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Research Article

Inequalities for Hyperbolic Functions and Their Applications

L. Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, China

Correspondence should be addressed to L. Zhu, zhuling0571@163.com

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A basic theorem is established and found to be a source of inequalities for hyperbolic functions, such as the ones of Cusa, Huygens, Wilker, Sandor-Bencze, Carlson, Shafer-Fink type inequality, and the one in the form of Oppenheim's problem. Furthermore, these inequalities described above will be extended by this basic theorem.

1. Introduction

In the study by Zhu in [1], a basic theorem is established and found to be a source of inequalities for circular functions, and these inequalities are extended by this basic theorem. In what follows we are going to present the counterpart of these results for the hyperbolic functions.

In this paper, we first establish the following Cusa-type inequalities in exponential type for hyperbolic functions described as Theorem 1.1. Then using the results of Theorem 1.1, we obtain Huygens, Wilker, Sandor-Bencze, Carlson, and Shafer-Fink-type inequalities in Sections 4, 5, 6, 7, 8, respectively.

Theorem 1.1 (Cusa-type inequalities). *Let $x > 0$. Then the following are considered.*

(i) *If $p \geq 4/5$, the double inequality*

$$(1-\lambda)+\lambda(\cosh x)^p < \left(\frac{\sinh x}{x}\right)^p < (1-\eta)+\eta(\cosh x)^p \tag{1.1}$$

holds if and only if $\eta \geq 1/3$ and $\lambda \leq 0$.

(ii) *If $p < 0$, the inequality*

$$\left(\frac{\sinh x}{x}\right)^p < (1-\eta)+\eta(\cosh x)^p \tag{1.2}$$

holds if and only if $\eta \leq 1/3$.

That is, let $\alpha > 0$, then the inequality

$$\left(\frac{x}{\sinh x}\right)^{\alpha} < (1-\eta) + \eta\left(\frac{1}{\cosh x}\right)^{\alpha} \tag{1.3}$$

holds if and only if $\eta \leq 1/3$.

2. Lemmas

Lemma 2.1 (see [2–18]). *Let $f, g : [a,b] \to \mathcal{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g' \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $(f(x)-f(b^-))/(g(x)-g(b^-))$ and $(f(x)-f(a^+))/(g(x)-g(a^+))$ are also increasing (or decreasing) on (a,b).*

Lemma 2.2. *Let $t \in (0,+\infty)$. Then the inequalities*

$$D_1(t) \triangleq \sinh^2 t \cosh t + t\sinh t - 2t^2\cosh t > 0, \tag{2.1}$$

$$D_2(t) \triangleq t^2\cosh t - t\cosh^2 t\sinh t - t\sinh t + \sinh^2 t\cosh t < 0, \tag{2.2}$$

$$D_3(t) \triangleq 9\sinh^2 t\cosh t + t\sinh t - 4t\cosh^2 t\sinh t - 6t^2\cosh t < 0 \tag{2.3}$$

hold.

Proof. Using the infinite series of $\sinh^3 x, \cosh^3 x, \sinh x$, and $\cosh x$, we have

$$\begin{aligned}
D_1(t) &= \frac{1}{4}(\cosh 3t - \cosh t) + t\sinh t - 2t^2\cosh t \\
&= \frac{1}{4}\sum_{n=1}^{\infty}\frac{3^{2n}-1}{(2n)!}t^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}t^{2n+2} - 2\sum_{n=0}^{\infty}\frac{1}{(2n)!}t^{2n+2} \\
&= \sum_{n=0}^{\infty}\left[\frac{3^{2n+2}-1+4(2n+2)}{4(2n+2)!} - \frac{2}{(2n)!}\right]t^{2n+2} \\
&= \sum_{n=2}^{\infty}\left[\frac{3^{2n+2}-1+4(2n+2)}{4(2n+2)!} - \frac{2}{(2n)!}\right]t^{2n+2} > 0,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
D_2(t) &= t^2\cosh t - \frac{t}{4}(\sinh 3t + \sinh t) - t\sinh t + \frac{1}{4}(\cosh 3t - \cosh t) \\
&= \sum_{n=0}^{\infty}\frac{t^{2n+2}}{(2n)!} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n+1}+1}{4(2n+1)!}t^{2n+2} - \sum_{n=0}^{\infty}\frac{t^{2n+2}}{(2n+1)!} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{3^{2n}-1}{(2n)!}t^{2n} \\
&= \sum_{n=0}^{\infty}\frac{(3-6n)3^{2n}+6n^2+14n-3}{4(2n+2)!}t^{2n+2} \\
&= \sum_{n=2}^{\infty}\frac{(3-6n)3^{2n}+6n^2+14n-3}{4(2n+2)!}t^{2n+2} < 0,
\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
D_3(t) &= \frac{9}{4}(\cosh 3t - \cosh t) + t\sinh t - t(\sinh 3t + \sinh t) - 6t^2\cosh t \\
&= \frac{9}{4}\sum_{n=1}^{\infty}\frac{3^{2n}-1}{(2n)!}t^{2n} - \sum_{n=1}^{\infty}\frac{3^{2n+1}}{(2n+1)!}t^{2n+2} - 6\sum_{n=1}^{\infty}\frac{1}{(2n)!}t^{2n+2} \\
&= \sum_{n=0}^{\infty}\frac{(57-24n)3^{2n} - 24(2n+2)(2n+1) - 9}{4(2n+2)!}t^{2n+2} \\
&= \sum_{n=3}^{\infty}\frac{(57-24n)3^{2n} - 24(2n+2)(2n+1) - 9}{4(2n+2)!}t^{2n+2} < 0.
\end{aligned} \tag{2.6}
$$

□

3. Proof of Theorem 1

Let $H(t) = ((\sinh t/t)^p - 1)/((\cosh t)^p - 1) = (f_1(t) - f_1(0^+))/(g_1(t) - g_1(0^+))$, where $f_1(t) = (\sinh t/t)^p$, and $g_1(t) = (\cosh t)^p$. Then

$$
\begin{aligned}
k(t) &\triangleq \frac{f_1'(t)}{g_1'(t)} = \left(\frac{\sinh t}{t\cosh t}\right)^{p-1}\frac{t\cosh t - \sinh t}{t^2\sinh t}, \\
k'(t) &= \left(\frac{\sinh t}{t\cosh t}\right)^{p-1}\frac{u(t)}{t^4\sinh t\cosh^2 t},
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
u(t) &= (p-1)(t - \sinh t\cosh t)(t\cosh t - \sinh t) \\
&\quad + \cosh t\left(2\sinh^2 t - t\sinh t\cosh t - t^2\right) \\
&= (p-1)\left(t^2\cosh t - t\cosh^2 t\sinh t - t\sinh t + \sinh^2 t\cosh t\right) \\
&\quad + 2\sinh^2 t\cosh t - t\cosh^2 t\sinh t - t^2\cosh t \\
&= (p-1)D_2(t) + 2\sinh^2 t\cosh t - t\cosh^2 t\sinh t - t^2\cosh t.
\end{aligned} \tag{3.2}
$$

We obtain results in the following two cases.

(a) When $p \geq 4/5$, by (3.2), (2.2), and (2.3) we have

$$
\begin{aligned}
u(t) &\leq -\frac{1}{5}\left(t^2\cosh t - t\cos h^2 t\sinh t - t\sinh t + \sinh^2 t\cosh t\right) \\
&\quad + 2\sinh^2 t\cosh t - t\cosh^2 t\sinh t - t^2\cosh t \\
&= \frac{1}{5}\left(9\sinh^2 t\cosh t + t\sinh t - 4t\cosh^2 t\sinh t - 6t^2\cosh t\right) \\
&= \frac{1}{5}D_3(t) < 0.
\end{aligned} \tag{3.3}
$$

So $k'(t) < 0$ and $f_1'(t)/g_1'(t)$ is decreasing on $(0,+\infty)$. This leads to that $H(t) = (f_1(t) - f_1(0^+))/(g_1(t) - g_1(0^+))$ is decreasing on $(0,+\infty)$ by Lemma 2.1. At the same time, using power series expansions, we have that $\lim_{t\to 0^+}H(t) = 1/3$, and rewriting $H(t)$ as $((\tanh t/t)^p - (1/\cosh t)^p)/(1-(1/\cosh t)^p)$, we see that $\lim_{t\to+\infty}H(t) = 0$. So the proof of (i) in Theorem 1.1 is complete.

(b) When $p < 0$, by (3.2), (2.2), and (2.1) we obtain

$$\begin{aligned} u(t) &> -\left(t^2\cosh t - t\cosh^2 t\sinh t - t\sinh t + \sinh^2 t\cosh t\right) \\ &\quad + 2\sinh^2 t\cosh t - t\cosh^2 t\sinh t - t^2\cosh t \\ &= \sinh^2 t\cosh t + t\sinh t - 2t^2\cosh t = D_1(t) > 0. \end{aligned} \tag{3.4}$$

So $k'(t) > 0$ and $(f_1'(t)/g_1'(t))$ is increasing on $(0,+\infty)$ and the function $H(x)$ is increasing on $(0,+\infty)$ by Lemma 2.1. At the same time, $\lim_{x\to 0^+}H(x) = 1/3$, but $\lim_{x\to(\pi/2)^-}H(x) = +\infty$. So the proof of (ii) in Theorem 1.1 is complete.

4. Huygens-Type Inequalities

Multiplying three functions by $(x/\sinh x)^p$ showed in (1.1) and (1.2), we can obtain the following results on Huygens-type inequalities for the hyperbolic functions.

Theorem 4.1. *Let $x > 0$. Then one has the following.*

(1) *When $p \geq 4/5$, the double inequality*

$$(1-\lambda)\left(\frac{x}{\sinh x}\right)^p + \lambda\left(\frac{x}{\tanh x}\right)^p < 1 < (1-\eta)\left(\frac{x}{\sinh x}\right)^p + \eta\left(\frac{x}{\tanh x}\right)^p \tag{4.1}$$

holds if and only if $\eta \geq 1/3$ and $\lambda \leq 0$.

(2) *When $p < 0$, the inequality*

$$(1-\eta)\left(\frac{x}{\sinh x}\right)^p + \eta\left(\frac{x}{\tanh x}\right)^p > 1 \tag{4.2}$$

holds if and only if $\eta \leq 1/3$.

Let $p = -\alpha, \alpha > 0$, then inequality (4.2) is equivalent to

$$(1-\eta)\left(\frac{\sinh x}{x}\right)^\alpha + \eta\left(\frac{\tanh x}{x}\right)^\alpha > 1 \tag{4.3}$$

and holds if and only if $\eta \leq 1/3$.

When letting $p = 1$ in (4.1) and $\alpha = 1$ in (4.3), one can obtain two results of Zhu [19].

Corollary 4.2 (see [19, Theorem 4]). *One has that*

$$(1-\lambda)\frac{x}{\sinh x}+\lambda\frac{x}{\tanh x}<1<(1-\eta)\frac{x}{\sinh x}+\eta\frac{x}{\tanh x} \tag{4.4}$$

holds for all $x \in (0,+\infty)$ if and only if $\eta \geq 1/3$ and $\lambda \leq 0$.

Corollary 4.3 (see [19, Theorem 2]). *One has that*

$$(1-\eta)\frac{\sinh x}{x}+\eta\frac{\tanh x}{x}>1 \tag{4.5}$$

holds for all $x \in (0,+\infty)$ if and only if $\eta \leq 1/3$.

When letting $\eta = 1/3$ in (4.4), one can obtain a result on Cusa-type inequality (see the study by Baricz and Zhu in [20]).

Corollary 4.4 (see [20, Theorem 1.3]). *One has that*

$$\frac{\sinh x}{x}<\frac{2}{3}+\frac{1}{3}\cosh x \tag{4.6}$$

or

$$\frac{3\sinh x}{2+\cosh x}<x \tag{4.7}$$

that is,

$$2\frac{x}{\sinh x}+\frac{x}{\tanh x}>3 \tag{4.8}$$

holds for all $x \in (0,+\infty)$.

Inequality (4.6) can deduce to the following one which is from the study by Baricz in [21]:

$$\frac{\sinh x}{x}<\frac{1}{2}+\frac{1}{2}\cosh x. \tag{4.9}$$

When letting $\eta = 1/3$ in (4.5), one can obtain a new result on Huygens-type inequality.

Corollary 4.5. *One has that*

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3 \tag{4.10}$$

holds for all $x \in (0,+\infty\ )$ if and only if $\eta \leq 1/3$.

Remark 4.6. Attention is drawn to the fact that, comparing Cusa-type inequality with Huygens-type inequality, Neuman and Sandor [21] obtained the following result:

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>2\frac{x}{\sinh x}+\frac{x}{\tanh x}>3. \tag{4.11}$$

5. Wilker-Type Inequalities

In this section, we obtain the following results on Wilker-type inequalities.

Theorem 5.1. *Let $x>0$. Then the following are considered.*

(i) *When $\alpha>0$, the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha}+\left(\frac{\tanh x}{x}\right)^{\alpha}>\left(\frac{x}{\sinh x}\right)^{2\alpha}+\left(\frac{x}{\tanh x}\right)^{\alpha} \tag{5.1}$$

holds.

(ii) *When $\alpha\geq 4/5$, then the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha}+\left(\frac{\tanh x}{x}\right)^{\alpha}>\left(\frac{x}{\sinh x}\right)^{2\alpha}+\left(\frac{x}{\tanh x}\right)^{\alpha}>2 \tag{5.2}$$

holds.

Proof. (i) The proof of (i) can be seen in [22, 23].

(ii) When $\alpha\geq 4/5$, we can obtain

$$1+\left(\frac{x}{\sinh x}\right)^{2\alpha}+\left(\frac{x}{\tanh x}\right)^{\alpha}\geq 2\left(\frac{x}{\sinh x}\right)^{\alpha}+\left(\frac{x}{\tan x}\right)^{\alpha}>3, \tag{5.3}$$

by the arithmetic mean-geometric mean inequality and the right of inequality (4.1). By (5.1), we have (5.2). □

One can obtain the following three results from Theorem 5.1.

Corollary 5.2 (First Wilker-type inequality, see [24]). *One has that*

$$\left(\frac{\sinh x}{x}\right)^{2}+\frac{\tanh x}{x}>2 \tag{5.4}$$

holds for all $x\in(0,+\infty)$.

Corollary 5.3 (Second Wilker-type inequality). *One has that*

$$\left(\frac{x}{\sinh x}\right)^{2}+\frac{x}{\tanh x}>2 \tag{5.5}$$

holds for all $x\in(0,+\infty)$.

Corollary 5.4. *One has that*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2 \tag{5.6}$$

holds for all $x \in (0, +\infty)$.

Remark 5.5. Inequality (5.2) is a generalization of a result of Zhu [22] since (5.2) holds for $\alpha \geq 4/5$ while it holds for $\alpha \geq 1$ in [22].

6. Sandor-Bencze-Type Inequalities

From Theorem 1.1, we can obtain some results on Sandor-Bencze-type inequalities (Sandor-Bencze inequalities for circular functions can be found in [25]).

Theorem 6.1. *Let* $x > 0$. *Then the following are considered.*

(1) *When* $\alpha \geq 4/5$, *one has*

$$\left(\frac{\sinh x}{x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}(\cosh x)^\alpha < \frac{(\cosh x)^\alpha + \sqrt{(\cosh x)^{2\alpha} + 8}}{4}. \tag{6.1}$$

(2) *When* $\alpha > 0$, *one has*

$$\left(\frac{x}{\sinh x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}\left(\frac{1}{\cosh x}\right)^\alpha < \frac{1 + \sqrt{8(\cosh x)^{2\alpha} + 1}}{4(\cosh x)^\alpha} < \left(\frac{1}{\cosh x}\right)^\alpha + 1. \tag{6.2}$$

7. Carlson-Type Inequalities

Let $\cosh^{-1} x = t$ for $x > 1$, then $x = \cosh t$ for $t > 0$, and

$$\sqrt{1+x} = \sqrt{2}\cosh\frac{t}{2}, \qquad \sqrt{x-1} = \sqrt{2}\sinh\frac{t}{2}. \tag{7.1}$$

Replacing x with $t/2$ and letting $\eta = 1/3$ in Theorem 1.1, we have the following.

Theorem 7.1. *Let* $x > 1$. *Then the following are considered.*

(1) *When* $p \geq 4/5$, *the double inequality*

$$\frac{3\left(2\sqrt{x-1}\right)^p}{\left(2\sqrt{2}\right)^p + \left(\sqrt{1+x}\right)^p} < \left(\cosh^{-1} x\right)^p < \frac{\left(4^{1/3}\sqrt{x-1}\right)^p}{(1+x)^{p/6}} \tag{7.2}$$

holds.

(2) *When* $p < 0$, *the left inequality of* (7.2) *holds too.*

When letting $p = 1$ in Theorem 7.1, one can obtain the following result.

Corollary 7.2. *Let $x > 1$. Then the double inequality*

$$\frac{6(x-1)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} < \cosh^{-1} x < \frac{4^{1/3}(x-1)^{1/2}}{(1+x)^{1/6}} \tag{7.3}$$

holds.

8. Shafer-Fink-Type Inequalities and an Extension of the Problem of Oppenheim

First, let $\sinh x = t$ and $\eta = 1/3$ in Theorem 1.1, then $t > 0, x = \sinh^{-1} t, \cosh x = \sqrt{1+t^2}$, and we have the following.

Theorem 8.1. *Let $t > 0$, $p \geq 4/5$ or $p < 0$. Then the inequality*

$$\frac{3t^p}{2 + \left(\sqrt{1+t^2}\right)^p} < \left(\sinh^{-1} t\right)^p \tag{8.1}$$

holds.

Theorem 8.1 can deduce to the following result.

Corollary 8.2 (see [26]). *Let $x > 0$. Then*

$$\frac{3x}{2 + \sqrt{1+x^2}} < \sinh^{-1} x. \tag{8.2}$$

Second, let $x = u/2$ for $x > 0$ and $\eta = 1/3$ in Theorem 1.1. Then let $t = \sinh u$ or $\sinh^{-1} t = u$. Since $(\sqrt{1+x^2} - 1)^{1/2} = \sqrt{2}\sinh(u/2) = \sqrt{2}\sinh x$ and $(\sqrt{1+x^2} + 1)^{1/2} = \sqrt{2}\cosh(u/2) = \sqrt{2}\cosh x$, one obtains the following result.

Theorem 8.3. *Let $t > 0$, $p \geq 4/5$ or $p < 0$. Then the inequality*

$$\frac{6 \cdot \left(\sqrt{2}\right)^p \left(\sqrt{1+t^2} - 1\right)^{p/2}}{4 + \left(\sqrt{2}\right)^{2-p} \left(\sqrt{1+t^2} + 1\right)^{p/2}} < \left(\sinh^{-1} t\right)^p \tag{8.3}$$

holds.

Theorem 8.3 can deduce to the following result.

Corollary 8.4 (see [26]). *Let $x > 0$. Then*

$$\frac{3x}{2+\sqrt{1+x^2}} < \frac{6\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{4+\sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} < \sinh^{-1}x. \tag{8.4}$$

Finally, Theorem 1.1 is equivalent to the following statement which modifies a problem of Oppenheim (a problem of Oppenheim for circular functions can be found in [20, 27–29]).

Theorem 8.5. *Let $x > 0, p \geq 4/5$ or $p < 0$. Then the inequality*

$$\frac{3}{2}\frac{\sinh^p x}{1+(1/2)\cosh^p x} < x^p \tag{8.5}$$

holds.

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