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Research Article

Stability of a Cauchy-Jensen Functional Equation in Quasi-Banach Spaces

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We obtain the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation $2f(x+y,(z+w)/2) = f(x,z)+f(x,w)+f(y,z)+f(y,w)$.

1. Introduction

In 1940, Ulam proposed the general Ulam stability problem (see [1]).

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [2] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability.

Throughout this paper, let X and Y be vector spaces. A mapping $g : X \to Y$ is called an *additive mapping* (respectively, an *affine mapping*) if g satisfies the Cauchy functional equation $g(x+y) = g(x)+g(y)$ (respectively, the Jensen functional equation $2g((x+y)/2) = g(x)+g(y)$). Aoki [3] and Rassias [4, 5] extended the Hyers-Ulam stability by considering variables for Cauchy equation. Using the method introduced in [3], Jung [6] obtained a result for Jensen equation. It also has been generalized to the function case by Găvruta [7] and Jung [8] for Cauchy equation, and by Lee and Jun [9] for Jensen equation.

Definition 1.1. A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* if f satisfies the system of equations

$$f(x+y,z) = f(x,z) + f(y,z),$$
$$2f\left(x, \frac{y+z}{2}\right) = f(x,y) + f(x,z). \tag{1.1}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy + bx$ is a solution of (1.1). In particular, letting $x = y$, we get a function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := f(x,x) = ax^2 + bx$.

For a mapping $f : X \times X \to Y$, consider the functional equation

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w). \tag{1.2}$$

Definition 1.2 (see [10, 11]). Let X be a real linear space. A *quasi-norm* is real-valued function on X satisfying the following.

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p \tag{1.3}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

The authors [12] obtained the solutions of (1.1) and (1.2) as follows.

Theorem A. *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if there exist a biadditive mapping $B : X \times X \to Y$ and an additive mapping $A : X \to Y$ such that $f(x,y) = B(x,y) + A(x)$ for all $x, y \in X$.*

Theorem B. *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if it satisfies* (1.2).

In this paper, we investigate the generalized Hyers-Ulam stability of (1.1) and (1.2).

2. Stability of (1.1) and (1.2)

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that Y is a p-Banach space with p-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$.

Let $\varphi : X \times X \times X \to [0,\infty)$ and $\psi : X \times X \times X \to [0,\infty)$ be two functions such that

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, z) = 0, \qquad \lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x, y, z) = 0, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{3^n}\varphi(x, y, 3^n z) = 0, \qquad \lim_{n\to\infty} \frac{1}{3^n}\psi(x, 3^n y, 3^n z) = 0 \tag{2.2}$$

for all $x, y, z \in X$, and

$$M(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^{pj}} \varphi\left(2^j x, 2^j y, z\right)^p < \infty, \tag{2.3}$$

$$N(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^{pj}} \psi\left(x, 3^j y, 3^j z\right)^p < \infty \tag{2.4}$$

for all $x, y, z \in X$.

Theorem 2.1. *Suppose that a mapping $f : X \times X \to Y$ satisfies the inequalities*

$$\|f(x+y,z) - f(x,z) - f(y,z)\|_Y \le \varphi(x,y,z), \tag{2.5}$$

$$\left\|2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z)\right\|_Y \le \psi(x,y,z) \tag{2.6}$$

for all $x, y, z \in X$. Then the limits

$$F_C(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f\left(2^j x, y\right), \qquad F_J(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f\left(x, 3^j y\right) \tag{2.7}$$

exist for all $x, y \in X$ and the mappings $F_C : X \times X \to Y$ and $F_J : X \times X \to Y$ are Cauchy-Jensen mappings satisfying

$$\|f(x,y) - F_C(x,y)\|_Y \le \frac{1}{2} M(x,x,y)^{1/p}, \tag{2.8}$$

$$\|f(x,y) - f(x,0) - F_J(x,y)\|_Y \le \frac{K}{3} N(x,y,y)^{1/p} \tag{2.9}$$

for all $x, y \in X$.

Proof. Letting $y = x$ and replacing z by y in (2.5) then,

$$\|f(2x,y) - 2f(x,y)\|_Y \le \varphi(x,x,y) \tag{2.10}$$

for all $x, y \in X$. Replacing x by $2^n x$ in the above inequality and dividing by 2^{n+1}, we get

$$\left\|\frac{1}{2^{n+1}} f\left(2^{n+1}x, y\right) - \frac{1}{2^n} f(2^n x, y)\right\|_Y \le \frac{1}{2^{n+1}} \varphi(2^n x, 2^n x, y) \tag{2.11}$$

for all $x, y \in X$ and all nonnegative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned}\left\|\frac{1}{2^{n+1}}f\left(2^{n+1}x,y\right)-\frac{1}{2^{m}}f(2^{m}x,y)\right\|_{Y}^{p}&\leq\sum_{j=m}^{n}\left\|\frac{1}{2^{j+1}}f(2^{j+1}x,y)-\frac{1}{2^{j}}f(2^{j}x,y)\right\|_{Y}^{p}\\&\leq\frac{1}{2^{p}}\sum_{j=m}^{n}\frac{1}{2^{pj}}\varphi\left(2^{j}x,2^{j}y,y\right)^{p}\end{aligned}\tag{2.12}$$

for all $x, y \in X$ and all nonnegative integers n and m with $n \geq m$. Therefore we conclude from (2.3) and (2.12) that the sequence $\{(1/2^n)f(2^n x, y)\}$ is a Cauchy sequence in Y for all $x, y \in X$. Since Y is complete, the sequence $\{(1/2^n)f(2^n x, y)\}$ converges in Y for all $x, y \in X$. So one can define the mapping $F_C : X \times X \to Y$ by

$$F_C(x,y) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x,y)\tag{2.13}$$

for all $x, y \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.12), we get (2.8). Now, we show that F_C is a Cauchy-Jensen mapping. It follows from (2.1), (2.11), and (2.13) that

$$\begin{aligned}\|F_C(2x,y)-2F_C(x,y)\|_Y&=\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^{n+1}x,y)-\frac{1}{2^{n-1}}f(2^n x,y)\right\|_Y\\&=2\lim_{n\to\infty}\left\|\frac{1}{2^{n+1}}f\left(2^{n+1}x,y\right)-\frac{1}{2^n}f(2^n x,y)\right\|_Y\\&\leq\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n x,y)=0\end{aligned}\tag{2.14}$$

for all $x, y \in X$. So $F_C(2x, y) = 2F_C(x, y)$ for all $x, y \in X$.

On the other hand it follows from (2.1), (2.5), (2.6), and (2.13) that

$$\begin{aligned}\|F_C(x+y,z)-F_C(x,z)-F_C(y,z)\|_Y&=\lim_{n\to\infty}\frac{1}{2^n}\|f(2^n x+2^n y,z)-f(2^n x,z)-f(2^n y,z)\|_Y\\&\leq\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n y,z)=0,\\\left\|2F_C\left(x,\frac{y+z}{2}\right)-F_C(x,y)-F_C(y,z)\right\|_Y&=\lim_{n\to\infty}\frac{1}{2^n}\left\|f\left(2^n x,\frac{y+z}{2}\right)-f(2^n x,y)-f(2^n y,z)\right\|_Y\\&\leq\lim_{n\to\infty}\frac{1}{2^n}\psi(2^n x,y,z)=0\end{aligned}\tag{2.15}$$

for all $x, y, z \in X$. Thus F_C is a Cauchy-Jensen mapping. Next, setting $z = -y$ in (2.6) and replacing y by $-y$ and z by $3y$ in (2.6), one can obtain that

$$\begin{aligned}\|2f(x,0)-f(x,y)-f(x,-y)\|_Y&\leq\psi(x,y,-y),\\\|2f(x,y)-f(x,-y)-f(x,3y)\|_Y&\leq\psi(x,-y,3y),\end{aligned}\tag{2.16}$$

respectively, for all $x, y \in X$. By two above inequalities,

$$\|3f(x,y) - 2f(x,0) - f(x,3y)\|_Y \le K(\varphi(x,y,-y) + \varphi(x,-y,3y)) \tag{2.17}$$

for all $x, y \in X$. By the same method as above, one can find a Cauchy-Jensen mapping F_J which satisfies (2.9). In fact, $F_J(x,y) := \lim_{j\to\infty}(1/3^j)f(x,3^j y)$ for all $x, y \in X$. □

From now on, let $\chi : X \times X \times X \times X \to [0,\infty)$ be a function such that

$$\lim_{n\to\infty}\frac{1}{6^n}\varphi(2^n x, 2^n y, 3^n z, 3^n w) = 0, \tag{2.18}$$

$$L(x,y,z,w) := \sum_{j=0}^{\infty}\frac{1}{6^{pj}}\chi\left(2^j x, 2^j y, 3^j z, 3^j w\right)^p < \infty \tag{2.19}$$

for all $x, y, z, w \in X$.

We will use the following lemma in order to prove Theorem 2.3.

Lemma 2.2 (see [13]). *Let $0 < p \le 1$ and let $x_1, x_2, \dots, x_n$ be nonnegative real numbers. Then*

$$\left(\sum_{j=1}^{n} x_j\right)^p \le \sum_{j=1}^{n} {x_j}^p. \tag{2.20}$$

Theorem 2.3. *Suppose that a mapping $f : X \times X \to Y$ satisfies $f(x,0) = f(0,x) = 0$ and the inequality*

$$\left\|2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\right\|_Y \le \chi(x,y,z,w) \tag{2.21}$$

for all $x, y, z, w \in X$. Then the limit $F(x,y) := \lim_{j\to\infty}(1/6^j)f(2^j x, 3^j y)$ exists for all $x, y \in X$ and the mapping $F : X \times X \to Y$ is the unique Cauchy-Jensen mapping satisfying

$$\|f(x,y) - F(x,y)\|_Y \le \frac{K}{6}\tilde{\chi}(x,y)^{1/p}, \tag{2.22}$$

where

$$\begin{aligned}
\tilde{\chi}(x,y) := \sum_{j=0}^{\infty}\frac{1}{6^{pj}}\Big[&3^p\chi\left(2^j x, 2^j x, 3^j y, -3^j y\right)^p \\
&+ K^{3p}\left(\chi\left(2^j x, 2^j x, -3^j y, 3^j y\right)^p + \chi\left(2^j x, 2^j x, 3^j y, 3^j y\right)^p\right) \\
&+ K^{2p}\chi\left(2^j x, 2^j x, -3^j y, 3^{j+1} y\right)^p + \frac{K^p}{2^p}\chi\left(2^j x, 2^j x, 3^{j+1} y, 3^{j+1} y\right)^p\Big]
\end{aligned} \tag{2.23}$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (2.21), we get

$$\left\|2f\left(2x, \frac{z+w}{2}\right) - 2f(x,z) - 2f(x,w)\right\|_Y \le \chi(x,x,z,w) \tag{2.24}$$

for all $x, z, w \in X$. Putting $z = y$ and $w = -y$ in (2.24), we get

$$\|2f(x,y) + 2f(x,-y)\|_Y \le \chi(x,x,y,-y) \tag{2.25}$$

for all $x, y \in X$. Replacing z by $-y$ and w by $-y$ in (2.24), we get

$$\|f(2x,-y) - 2f(x,-y)\|_Y \le \frac{1}{2}\chi(x,x,-y,-y) \tag{2.26}$$

for all $x, y \in X$. By (2.25) and (2.26), we have

$$\|2f(x,y) + f(2x,-y)\|_Y \le K\left(\chi(x,x,y,-y) + \frac{1}{2}\chi(x,x,-y,-y)\right) \tag{2.27}$$

for all $x, y \in X$. Setting $z = y$ and $w = -3y$ in (2.24), we get

$$\|f(2x,-y) - f(x,y) - f(x,-3y)\|_Y \le \frac{1}{2}\chi(x,x,y,-3y) \tag{2.28}$$

for all $x, y \in X$. By (2.27) and the above inequality, we get

$$\|3f(x,y) + f(x,-3y)\|_Y \le K^2\left(\chi(x,x,y,-y) + \frac{1}{2}\chi(x,x,-y,-y)\right) + \frac{K}{2}\chi(x,x,y,-3y) \tag{2.29}$$

for all $x, y \in X$. Replacing y by $3y$ in (2.26), we get

$$\|f(2x,-3y) - 2f(x,-3y)\|_Y \le \frac{1}{2}\chi(x,x,-3y,-3y) \tag{2.30}$$

for all $x, y \in X$. By (2.29) and the above inequality, we have

$$\begin{aligned}\|6f(x,y) + f(2x,-3y)\|_Y \le{}& K^3(2\chi(x,x,y,-y) + \chi(x,x,-y,-y)) + K^2\chi(x,x,y,-3y)\\ &+ \frac{K}{2}\chi(x,x,-3y,-3y)\end{aligned} \tag{2.31}$$

for all $x, y \in X$. Replacing y by $-y$ in the above inequality, we get

$$\begin{aligned}\|6f(x,-y)+f(2x,3y)\|_Y \le K^3(2\chi(x,x,-y,y)+\chi(x,x,y,y))+K^2\chi(x,x,-y,3y)\\ +\frac{K}{2}\chi(x,x,3y,3y)\end{aligned} \tag{2.32}$$

for all $x, y \in X$. By (2.25) and the above inequality, we get

$$\|6f(x,y)-f(2x,3y)\|_Y \le \chi_*(x,y), \tag{2.33}$$

where

$$\begin{aligned}\chi_*(x,y) := 3K\chi(x,x,y,-y)+K^4(2\chi(x,x,-y,y)+\chi(x,x,y,y))+K^3\chi(x,x,-y,3y)\\ +\frac{K^2}{2}\chi(x,x,3y,3y)\end{aligned} \tag{2.34}$$

for all $x, y \in X$. Replacing x by $2^n x$ and y by $3^n y$ in the above inequality and dividing 6^{n+1},we get

$$\left\|\frac{1}{6^n}f(2^nx,3^ny)-\frac{1}{6^{n+1}}f\left(2^{n+1}x,3^{n+1}y\right)\right\|_Y \le \frac{1}{6^{n+1}}\chi_*(2^nx,3^ny) \tag{2.35}$$

for all $x, y \in X$ and all nonnegative integers n. Since $\|\cdot\|_Y$ is a p-norm, we have

$$\begin{aligned}\left\|\frac{1}{6^{n+1}}f(2^{n+1}x,3^{n+1}y)-\frac{1}{6^m}f(2^mx,3^my)\right\|_Y^p &\le \sum_{j=m}^{n}\left\|\frac{1}{6^{j+1}}f\left(2^{j+1}x,3^{j+1}y\right)-\frac{1}{6^j}f(2^jx,3^jy)\right\|_Y^p\\ &\le \frac{1}{6^p}\sum_{j=m}^{n}\frac{1}{6^{pj}}\chi_*\left(2^jx,3^jy\right)^p\end{aligned} \tag{2.36}$$

for all $x, y \in X$ and all nonnegative integers n and m with $n \ge m$. Therefore we conclude from (2.18) and (2.36) that the sequence $\{(1/6^n)f(2^nx,3^ny)\}$ is a Cauchy sequence in Y for all $x, y \in X$. Since Y is complete, the sequence $\{(1/6^n)f(2^nx,3^ny)\}$ converges in Y for all $x, y \in X$. So one can define the mapping $F : X \times X \to Y$ by

$$F(x,y) := \lim_{n\to\infty}\frac{1}{6^n}f(2^nx,3^ny) \tag{2.37}$$

for all $x, y \in X$. Letting $m = 0$, passing the limit $n \to \infty$ in (2.36), and applying lemma, we get (2.22). Now, we show that F is a Cauchy-Jensen mapping. By lemma, we infer that

$\lim_{n\to\infty}(1/6^n)\chi_*(2^n x, 3^n y) = 0$ for all $x, y \in X$. It follows from (2.18), (2.35), and the above equality that

$$\begin{aligned}\|F(2x,3y)-6F(x,y)\|_Y &= \lim_{n\to\infty}\left\|\frac{1}{6^n}f(2^{n+1}x,3^{n+1}y)-\frac{1}{6^{n-1}}f(2^n x,3^n y)\right\|_Y \\ &= 6\lim_{n\to\infty}\left\|\frac{1}{6^{n+1}}f(2^{n+1}x,3^{n+1}y)-\frac{1}{6^n}f(2^n x,3^n y)\right\|_Y \\ &\le \lim_{n\to\infty}\frac{1}{6^n}\chi_*(2^n x,3^n y)=0\end{aligned} \tag{2.38}$$

for all $x, y \in X$. So $F(2x, 3y) = 6F(x, y)$ for all $x, y \in X$.

On the other hand it follows from (2.18), (2.21), and (2.37) that

$$\begin{aligned}&\left\|2F\left(x+y,\frac{z+w}{2}\right)-F(x,z)-F(x,w)-F(y,z)-F(y,w)\right\|_Y \\ &\quad= \lim_{n\to\infty}\frac{1}{6^n}\left\|f\left(2^n x+2^n y,\frac{3^n z+3^n w}{2}\right)-f(2^n x,3^n z)-f(2^n x,3^n w)\right. \\ &\qquad\left.-f(2^n y,3^n z)-f(2^n y,3^n w)\right\|_Y \\ &\quad= \lim_{n\to\infty}\frac{1}{6^n}\chi(2^n x,2^n y,3^n z,3^n w)=0\end{aligned} \tag{2.39}$$

for all $x, y, z, w \in X$. Hence the mapping F satisfies (1.2). To prove the uniqueness of F, let $G : X \to Y$ be another Cauchy-Jensen mapping satisfying (2.22). It follows from (2.19) that

$$\lim_{n\to\infty}\frac{1}{6^{pn}}L(2^n x,2^n y,3^n z,3^n w)=\lim_{n\to\infty}\sum_{j=n}^{\infty}\frac{1}{6^{pj}}\chi\left(2^j x,2^j y,3^j z,3^j w\right)^p=0 \tag{2.40}$$

for all $x, y, z, w \in X$. Hence $\lim_{n\to\infty}\frac{1}{6^{pn}}\widetilde{\chi}(2^n x, 3^n y) = 0$ for all $x, y \in X$. So it follows from (2.22) and (2.37) that

$$\begin{aligned}\|F(x,y)-G(x,y)\|_Y^p &= \lim_{n\to\infty}\frac{1}{6^{pn}}\|f(2^n x,3^n y)-G(2^n x,3^n y)\|_Y^p \\ &\le \frac{K^p}{6^p}\lim_{n\to\infty}\frac{1}{6^{pn}}\widetilde{\chi}(2^n x,3^n y)=0\end{aligned} \tag{2.41}$$

for all $x, y \in X$. So $F = G$. □

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