\tau_X} z \in X$. Moreover, by (4.20), it is easy to see that $\lim_{n \rightarrow \infty} F_{y_n}(t) = 1$ for all $t > 0$ and so $y_n \xrightarrow{\tau_Y} \theta$. Since S_i and S_j are τ -closed, it follows from (4.19) and the assumption (i) that

$$\begin{cases} \theta \in S_i(z), \\ \theta \in S_j(z), \quad \forall i, j \geq 1, \end{cases} \tag{4.30}$$

that is, z is a solution of (4.24) and $\{f(z)\} = \{g(z)\} \subset \bigcap_{i=1}^\infty S_i(z)$. This completes the proof. \square

From Theorem 4.5, we have the following.

Corollary 4.6. *Let $(X, \widehat{\mathcal{F}}, T)$ be an N.A. Menger PN-space with a t -norm T and (Y, \mathcal{F}, T_Y) a Menger PN-space with a t -norm T_Y . Let f, g be two operators from X into itself, $\{\tilde{S}_n\}_{n=1}^\infty$ a sequence of fuzzy operators from X into $W(X)$ satisfying the condition (I), and S_n the τ -closed set-valued operators induced by the fuzzy operators \tilde{S}_n for all $n \geq 1$. Let $\Gamma_1, \Gamma_2 : X \rightarrow L(Y, X)$ and $\Psi : [0, 1] \rightarrow [0, 1]$ satisfy the condition (Ψ) . Suppose that the conditions (i)-(iv) in Algorithm 1 and (v)-(vi) of Theorem 4.5 are satisfied. If t -norm T is of H -type, then the conclusions of Theorem 4.5 still hold.*

Proof. By Proposition 3.2, we know that all the conditions of Theorem 4.5 are satisfied. Thus the conclusions of Theorem 4.5 still hold. \square

Corollary 4.7. *Let $(X, \widehat{\mathcal{F}}, (T_\lambda^D))$ for some $\lambda > 0$ be a N.A. Menger PN-space and (Y, \mathcal{F}, T_Y) be a Menger PN-space. Let f, g be two operators from X into itself, $\{\tilde{S}_n\}_{n=1}^\infty$ be a sequence of fuzzy operators from X into $W(X)$ satisfying the condition (I) and S_n be the τ -closed set-valued operators induced by the fuzzy operators \tilde{S}_n for all $n \geq 1$. Let $\Gamma_1, \Gamma_2 : X \rightarrow L(Y, X)$ and $\Psi : [0, 1] \rightarrow [0, 1]$ satisfy the condition (Ψ) . Suppose that the conditions (i)-(iv) in Algorithm 1 and (v)-(vi) of Theorem 4.5 are satisfied. If there exist $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$ for some $j \in \mathbb{N}$ such that $\sum_{n=1}^\infty (1 - \Psi^n(F_{y_0}(t/\mu M)))^\lambda < \infty$ for all $t \geq 0$, where $\mu > 0$ is a constant, then the conclusions of Theorem 4.5 still hold.*

Proof. From the equivalence (3.10), we have

$$\sum_{n=1}^{\infty} \left(1 - \Psi^n \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) \right)^\lambda < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^D \right)_{i=n}^\infty \Psi^i \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) = 1. \tag{4.31}$$

Corollary 4.8. Let $(X, \widehat{\mathcal{F}}, (T_\lambda^{SW}))$ for some $\lambda > -1$ be an N.A. Menger PN-space. Let (Y, \mathcal{F}, T_Y) , f , g , $\{\tilde{S}_n\}_{n=1}^\infty$, S_n , Γ_1 , Γ_2 , and Ψ be the same as in Theorem 4.5. Suppose that the conditions (i)–(iv) in Algorithm 1 and (v)–(vi) in Theorem 4.5 are satisfied. If there exist $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$ for some $j \in \mathbb{N}$ such that $\sum_{n=1}^\infty (1 - \Psi^n(F_{y_0}(t/\mu M))) < \infty$ for all $t \geq 0$, where $\mu > 0$ is a constant, then the conclusions of Theorem 4.5 still hold.

Proof. From the equivalence (3.11), we have

$$\sum_{n=1}^{\infty} \left(1 - \Psi^n \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) \right) < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^{SW} \right)_{i=n}^\infty \Psi^i \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) = 1. \tag{4.32}$$

Remark 4.9. Since

$$T_L \in \bigcup_{\lambda \in (-1, \infty)} T_\lambda^{SW}, \tag{4.33}$$

it is easy to see that Corollary 4.8 is a generalization of the corresponding result in Fang [9].

Corollary 4.10. Let $(X, \widehat{\mathcal{F}}, (T_\lambda^{AA}))$ for some $\lambda > 0$ be an N.A. Menger PN-space with a t -norm T . Let (Y, \mathcal{F}, T_Y) , f , g , $\{\tilde{S}_n\}_{n=1}^\infty$, S_n , Γ_1 , Γ_2 , and Ψ be the same as in Theorem 4.5. Suppose that the conditions (i)–(iv) in Algorithm 1 and (v)–(vi) of Theorem 4.5 are satisfied. If there exist $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$ for some $j \geq 1$ such that $\sum_{n=1}^\infty (1 - \Psi^n(F_{y_0}(t/\mu M)))^\lambda < \infty$ for all $t \geq 0$, where $\mu > 0$ is a constant, then the conclusions of Theorem 4.5 still hold.

Proof. From the equivalence (3.12), we have

$$\sum_{n=1}^{\infty} \left(1 - \Psi^n \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) \right)^\lambda < \infty \iff \lim_{n \rightarrow \infty} \left(T_\lambda^{AA} \right)_{i=n}^\infty \Psi^i \left(F_{y_0} \left(\frac{t}{\mu M} \right) \right) = 1. \tag{4.34}$$

Corollary 4.11. Let $(X, \widehat{\mathcal{F}}, T)$ be an N.A. Menger PN-space and $L : X \rightarrow \Omega_X$ satisfy the following condition:

$$F_{Lx, Ly}(t) \geq \Psi(\min\{F_{x-y}(t), F_{x-L(x)}(t), F_{y-L(y)}(t)\}), \quad \forall t \geq 0, x, y \in X, \tag{4.35}$$

where a mapping $\Psi : [0, 1] \rightarrow [0, 1]$ satisfies the condition (Ψ) . Suppose that the conditions (i) in Algorithm 1 and (v) in Theorem 4.5 are satisfied and there exists $x_0 \in X$ and $y_0 \in x_0 - L(x_0)$ such that t -norm T satisfies the following condition:

$$\lim_{n \rightarrow \infty} T_{i=n}^\infty \Psi^i(F_{y_0}(t)) = 1, \quad \forall t \geq 0, \tag{4.36}$$

and, for all $x \in X$ and $y \in x - L(x)$, there exists $v \in x + y - L(x + y)$ such that

$$F_v(t) \geq F_{x+y-L(x+y),x-L(x)-y}(t), \quad \forall t \geq 0. \tag{4.37}$$

Then there exists a point $x^* \in X$ such that $x^* \in Lx^*$, that is, x^* is a fixed point of L .

Proof. Putting $S_i(x) = S_j(x) = x - L(x)$ for any fixed $i, j \geq 1$ and $\Gamma_1(x) = \Gamma_2(x) = I_X$, the mappings S_i, S_j, Γ_1 and Γ_2 satisfy all the hypotheses of Theorem 4.5. Therefore, there exists a point $x^* \in X$ such that $\theta \in S_i(x^*) = S_j(x^*) = x^* - L(x^*)$, which means that x^* is a fixed point of T . This completes the proof. \square

Theorem 4.12. Let $(X, \widehat{\mathcal{F}}, T)$ be an N.A. Menger PN-space with a t -norm T and (Y, \mathcal{F}, T_Y) be a Menger PN-space with a t -norm T_Y . Let $A, V, \{\tilde{S}_n\}_{n=1}^\infty, S_n, \Gamma_1, \Gamma_2$, and Ψ be the same as in Algorithm 2. Suppose that the conditions (ii)–(iv) in Algorithm 1 and (i)' in Algorithm 2 are satisfied. If

- (v)' $X - V(X)$ is x_0 -joint orbitally complete for some $x_0 \in X$,
- (vi)' there exists a constant $M > 0$ such that

$$\begin{cases} \widehat{F}_{\Gamma_1(x)y}(t) \geq F_y\left(\frac{t}{M}\right), & \forall x \in D(S_j), y \in Y, t \geq 0, \\ \widehat{F}_{\Gamma_2(x)y}(t) \geq F_y\left(\frac{t}{M}\right), & \forall x \in D(S_i), y \in Y, t \geq 0, \end{cases} \tag{4.38}$$

- (vii)' there exist $x_0 \in D(S_j)$ and $y_0 \in S_j(x_0)$ such that the t -norm T satisfies the following condition:

$$\lim_{n \rightarrow \infty} T_{i=n}^\infty \Psi^i\left(F_{y_0}\left(\frac{t}{M}\right)\right) = 1, \quad \forall t \geq 0, \tag{4.39}$$

then the following system of nonlinear operator equations:

$$\begin{cases} x = A(x), \\ x = V(x) \end{cases} \tag{4.40}$$

has a solution z such that $\{z - A(z)\} = \{z - V(z)\} \subset \bigcap_{i=1}^\infty S_i(z)$. Further, $\{x_n\}$ τ -converges to a solution of (4.40) and $\{y_n\}$ τ -converges to θ , where the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y are defined by Algorithm 2.

Proof. Let $f(x) = x - A(x)$ and $g(x) = x - V(x)$ for all $x \in X$. It is obvious that all the conditions of Theorem 4.5 are satisfied. Therefore, the conclusion of Theorem 4.12 follows from Theorem 4.5 immediately. \square

Remark 4.13. Similarly, we can obtain the conclusions of Theorem 4.12 if we replace the condition (iii) in Theorem 4.12 by the corresponding condition in Proposition 3.2 and the equivalences (3.10)–(3.12), respectively.

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