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Research Article

An Optimal Double Inequality between Power-Type Heron and Seiffert Means

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For $k \in [0, +\infty)$, the power-type Heron mean $H_k(a,b)$ and the Seiffert mean $T(a,b)$ of two positive real numbers a and b are defined by $H_k(a,b) = ((a^k + (ab)^{k/2} + b^k)/3)^{1/k}$, $k \neq 0$; $H_k(a,b) = \sqrt{ab}$, $k = 0$ and $T(a,b) = (a-b)/2\arctan((a-b)/(a+b))$, $a \neq b$; $T(a,b) = a$, $a = b$, respectively. In this paper, we find the greatest value p and the least value q such that the double inequality $H_p(a,b) < T(a,b) < H_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

1. Introduction

For $k \in [0, +\infty)$, the power-type Heron mean $H_k(a,b)$ and the Seiffert mean $T(a,b)$ of two positive real numbers a and b are defined by

$$H_k(a,b) = \begin{cases} \left(\dfrac{a^k + (ab)^{k/2} + b^k}{3}\right)^{1/k}, & k \neq 0, \\ \sqrt{ab}, & k = 0, \end{cases} \tag{1.1}$$

$$T(a,b) = \begin{cases} \dfrac{a-b}{2\arctan((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

Recently, the means of two variables have been the subject of intensive research [1–15]. In particular, many remarkable inequalities for $H_k(a,b)$ and $T(a,b)$ can be found in the literature [16–20].

It is well known that $H_k(a,b)$ is continuous and strictly increasing with respect to $k \in [0,+\infty)$ for fixed $a,b>0$ with $a \neq b$. Let $A(a,b)=(a+b)/2$, $I(a,b)=(1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b)=(b-a)/(\log b-\log a)$, $G(a,b)=\sqrt{ab}$, and $H(a,b)=2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}. \tag{1.3}$$

For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.4}$$

The main properties for power mean are given in [21].

In [16], Jia and Cao presented the inequalities

$$\begin{gathered} H_0(a,b) = G(a,b) < L(a,b) < H_p(a,b) < M_q(a,b), \\ A(a,b) < H_{\log 3/\log 2}(a,b) \end{gathered} \tag{1.5}$$

for all $a,b>0$ with $a \neq b$, $p \geq 1/2$, and $q \geq (2/3)p$.

Sándor [22] proved that

$$I(a,b) > H_1(a,b) \tag{1.6}$$

for all $a,b>0$ with $a \neq b$.

In [19], Seiffert established that

$$M_1(a,b) < T(a,b) < M_2(a,b) \tag{1.7}$$

for all $a,b>0$ with $a \neq b$.

The purpose of this paper is to present the optimal upper and lower power-type Heron mean bounds for the Seiffert mean $T(a,b)$. Our main result is the following Theorem 1.1.

Theorem 1.1. *For all $a,b>0$ with $a \neq b$, one has*

$$H_{\log 3/\log(\pi/2)}(a,b) < T(a,b) < H_{5/2}(a,b), \tag{1.8}$$

and $H_{\log 3/\log(\pi/2)}(a,b)$ and $H_{5/2}(a,b)$ are the best possible lower and upper power-type Heron mean bounds for the Seiffert mean $T(a,b)$, respectively.

2. Lemmas

In order to prove our main result, Theorem 1.1, we need two lemmas which we present in this section.

Lemma 2.1. *If* $k = \log 3/\log(\pi/2) = 2.43\ldots$ *and* $t > 1$, *then*

$$\begin{aligned}&-24(k-2)(k+3)(k+4)(3k+2)t^{k+8} + 48k(k-1)(k+3)(3k-2)t^{k+6}\\&\quad-(k+4)(k+6)(k+8)(4k-7)t^{8} < 0.\end{aligned}\tag{2.1}$$

Proof. For $t > 1$, we clearly see that

$$\begin{aligned}&-24(k-2)(k+3)(k+4)(3k+2)t^{k+8} + 48k(k-1)(k+3)\\&\quad\times(3k-2)t^{k+6} - (k+4)(k+6)(k+8)(4k-7)t^{8}\\&\quad< t^{8}\Big[-24(k-2)(k+3)(k+4)(3k+2)t^{2} + 48k(k-1)(k+3)\\&\qquad\times(3k-2)t - (k+4)(k+6)(k+8)(4k-7)\Big].\end{aligned}\tag{2.2}$$

Let

$$\begin{aligned}h(t) &= -24(k-2)(k+3)(k+4)(3k+2)t^{2} + 48k(k-1)(k+3)\\&\quad\times(3k-2)t - (k+4)(k+6)(k+8)(4k-7).\end{aligned}\tag{2.3}$$

Then

$$h(1) = 68k^{4} - 281k^{3} - 1010k^{2} + 2072k + 2496 = -104.992\ldots < 0\tag{2.4}$$

and $h(t)$ is strictly decreasing in $[1,+\infty)$ because of $k(k-1)(k+3)(3k-2)/(k-2)(k+3)(k+4)(3k+2) < 1$ for $k = \log 3/\log(\pi/2)$.

Therefore, Lemma 2.1 follows from (2.2)–(2.4) together with the monotonicity of $h(t)$. □

Lemma 2.2. *If* $k = \log 3/\log(\pi/2) = 2.43\ldots$, $t \in [1,+\infty)$, *and* $g(t) = -8t^{4k-4} + 8t^{4k-6} + (2-k)t^{3k+2} + 2kt^{3k} - 2(k+2)t^{3k-2} + 2(k-4)t^{3k-4} + (10-k)t^{3k-6} + (7-4k)t^{2k+2} + 2(4k-1)t^{2k} - 2(4k+5)t^{2k-2} + 2(4k-1)t^{2k-4} + (7-4k)t^{2k-6} + (10-k)t^{k+2} + 2(k-4)t^{k} - 2(k+2)t^{k-2} + 2kt^{k-4} + (2-k)t^{k-6} + 8t^{2} - 8$, *then there exists* $\lambda \in (1,+\infty)$ *such that* $g(t) > 0$ *for* $t \in (1,\lambda)$ *and* $g(t) < 0$ *for* $t \in (\lambda,\infty)$.

Proof. Let $g_1(t) = g'(t)/t$, $g_2(t) = t^{9-k}g_1'(t)$, $g_3(t) = g_2'(t)/2t$, $g_4(t) = g_3'(t)/2t$, $g_5(t) = g_4'(t)/kt$, $g_6(t) = g_5'(t)/t$, and $g_7(t) = t^{9-k}g_6'(t)$. Then elaborated computations lead to

$$g(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} g(t) = -\infty, \tag{2.6}$$

$$\begin{aligned} g_1(t) &= -32(k-1)t^{4k-6} + 16(2k-3)t^{4k-8} + (2-k)(3k+2)t^{3k} \\ &\quad +6k^2t^{3k-2} - 2(k+2)(3k-2)t^{3k-4} + 2(k-4)(3k-4)t^{3k-6} \\ &\quad +3(k-2)(10-k)t^{3k-8} + 2(k+1)(7-4k)t^{2k} + 4k(4k-1)t^{2k-2} \\ &\quad -4(k-1)(4k+5)t^{2k-4} + 4(k-2)(4k-1)t^{2k-6} + 2(7-4k) \\ &\quad \times(k-3)t^{2k-8} + (k+2)(10-k)t^k + 2k(k-4)t^{k-2} \\ &\quad -2(k-2)(k+2)t^{k-4} + 2k(k-4)t^{k-6} + (2-k)(k-6)t^{k-8} + 16, \end{aligned} \tag{2.7}$$

$$g_1(1) = 0, \tag{2.8}$$

$$\lim_{t\to+\infty} g_1(t) = -\infty, \tag{2.9}$$

$$\begin{aligned} g_2(t) &= -64(k-1)(2k-3)t^{3k+2} + 64(k-2)(2k-3)t^{3k} + 3k(2-k) \\ &\quad \times(3k+2)t^{2k+8} + 6k^2(3k-2)t^{2k+6} - 2(k+2)(3k-2)(3k-4)t^{2k+4} \\ &\quad +6(k-2)(k-4)(3k-4)t^{2k+2} + 3(k-2)(10-k)(3k-8)t^{2k} \\ &\quad +4k(k+1)(7-4k)t^{k+8} + 8k(k-1)(4k-1)t^{k+6} - 8(k-1) \\ &\quad \times(k-2)(4k+5)t^{k+4} + 8(k-2)(k-3)(4k-1)t^{k+2} + 4(k-4) \\ &\quad \times(k-3)(7-4k)t^k + k(k+2)(10-k)t^8 + 2k(k-2)(k-4)t^6 \\ &\quad -2(k-2)(k-4)(k+2)t^4 + 2k(k-4)(k-6)t^2 + (2-k)(k-6)(k-8), \end{aligned} \tag{2.10}$$

$$g_2(1) = 144(5-2k) > 0, \tag{2.11}$$

$$\lim_{t\to+\infty} g_2(t) = -\infty, \tag{2.12}$$

$$\begin{aligned} g_3(t) &= -32(k-1)(2k-3)(3k+2)t^{3k} + 96k(k-2)(2k-3)t^{3k-2} \\ &\quad +3k(2-k)(k+4)(3k+2)t^{2k+6} + 6k^2(k+3)(3k-2)t^{2k+4} \\ &\quad -2(k+2)^2(3k-2)(3k-4)t^{2k+2} + 6(k+1)(k-2)(k-4) \\ &\quad \times(3k-4)t^{2k} + 3k(k-2)(10-k)(3k-8)t^{2k-2} + 2k(k+1) \\ &\quad \times(7-4k)(k+8)t^{k+6} + 4k(k-1)(k+6)(4k-1)t^{k+4} \\ &\quad -4(k-1)(k-2)(k+4)(4k+5)t^{k+2} + 4(k-2)(k-3)(k+2) \\ &\quad \times(4k-1)t^k + 2k(k-4)(k-3)(7-4k)t^{k-2} + 4k(k+2) \\ &\quad \times(10-k)t^6 + 6k(k-2)(k-4)t^4 - 4(k-2)(k-4)(k+2)t^2 \\ &\quad +2k(k-4)(k-6), \end{aligned} \tag{2.13}$$

$$g_3(1) = 72(5k-2)(5-2k) > 0, \tag{2.14}$$

$$\lim_{t\to+\infty} g_3(t) = -\infty, \tag{2.15}$$

$$\begin{aligned} g_4(t) &= -48k(k-1)(2k-3)(3k+2)t^{3k-2} + 48k(k-2)(2k-3) \\ &\quad \times(3k-2)t^{3k-4} + 3k(2-k)(k+3)(k+4)(3k+2)t^{2k+4} \\ &\quad +6k^2(k+2)(k+3)(3k-2)t^{2k+2} - 2(k+1)(k+2)^2(3k-2) \\ &\quad \times(3k-4)t^{2k} + 6k(k+1)(k-2)(k-4)(3k-4)t^{2k-2} \\ &\quad +3k(k-1)(k-2)(10-k)(3k-8)t^{2k-4} + k(k+1)(k+6) \\ &\quad \times(7-4k)(k+8)t^{k+4} + 2k(k-1)(k+4)(k+6)(4k-1)t^{k+2} \\ &\quad -2(k-1)(k-2)(k+2)(k+4)(4k+5)t^{k} + 2k(k-2)(k-3) \\ &\quad \times(k+2)(4k-1)t^{k-2} + k(k-2)(k-3)(k-4)(7-4k)t^{k-4} \\ &\quad +12k(k+2)(10-k)t^4 + 12k(k-2)(k-4)t^2 - 4(k-2)(k-4)(k+2), \end{aligned} \tag{2.16}$$

$$g_4(1) = 4\left(-318k^3 + 885k^2 - 210k - 72\right) = 304.99\ldots > 0, \tag{2.17}$$

$$\lim_{t\to+\infty} g_4(t) = -\infty, \tag{2.18}$$

$$\begin{aligned} g_5(t) &= -48(k-1)(2k-3)(3k-2)(3k+2)t^{3k-4} + 48(k-2)(2k-3) \\ &\quad \times(3k-2)(3k-4)t^{3k-6} + 6(2-k)(k+2)(k+3)(k+4)(3k+2) \\ &\quad \times t^{2k+2} + 12k(k+1)(k+2)(k+3)(3k-2)t^{2k} - 4(k+1)(k+2)^2 \\ &\quad \times(3k-2)(3k-4)t^{2k-2} + 12(k-1)(k-2)(k-4)(k+1)(3k-4) \\ &\quad \times t^{2k-4} + 6(k-1)(k-2)^2(10-k)(3k-8)t^{2k-6} + (k+1)(k+4) \\ &\quad \times(k+6)(k+8)(7-4k)t^{k+2} + 2(k-1)(k+2)(k+4)(k+6) \\ &\quad \times(4k-1)t^{k} - 2(k-1)(k-2)(k+2)(k+4)(4k+5)t^{k-2} \\ &\quad +2(k-2)^2(k-3)(k+2)(4k-1)t^{k-4} + (k-2)(k-3)(k-4)^2 \\ &\quad \times(7-4k)t^{k-6} + 48(k+2)(10-k)t^2 + 24(k-2)(k-4), \end{aligned} \tag{2.19}$$

$$g_5(1) = 4\left(-1038k^3 + 3549k^2 - 3360k + 2196\right) = 323.50\ldots > 0, \tag{2.20}$$

$$\lim_{t\to+\infty} g_5(t) = -\infty, \tag{2.21}$$

$$\begin{aligned}
g_6(t) &= -48(k-1)(2k-3)(3k-2)(3k-4)(3k+2)t^{3k-6} + 144(k-2)^2 \\
&\quad \times (2k-3)(3k-2)(3k-4)t^{3k-8} + 12(2-k)(k+1)(k+2) \\
&\quad \times (k+3)(k+4)(3k+2)t^{2k} + 24k^2(k+1)(k+2)(k+3) \\
&\quad \times (3k-2)t^{2k-2} - 8(k-1)(k+1)(k+2)^2(3k-2)(3k-4)t^{2k-4} \\
&\quad + 24(k-1)(k-2)^2(k-4)(k+1)(3k-4)t^{2k-6} + 12(k-1) \\
&\quad \times (k-2)^2(k-3)(10-k)(3k-8)t^{2k-8} + (k+1)(k+2) \\
&\quad \times (k+4)(k+6)(k+8)(7-4k)t^k + 2k(k-1)(k+2)(k+4) \\
&\quad \times (k+6)(4k-1)t^{k-2} - 2(k-1)(k-2)^2(k+2)(k+4) \\
&\quad \times (4k+5)t^{k-4} + 2(k-2)^2(k-3)(k-4)(k+2)(4k-1)t^{k-6} \\
&\quad + (k-2)(k-3)(k-4)^2(k-6)(7-4k)t^{k-8} + 96(k+2)(10-k),
\end{aligned}$$

$$g_6(1) = 4\left(-3348k^4 + 16233k^3 - 30204k^2 + 28092k - 6768\right) = -2933.37\ldots < 0, \tag{2.22}$$

$$\begin{aligned}
g_7(t) &= -144(k-1)(k-2)(2k-3)(3k-2)(3k-4)(3k+2)t^{2k+2} \\
&\quad -144(k-2)^2(2k-3)(3k-2)(3k-4)(8-3k)t^{2k} - 24k(k-2) \\
&\quad \times (k+1)(k+2)(k+3)(k+4)(3k+2)t^{k+8} + 48k^2(k-1)(k+1) \\
&\quad \times (k+2)(k+3)(3k-2)t^{k+6} - 16(k-1)(k-2)(k+1)(k+2)^2 \\
&\quad \times (3k-2)(3k-4)t^{k+4} + 48(k-1)(k-2)^2(3-k)(4-k)(k+1) \\
&\quad \times (3k-4)t^{k+2} - 24(k-1)(k-2)^2(3-k)(4-k)(10-k)(8-3k) \\
&\quad \times t^k - k(k+1)(k+2)(k+4)(k+6)(k+8)(4k-7)t^8 + 2k \\
&\quad \times (k-1)(k-2)(k+2)(k+4)(k+6)(4k-1)t^6 + 2(k-1) \\
&\quad \times (k-2)^2(4-k)(k+2)(k+4)(4k+5)t^4 - 2(k-2)^2(3-k) \\
&\quad \times (4-k)(6-k)(k+2)(4k-1)t^2 + (k-2)(3-k)(k-4)^2 \\
&\quad \times (6-k)(4k-7)(8-k).
\end{aligned} \tag{2.23}$$

From the expression of $g_7(t)$ and Lemma 2.1, we get

$$\begin{aligned}
g_7(t) < \Big[&- 144(k-1)(k-2)(2k-3)(3k-2)(3k-4)(3k+2) + 48(k-1) \\
&\times (k-2)^2(3-k)(4-k)(k+1)(3k-4) + 2k(k-1)(k-2)(k+2) \\
&\times (k+4)(k+6)(4k-1) + 2(k-1)(k-2)^2(4-k)(k+2)(k+4) \\
&\times (4k+5) + (k-2)(3-k)(k-4)^2(6-k)(8-k)(4k-7)\Big] t^{2k+2} \\
&+ k(k+1)(k+2)\Big[-24(k-2)(k+3)(k+4)(3k+2)t^{k+8} + 48k \\
&\qquad \times (k-1)(k+3)(3k-2)t^{k+6} - (k+4)(k+6)(k+8)(4k-7)t^8\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \Big(140k^7 - 9353k^6 + 52543k^5 - 103636k^4 + 51700k^3 + 88448k^2 \\
&\quad -131968k + 54016\Big)t^{2k+2} + k(k+1)(k+2) \\
&\quad \times \Big[-24(k-2)(k+3)(k+4)(3k+2)t^{k+8} + 48k(k-1)(k+3)(3k-2)t^{k+6} \\
&\quad -(k+4)(k+6)(k+8)(4k-7)t^8\Big] \\
&= (-20221.36\ldots)t^{2k+2} + k(k+1)(k+2) \\
&\quad \times \Big[-24(k-2)(k+3)(k+4)(3k+2)t^{k+8} + 48k(k-1)(k+3)(3k-2)t^{k+6} \\
&\quad -(k+4)(k+6)(k+8)(4k-7)t^8\Big] \\
&< 0.
\end{aligned}
\tag{2.24}
$$

From (2.24), we know that $g_6(t)$ is strictly decreasing in $[1,\infty)$. Then (2.22) implies that $g_5(t)$ is strictly decreasing in $[1,\infty)$.

From (2.20) and (2.21) together with the monotonicity of $g_5(t)$, we clearly see that there exists $\lambda_1 \in (1,\infty)$ such that $g_4(t)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,\infty)$.

Inequality (2.17) and (2.18) together with the piecewise monotonicity of $g_4(t)$ imply that there exists $\lambda_2 \in (1,\infty)$ such that $g_3(t)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,\infty)$.

The piecewise monotonicity of $g_3(t)$ together with (2.14) and (2.15) leads to the fact that there exists $\lambda_3 \in (1,\infty)$ such that $g_2(t)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,\infty)$.

From (2.11) and (2.12) together with the piecewise monotonicity of $g_2(t)$, we conclude that there exists $\lambda_4 \in (1,\infty)$ such that $g_1(t)$ is strictly increasing in $[1,\lambda_4]$ and strictly decreasing in $[\lambda_4,\infty)$.

Equations (2.8) and (2.9) together with the piecewise monotonicity of $g_1(t)$ imply that there exists $\lambda_5 \in (1,\infty)$ such that $g(t)$ is strictly increasing in $[1,\lambda_5]$ and strictly decreasing in $[\lambda_5,\infty)$.

Therefore, Lemma 2.2 follows from (2.5) and (2.6) together with the piecewise monotonicity of $g(t)$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Without loss of generality, we assume that $a > b$. We first prove that $T(a,b) < H_{5/2}(a,b)$. Let $t = \sqrt[4]{a/b} > 1$, then from (1.1) and (1.2) we have

$$
\log T(a,b) - \log H_{5/2}(a,b) = \log \frac{t^4-1}{2\arctan\big((t^4-1)/(t^4+1)\big)} - \frac{2}{5}\log\frac{t^{10}+t^5+1}{3}. \tag{3.1}
$$

Let

$$f(t) = \log \frac{t^4 - 1}{2\arctan\left((t^4 - 1)/(t^4 + 1)\right)} - \frac{2}{5}\log \frac{t^{10} + t^5 + 1}{3}. \tag{3.2}$$

Then simple computations lead to

$$\begin{aligned} &\lim_{t\to 1} f(t) = 0, \\ &f'(t) = \frac{2t^3\left(2t^6 + t^5 + t + 2\right)}{(t^4 - 1)\left(t^{10} + t^5 + 1\right)\left(\arctan\left((t^4 - 1)/(t^4 + 1)\right)\right)} f_1(t), \end{aligned} \tag{3.3}$$

where $f_1(t) = \arctan((t^4 - 1)/(t^4 + 1)) - 2(t^4 - 1)(t^{10} + t^5 + 1)/(t^8 + 1)(2t^6 + t^5 + t + 2)$. Note that

$$\begin{aligned} &\lim_{t\to 1} f_1(t) = 0, \\ &f_1'(t) = -\frac{2\left(t^2 + 1\right)(t + 1)^2(t - 1)^4}{(1 + t^8)^2(2t^6 + t^5 + t + 2)^2} f_2(t), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} f_2(t) = {} & t^{18} + 2t^{17} + 4t^{16} + 6t^{15} - 8t^{12} + 6t^{11} + 21t^{10} + 28t^9 \\ & + 21t^8 + 6t^7 - 8t^6 + 6t^3 + 4t^2 + 2t + 1 > 0 \end{aligned} \tag{3.5}$$

for $t > 1$.

Therefore, $T(a,b) < H_{5/2}(a,b)$ follows from (3.1)–(3.5).

Next, we prove that $T(a,b) > H_{\log 3/\log(\pi/2)}(a,b)$. Let $k = \log 3/\log(\pi/2) = 2.43\ldots$ and $t = \sqrt{a/b} > 1$, then (1.1) and (1.2) lead to

$$\log T(a,b) - \log H_k(a,b) = \log \frac{t^2 - 1}{2\arctan((t^2 - 1)/(t^2 + 1))} - \frac{1}{k}\log \frac{t^{2k} + t^k + 1}{3}. \tag{3.6}$$

Let

$$F(t) = \log \frac{t^2 - 1}{2\arctan((t^2 - 1)/(t^2 + 1))} - \frac{1}{k}\log \frac{t^{2k} + t^k + 1}{3}. \tag{3.7}$$

Then simple computations lead to

$$\lim_{t\to 1} F(t) = \lim_{t\to +\infty} F(t) = 0, \tag{3.8}$$

$$F'(t) = \frac{2t^{2k-1} + t^{k+1} + t^{k-1} + 2t}{(t^2 - 1)\left(t^{2k} + t^k + 1\right)\arctan((t^2 - 1)/(t^2 + 1))} F_1(t), \tag{3.9}$$

where $F_1(t) = \arctan((t^2-1)/(t^2+1)) - 2(t^2-1)(t^{2k}+t^k+1)/(t^4+1)(2t^{2k-2}+t^k+t^{k-2}+2)$. Note that

$$\lim_{t\to 1} F_1(t) = 0, \tag{3.10}$$

$$\lim_{t\to +\infty} F_1(t) = \frac{\pi}{4} - 1 < 0, \tag{3.11}$$

$$F'(t) = \frac{2t^3}{(t^4+1)^2(2t^{2k-2}+t^k+t^{k-2}+2)^2} F_2(t), \tag{3.12}$$

where

$$\begin{aligned} F_2(t) = {} & -8t^{4k-4} + 8t^{4k-6} + (2-k)t^{3k+2} + 2kt^{3k} - 2(k+2)t^{3k-2} \\ & + 2(k-4)t^{3k-4} + (10-k)t^{3k-6} + (7-4k)t^{2k+2} \\ & + 2(4k-1)t^{2k} - 2(4k+5)t^{2k-2} + 2(4k-1)t^{2k-4} \\ & + (7-4k)t^{2k-6} + (10-k)t^{k+2} + 2(k-4)t^k \\ & - 2(k+2)t^{k-2} + 2kt^{k-4} + (2-k)t^{k-6} + 8t^2 - 8. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) together with Lemma 2.2, we clearly see that there exists $\lambda \in (1,\infty)$ such that $F_1(t)$ is strictly increasing in $[1,\lambda]$ and strictly decreasing in $[\lambda,\infty)$.

Equations (3.9)–(3.11) and the piecewise monotonicity of $F_1(t)$ imply that there exists $\mu \in (1,\infty)$ such that $F(t)$ is strictly increasing in $[1,\mu]$ and strictly decreasing in $[\mu,\infty)$. Then from (3.8) we get

$$F(t) > 0 \tag{3.14}$$

for $t > 1$.

Therefore, $T(a,b) > H_{\log 3/\log(\pi/2)}(a,b)$ follows from (3.6) and (3.7) together with (3.14).

At last, we prove that $H_{\log 3/\log(\pi/2)}(a,b)$ and $H_{5/2}(a,b)$ are the best possible lower and upper power-type Heron mean bounds for the Seiffert mean $T(a,b)$, respectively.

For any $0 < \varepsilon < k = \log 3/\log(\pi/2) = 2.43\ldots$ and $x > 0$, from (1.1) and (1.2), one has

$$[T(1,1+x)]^{5/2-\varepsilon} - [H_{5/2-\varepsilon}(1,1+x)]^{5/2-\varepsilon} = \frac{J(x)}{3(2^{5/2-\varepsilon})(\arctan(x/(x+2)))^{5/2-\varepsilon}}, \tag{3.15}$$

$$\lim_{x\to +\infty} \frac{H_{k+\varepsilon}(x,1)}{T(x,1)} = \frac{\pi}{2}\cdot 3^{-1/(k+\varepsilon)} > \frac{\pi}{2}\cdot 3^{-1/k} = 1, \tag{3.16}$$

where $J(x) = 3x^{5/2-\varepsilon} - [(1+x)^{5/2-\varepsilon} + (1+x)^{5/4-\varepsilon/2} + 1][2\arctan(x/(x+2))]^{5/2-\varepsilon}$.

Let $x \to 0$, making use of Taylor extension, we get

$$J(x) = 3x^{5/2-\varepsilon} - 2^{5/2-\varepsilon}\left[\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} + o\left(x^3\right)\right]^{5/2-\varepsilon} \times \left[3 + \left(\frac{15}{4} - \frac{3}{2}\varepsilon\right)x + \left(\frac{13}{8} - \frac{5}{4}\varepsilon\right)\left(\frac{5}{4} - \frac{\varepsilon}{2}\right)x^2 + o\left(x^2\right)\right] = \frac{1}{8}\varepsilon(5-2\varepsilon)x^{9/2-\varepsilon} + o\left(x^{9/2-\varepsilon}\right). \tag{3.17}$$

Equations (3.15) and (3.17) together with inequality (3.16) imply that for any $0 < \varepsilon < \log 3/\log(\pi/2)$, there exist $\delta = \delta(\varepsilon) > 0$ and $X = X(\varepsilon) > 1$ such that $T(1, 1+x) > H_{5/2-\varepsilon}(1, 1+x)$ for $x \in (0, \delta)$ and $H_{\log 3/\log(\pi/2)+\varepsilon}(1, x) > T(1, x)$ for $x \in (X, +\infty)$. □

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