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Research Article

Almost Sure Central Limit Theorem for a Nonstationary Gaussian Sequence

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Let $\{X_n;\ n \geq 1\}$ be a standardized non-stationary Gaussian sequence, and let denote $S_n = \sum_{k=1}^{n} X_k$, $\sigma_n = \sqrt{\mathrm{Var}(S_n)}$. Under some additional condition, let the constants $\{u_{ni};\ 1 \leq i \leq n, n \geq 1\}$ satisfy $\sum_{i=1}^{n}(1-\Phi(u_{ni})) \to \tau$ as $n \to \infty$ for some $\tau \geq 0$ and $\min_{1\leq i\leq n} u_{ni} \geq c(\log n)^{1/2}$, for some $c > 0$, then, we have $\lim_{n\to\infty}(1/\log n)\sum_{k=1}^{n}(1/k)I\{\cap_{i=1}^{k}(X_i \leq u_{ki}), S_k/\sigma_k \leq x\} = e^{-\tau}\Phi(x)$ almost surely for any $x \in R$, where $I(A)$ is the indicator function of the event A and $\Phi(x)$ stands for the standard normal distribution function.

1. Introduction

When $\{X, X_n; n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables and $S_n = \sum_{k=1}^{n} X_k, n \geq 1,\ M_n = \max_{1\leq k\leq n} X_k$ for $n \geq 1$. If $E(X) = 0,\ \mathrm{Var}(X) = 1$, the so-called almost sure central limit theorem (ASCLT) has the simplest form as follows:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_k}{\sqrt{k}} \leq x\right\} = \Phi(x), \tag{1.1}$$

almost surely for all $x \in R$, where $I(A)$ is the indicator function of the event A and $\Phi(x)$ stands for the standard normal distribution function. This result was first proved independently by Brosamler [1] and Schatte [2] under a stronger moment condition; since then, this type of almost sure version was extended to different directions. For example, Fahrner and Stadtmüller [3] and Cheng et al. [4] extended this almost sure convergence for partial sums to the case of maxima of i.i.d. random variables. Under some natural conditions, they proved as follows:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{M_k - b_k}{a_k} \leq x\right\} = G(x) \quad \text{a.s.} \tag{1.2}$$

for all $x \in R$, where $a_k > 0$ and $b_k \in R$ satisfy

$$P\left(\frac{M_k - b_k}{a_k} \leq x\right) \longrightarrow G(x), \quad \text{as } k \longrightarrow \infty \tag{1.3}$$

for any continuity point x of G.

In a related work, Csáki and Gonchigdanzan [5] investigated the validity of (1.2) for maxima of stationary Gaussian sequences under some mild condition whereas Chen and Lin [6] extended it to non-stationary Gaussian sequences. Recently, Dudziński [7] obtained two-dimensional version for a standardized stationary Gaussian sequence. In this paper, inspired by the above results, we further study ASCLT in the joint version for a non-stationary Gaussian sequence.

2. Main Result

Throughout this paper, let $\{X_n; n \geq 1\}$ be a non-stationary standardized normal sequence, and $\sigma_n = \sqrt{\operatorname{Var}(S_n)}$. Here $a \ll b$ and $a \sim b$ stand for $a = O(b)$ and $a/b \to 1$, respectively. $\Phi(x)$ is the standard normal distribution function, and $\phi(x)$ is its density function; C will denote a positive constant although its value may change from one appearance to the next. Now, we state our main result as follows.

Theorem 2.1. *Let $\{X_n; n \geq 1\}$ be a sequence of non-stationary standardized Gaussian variables with covariance matrix (r_{ij}) such that $0 \leq r_{ij} \leq \rho_{|i-j|}$ for $i \neq j$, where $\rho_n \leq 1$ for all $n \geq 1$ and $\sup_{s\geq n} \sum_{i=s-n}^{s-1} \rho_i \ll (\log n)^{1/2}/(\log\log n)^{1+\varepsilon}$, $\varepsilon > 0$. If the constants $\{u_{ni}; 1 \leq i \leq n, n \geq 1\}$ satisfy $\sum_{i=1}^{n}(1-\Phi(u_{ni})) \to \tau$ as $n \to \infty$ for some $\tau \geq 0$ and $\min_{1\leq i\leq n} u_{ni} \geq c(\log n)^{1/2}$, for some $c > 0$, then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\bigcap_{i=1}^{k} (X_i \leq u_{ki}), \frac{S_k}{\sigma_k} \leq x\right\} = e^{-\tau}\Phi(x), \tag{2.1}$$

almost surely for any $x \in R$.

Remark 2.2. The condition $\sup_{s\geq n} \sum_{i=s-n}^{s-1} \rho_i \ll (\log n)^{1/2}/(\log\log n)^{1+\varepsilon}, \varepsilon > 0$ is inspired by (a1) in Dudziński [8], which is much more weaker.

3. Proof

First, we introduce the following lemmas which will be used to prove our main result.

Lemma 3.1. *Under the assumptions of Theorem 2.1, one has*

$$\sum_{1\leq i<j\leq n} r_{ij} \exp\left(-\frac{u_{ni}^2 + u_{nj}^2}{2(1+r_{ij})}\right) \leq \frac{C}{(\log\log n)^{1+\varepsilon}}. \tag{3.1}$$

Proof. This lemma comes from Chen and Lin [6]. □

The following lemma is Theorem 2.1 and Corollary 2.1 in Li and Shao [9].

Lemma 3.2. *(1) Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of standard Gaussian variables with covariance matrices $R^1 = (r_{ij}^1)$ and $R^0 = (r_{ij}^0)$, respectively. Put $\rho_{ij} = \max(|r_{ij}^1|, |r_{ij}^0|)$. Then one has*

$$\begin{aligned} &P\left(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\right) - P\left(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\right) \\ &\quad \le \frac{1}{2\pi}\sum_{1\le i<j\le n}\left(\arcsin\left(r_{ij}^1\right) - \arcsin\left(r_{ij}^0\right)\right)^+ \exp\left(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\right), \end{aligned} \tag{3.2}$$

for any real numbers u_i, $i = 1, 2, \ldots, n$.

(2) Let $\{\xi_n; n \ge 1\}$ be standard Gaussian variables with $r_{ij} = Cov(\xi_i, \xi_j)$. Then

$$\left|P\left(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\right) - \prod_{j=1}^{n}P(\xi_j \le u_j)\right| \le \frac{1}{4}\sum_{1\le i<j\le n}|r_{ij}|\exp\left(-\frac{u_i^2+u_j^2}{2(1+|r_{ij}|)}\right), \tag{3.3}$$

for any real numbers u_i, $i = 1, 2, \ldots, n$.

Lemma 3.3. *Let $\{X_n\}$ be a sequence of standard Gaussian variables and satisfy the conditions of Theorem 2.1, then for $1 \le k < n$, one has*

$$P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) - P\left(\bigcap_{i=1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) \le \frac{k}{n} + \frac{C}{(\log\log n)^{1+\varepsilon}} \tag{3.4}$$

for any $y \in R$.

Proof. By the conditions of Theorem 2.1, we have

$$\sigma_n = \sqrt{n + 2\sum_{1\le i<j\le n} r_{ij}} \ge \sqrt{n}, \tag{3.5}$$

then, for $1 \le i \le n$, by $\sup_{s\ge n}\sum_{i=s-n}^{s-1}\rho_i \ll (\log n)^{1/2}/(\log\log n)^{1+\varepsilon}$, $\varepsilon > 0$, it follows that

$$\operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right) \le \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\rho_k \ll \frac{(\log n)^{1/2}}{\sqrt{n}(\log\log n)^{1+\varepsilon}}. \tag{3.6}$$

Then, there exist numbers δ, n_0, such that, for any $n > n_0$, we have

$$\sup_{1\le i\le n}\operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right) < \delta < \frac{1}{2}. \tag{3.7}$$

We can write that

$$
\begin{aligned}
L &:= P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) - P\left(\bigcap_{i=1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) \\
&\le \left|P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) - P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}\right)P(Y_n \le y)\right| \\
&\quad + \left|P\left(\bigcap_{i=1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right) - P\left(\bigcap_{i=1}^{n}\{X_i \le u_{ni}\}\right)P(Y_n \le y)\right| \\
&\quad + \left(P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}\right) - P\left(\bigcap_{i=1}^{n}\{X_i \le u_{ni}\}\right)\right) \\
&=: L_1 + L_2 + L_3,
\end{aligned}
\tag{3.8}
$$

where $\{Y_n\}$ is a random variable, which has the same distribution as $\{S_n/\sigma_n\}$, but it is independent of $(X_1, X_2, \dots, X_n)$. For L_1, L_2, apply Lemma 3.2 (1) with $(\xi_i = X_i,\ i = 1, \dots, n; \xi_{n+1} = S_n/\sigma_n)$, $(\eta_j = X_j,\ j = 1, \dots, n;\ \eta_{n+1} = Y_n)$. Then $r_{ij}^1 = r_{ij}^0 = r_{ij}$ for $1 \le i < j \le n$ and $r_{ij}^1 = \operatorname{Cov}(X_i, S_n/\sigma_n), r_{ij}^0 = 0$ for $1 \le i \le n, j = n+1$. Thus, we have (for $i = 1, 2$)

$$
L_i \ll \sum_{i=1}^{n} \operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right) \exp\left(-\frac{u_{ni}^2 + y^2}{2(1 + \operatorname{Cov}(X_i, S_n/\sigma_n))}\right).
\tag{3.9}
$$

Since (3.5), (3.7) hold, we obtain

$$
L_i \ll \frac{(\log n)^{1/2}}{\sqrt{n}(\log\log n)^{1+\varepsilon}} \sum_{i=1}^{n} \exp\left(-\frac{u_{ni}^2}{2(1+\delta)}\right).
\tag{3.10}
$$

Now define u_n by $1 - \Phi(u_n) = 1/n$. By the well-known fact

$$
1 - \Phi(x) \sim \frac{\phi(x)}{x}, \quad x \longrightarrow \infty,
\tag{3.11}
$$

it is easy to see that

$$
\exp\left(-\frac{u_n^2}{2}\right) \sim \frac{\sqrt{2\pi}u_n}{n}, \qquad u_n \sim \sqrt{2\log n}.
\tag{3.12}
$$

Thus, according to the assumption $\min_{1\le i\le n} u_{ni} \ge c(\log n)^{1/2}$, we have $u_{ni} \ge c u_n$ for some $c > 0$. Hence

$$\begin{aligned}
L_i &\le \frac{(\log n)^{1/2}}{\sqrt{n}(\log\log n)^{1+\varepsilon}} \sum_{1\le i\le n} \exp\left(-\frac{u_{ni}^2}{2(1+\delta)}\right) \\
&\le \frac{\sqrt{n}(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} \exp\left(-\frac{u_n^2}{2(1+\delta)}\right) \\
&\ll \frac{\sqrt{n}\left(\sqrt{2\log n}\right)^{(2+\delta)/(1+\delta)}}{n^{1/(1+\delta)}(\log\log n)^{1+\varepsilon}} \\
&\ll \frac{\left(\sqrt{\log n}\right)^{(2+\delta)/(1+\delta)}}{n^{1/(1+\delta)-(1/2)}} \\
&\ll \frac{1}{n^{\delta'}}, \qquad \delta' > 0.
\end{aligned} \tag{3.13}$$

Now, we are in a position to estimate L_3. Observe that

$$\begin{aligned}
L_3 &= P\left(\bigcap_{i=k+1}^{n} \{X_i \le u_{ni}\}\right) - P\left(\bigcap_{i=1}^{n} \{X_i \le u_{ni}\}\right) \\
&\le \left| P\left(\bigcap_{i=k+1}^{n} \{X_i \le u_{ni}\}\right) - \prod_{i=k+1}^{n} \Phi(u_{ni}) \right| + \left| P\left(\bigcap_{i=1}^{n} \{X_i \le u_{ni}\}\right) - \prod_{i=1}^{n} \Phi(u_{ni}) \right| \\
&\quad + \left| \prod_{i=k+1}^{n} \Phi(u_{ni}) - \prod_{i=1}^{n} \Phi(u_{ni}) \right| \\
&=: L_{31} + L_{32} + L_{33}.
\end{aligned} \tag{3.14}$$

For L_{33}, it follows that

$$\begin{aligned}
L_{33} &= \prod_{i=k+1}^{n} \Phi(u_{ni}) \left(1 - \prod_{i=1}^{k} \Phi(u_{ni})\right) \\
&\ll 1 - \Phi^k(u_n) \\
&= 1 - \left(1 - \frac{1}{n}\right)^k \le \frac{k}{n}.
\end{aligned} \tag{3.15}$$

By Lemma 3.2 (2), we have

$$L_{3i} \le \frac{1}{4} \sum_{1\le i<j\le n} r_{ij} \exp\left(-\frac{u_{ni}^2 + u_{nj}^2}{2(1+r_{ij})}\right), \quad i = 1, 2. \tag{3.16}$$

Thus by Lemma 3.1 we obtain the desired result. □

Lemma 3.4. *Let $\{X_n\}$ be a sequence of standard Gaussian variables satisfying the conditions of Theorem 2.1, then for $1 \le k < n$, any $y \in R$, one has*

$$\begin{aligned}&\left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le y\right), I\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right)\right)\right| \\ &\quad \ll \sqrt{\frac{k}{n}}\frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} + \frac{1}{(\log\log n)^{1+\varepsilon}}.\end{aligned} \tag{3.17}$$

Proof. Apply Lemma 3.2 (1) with ($\xi_i = X_i,\ 1 \le i \le k,\ \xi_{k+1} = S_k/\sigma_k,\ \xi_{i+1} = X_i,\ k+1 \le i \le n,\ \xi_{n+2} = S_n/\sigma_n$), ($\eta_j = \xi_j,\ 1 \le j \le k+1,\ \eta_j = \overline{\xi}_j,\ k+2 \le j \le n+2$), where $(\overline{\xi}_{k+2}, \ldots, \overline{\xi}_{n+2})$ has the same distribution as $(\xi_{k+2}, \ldots, \xi_{n+2})$, but it is independent of $(\xi_{k+2}, \ldots, \xi_{n+2})$. Then,

$$\begin{aligned}&r_{ij}^1 = r_{ij}^0 \quad \text{for } 1 \le i < j \le k+1 \quad \text{or} \quad k+2 \le i < j \le n+2; \\ &r_{ij}^1 = r_{i(j-1)}, \quad r_{ij}^0 = 0 \quad \text{for } 1 \le i \le k,\ k+2 \le j \le n+1; \\ &r_{ij}^1 = \operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right), \quad r_{ij}^0 = 0 \text{ for } 1 \le i \le k,\ j = n+2; \\ &r_{ij}^1 = \operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right), \quad r_{ij}^0 = 0 \quad \text{for } k+1 \le i \le n,\ j = k+1; \\ &r_{ij}^1 = \operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_n}{\sigma_n}\right), \quad r_{ij}^0 = 0 \quad \text{for } i = k+1,\ j = n+2.\end{aligned} \tag{3.18}$$

Thus, combined with (3.5), (3.7), it follows that

$$\begin{aligned}&\left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le y\right), I\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right)\right)\right| \\ &\quad = \left|P\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_k}{\sigma_k} \le y, \frac{S_n}{\sigma_n} \le y\right)\right. \\ &\qquad \left. - P\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le y\right) P\left(\bigcap_{i=k+1}^{n}\{X_i \le u_{ni}\}, \frac{S_n}{\sigma_n} \le y\right)\right| \\ &\quad \le \frac{1}{4}\sum_{1\le i\le k}\sum_{k+1\le j\le n} r_{ij}\exp\left(-\frac{u_{ki}^2 + u_{nj}^2}{2(1+r_{ij})}\right) + \frac{1}{4}\sum_{i=1}^{k}\operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right)\exp\left(-\frac{u_{ki}^2 + y^2}{2(1+Cov(X_i, S_n/\sigma_n))}\right) \\ &\qquad + \frac{1}{4}\sum_{i=k+1}^{n}\operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\exp\left(-\frac{u_{ni}^2 + y^2}{2(1+Cov(X_i, S_k/\sigma_k))}\right) + \frac{1}{4}\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_n}{\sigma_n}\right) \\ &\quad \le \frac{1}{4}\sum_{1\le i\le k}\sum_{k+1\le j\le n} r_{ij}\exp\left(-\frac{u_{ki}^2 + u_{nj}^2}{2(1+r_{ij})}\right) + \frac{1}{4}\sum_{i=1}^{k}\operatorname{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right)\exp\left(-\frac{u_{ki}^2}{2(1+\delta)}\right) \\ &\qquad + \frac{1}{4}\sum_{i=k+1}^{n}\operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\exp\left(-\frac{u_{ni}^2}{2(1+\delta)}\right) + \frac{1}{4}\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_n}{\sigma_n}\right) \\ &\quad =: T_1 + T_2 + T_3 + T_4.\end{aligned} \tag{3.19}$$

Using Lemma 3.1, we have

$$T_1 \leq \frac{C}{(\log\log n)^{1+\varepsilon}}, \quad \varepsilon > 0. \tag{3.20}$$

By the similar technique that was applied to prove (3.10), we obtain

$$T_2 \ll \frac{1}{n^{\alpha}}, \quad \alpha > 0. \tag{3.21}$$

For T_3, by $\sup_{s\geq n}\sum_{i=s-n}^{s-1}\rho_i \ll (\log n)^{1/2}/(\log\log n)^{1+\varepsilon}, \varepsilon > 0$, and (3.12), we have

$$\begin{aligned}
T_3 &\ll \exp\left(-\frac{u_n^2}{2(1+\delta)}\right)\sum_{i=k+1}^{n}\mathrm{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right) \\
&\ll \frac{1}{n^{1/(1+\delta)}}\sum_{i=k+1}^{n}\mathrm{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right) \\
&\ll \frac{1}{n^{1/(1+\delta)}}\frac{1}{\sqrt{k}}\sum_{i=k+1}^{n}\mathrm{Cov}(X_i, S_k) \\
&\ll \frac{1}{n^{1/(1+\delta)}}\frac{1}{\sqrt{k}}\sum_{j=1}^{k}\sum_{i=k+1}^{n}\mathrm{Cov}(X_i, X_j) \\
&\ll \frac{1}{n^{1/(1+\delta)}}\frac{1}{\sqrt{k}}\sum_{j=1}^{k}\sum_{i=1}^{n}\rho_i \\
&\ll \frac{\sqrt{k}}{n^{1/(1+\delta)}}\frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} \\
&\ll \frac{1}{n^{\beta}}, \quad \beta > 0.
\end{aligned} \tag{3.22}$$

As to T_4, by (3.5) and (3.6), we have

$$T_4 \ll \frac{1}{\sigma_k}\sum_{i=1}^{k}\mathrm{Cov}\left(X_i, \frac{S_n}{\sigma_n}\right) \ll \sqrt{\frac{k}{n}}\frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}}. \tag{3.23}$$

Thus the proof of this lemma is completed. □

Proof of Theorem 2.1. First, by assumptions and Theorem 6.1.3 in Leadbetter et al. [10], we have

$$P\left\{\bigcap_{i=1}^{n}(X_i \leq u_{ni})\right\} \longrightarrow e^{-\tau}. \tag{3.24}$$

Let Y_n denote a random variable which has the same distribution as S_n/σ_n, but it is independent of $(X_1, X_2, \ldots, X_n)$, then by (3.10), we derive

$$P\left\{\bigcap_{i=1}^{n}(X_i \le u_{ni}), \frac{S_n}{\sigma_n} \le y\right\} - P\left\{\bigcap_{i=1}^{n}(X_i \le u_{ni})\right\} P\{Y_n \le y\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

Thus, by the standard normal property of Y_n, we have

$$\lim_{n\to\infty} P\left\{\bigcap_{i=1}^{n}(X_i \le u_{ni}), \frac{S_n}{\sigma_n} \le y\right\} = e^{-\tau}\Phi(y), \quad y \in R. \tag{3.26}$$

Hence, to complete the proof, it is sufficient to show

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left(I\left\{\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le x\right\} - P\left\{\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le x\right\}\right) = 0 \quad \text{a.s.} \tag{3.27}$$

In order to show this, by Lemma 3.1 in Csáki and Gonchigdanzan [5], we only need to prove

$$\operatorname{Var}\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k} I\left\{\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le x\right\}\right) \ll \frac{1}{(\log\log n)^{1+\varepsilon}}, \tag{3.28}$$

for $\varepsilon > 0$ and any $x \in R$. Let $\eta_k = I\{\bigcap_{i=1}^{k}(X_i \le u_{ki}), S_k/\sigma_k \le x\} - P\{\bigcap_{i=1}^{k}(X_i \le u_{ki}),\ S_k/\sigma_k \le x\}$. Then

$$\begin{aligned}
\operatorname{Var}\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k} I\left\{\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le x\right\}\right) &= E\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\eta_k\right)^2 \\
&= \frac{1}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k^2}E|\eta_k|^2 + \frac{2}{\log^2 n}\sum_{1\le k<l\le n}\frac{|E(\eta_k\eta_l)|}{kl} \\
&=: S_1 + S_2.
\end{aligned} \tag{3.29}$$

Since $|\eta_k| \le 2$, it follows that

$$S_1 \ll \frac{1}{\log^2 n}. \tag{3.30}$$

Now, we turn to estimate S_2. Observe that for $l > k$

$$\begin{aligned}
\left|E(\eta_k\eta_l)\right| &= \left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le x\right), I\left(\bigcap_{i=1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right)\right| \\
&\le \left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le x\right), I\left(\bigcap_{i=1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right.\right. \\
&\qquad \left.\left. - I\left(\bigcap_{i=k+1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right)\right| \\
&\quad + \left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le x\right), I\left(\bigcap_{i=k+1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right)\right| \\
&\le E\left|I\left(\bigcap_{i=1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right) - I\left(\bigcap_{i=k+1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right| \\
&\quad + \left|\operatorname{Cov}\left(I\left(\bigcap_{i=1}^{k}\{X_i \le u_{ki}\}, \frac{S_k}{\sigma_k} \le x\right), I\left(\bigcap_{i=k+1}^{l}\{X_i \le u_{li}\}, \frac{S_l}{\sigma_l} \le x\right)\right)\right| \\
&=: S_{21} + S_{22}.
\end{aligned} \tag{3.31}$$

By Lemma 3.3, we have

$$S_{21} \le \frac{k}{l} + \frac{C}{(\log\log l)^{1+\varepsilon}}. \tag{3.32}$$

Using Lemma 3.4, it follows that

$$S_{22} \le \sqrt{\frac{k}{l}}\frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} + \frac{C}{(\log\log l)^{1+\varepsilon}}. \tag{3.33}$$

Hence for $l > k$, we have

$$\left|E(\eta_k\eta_l)\right| \le \frac{k}{l} + \frac{C}{(\log\log l)^{1+\varepsilon}} + \sqrt{\frac{k}{l}}\frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}}. \tag{3.34}$$

Consequently

$$
\begin{aligned}
S_2 &\ll \frac{1}{\log^2 n}\left(\sum_{1\le k<l\le n}\frac{1}{kl}\left(\frac{k}{l}+\sqrt{\frac{k}{l}}\frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}}\right)\right)+\sum_{1\le k<l\le n}\frac{1}{kl(\log\log l)^{1+\varepsilon}} \\
&\ll \frac{1}{\log^2 n}\sum_{1\le k<l\le n}\frac{1}{l^2}+\frac{1}{\log^2 n}\frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}}\sum_{l=2}^{n}\frac{1}{l^{3/2}}\sum_{k=1}^{l-1}\frac{1}{\sqrt{k}} \\
&\quad+\frac{1}{\log^2 n}\sum_{l=3}^{n}\frac{1}{l(\log\log l)^{1+\varepsilon}}\sum_{k=1}^{l-1}\frac{1}{k} \\
&\ll \frac{1}{\log n}+\frac{1}{\sqrt{\log n}(\log\log n)^{1+\varepsilon}}+\frac{1}{\log^2 n}\sum_{l=3}^{n}\frac{\log l}{l(\log\log l)^{1+\varepsilon}} \\
&\ll \frac{1}{\log n}+\frac{1}{(\log\log n)^{1+\varepsilon}}.
\end{aligned}
\tag{3.35}
$$

Thus, we complete the proof of (3.28) by (3.30) and (3.35). Further, our main result is proved. □

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