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Research Article

More About Hermite-Hadamard Inequalities, Cauchy's Means, and Superquadracity

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New results associated with Hermite-Hadamard inequalities for superquadratic functions are given. A set of Cauchy's type means is derived from these Hermite-Hadamard-type inequalities, and its log-convexity and monotonicity are proved.

1. Introduction

The following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is holding for any convex function, that is, well known in the literature as the Hermite-Hadamard inequality (see [1, page 137]). In many areas of analysis applications of Hermite-Hadamard inequality appear for different classes of functions with and without weights; see for convex functions, for example, [2, 3]. Also some useful mappings are defined connected to this inequality see in [4–6]. Here we focus on a class of functions which are superquadratic and analogs and refinements of (1.1) are applied to obtain results useful in analysis.

Now we present definitions, theorems, and results that we use in this paper.

The following definition is given in [7].

Definition A. A function $\varphi : [0,\infty) \to R$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C(x)$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C(x)(y-x) \tag{1.2}$$

for all $y \geq 0$. One says that φ is subquadratic if $-\varphi$ is a superquadratic function.

The followings theorem is given in [8] and is used in our main results:

Theorem 1.1. *Let $\varphi : [0,\infty) \to R$ be an integrable superquadratic function; then for $0 \leq a < b$ one has*

$$\varphi\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right)dx \leq \frac{1}{b-a}\int_a^b \varphi(x)dx, \tag{1.3}$$

$$\frac{1}{b-a}\int_a^b \varphi(x)dx \leq \frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx. \tag{1.4}$$

Definition A_1 (see [9, Definition 1]). A function $h : (a,b) \to R$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h(x_i + x_j) \geq 0 \tag{1.5}$$

for all $n \in \mathbb{N}$ and all choices $u_i \in R,\ i = 1,2,\ldots,n$ and $x_i \in (a,b)$, such that $x_i + x_j \in (a,b),\ 1 \leq i,j \leq n$.

Proposition 1.2 (see [9, Proposition 1]). *Let $h : (a,b) \to R$. The following are equivalent:*

(i) *h is exponentially convex,*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} u_i u_j h\left(\frac{x_i + x_j}{2}\right) \geq 0 \tag{1.6}$$

for every $u_i \in R$ and every $x_i, x_j \in (a,b), 1 \leq i,\ j \leq n$,

(iii) *h is continuous and*

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0, \quad 1 \leq m \leq n, \tag{1.7}$$

for every $x_i \in (a,b),\ i = 1,2,\ldots,n$.

Corollary 1.3 (see [8, 9]). *If $h : (a,b) \to (0,\infty)$ is exponentially convex function, then h is a log-convex function:*

$$h\left(\frac{x+y}{2}\right) \leq \sqrt{h(x)h(y)} \tag{1.8}$$

for all $x, y \in (a,b)$.

Remark 1.4. In Definition A_1 and Proposition 1.2 it is sufficient to require measurability and finiteness almost every where in place of continuity because of the following theorem (see [10, page 105, Theorem 9.1b] and [11]): if the function $h : (a,b) \to R$ is measurable and finite almost everywhere and if in addition

$$\begin{gathered} -\infty < h(x) \leq \infty, \\ h\left(\frac{x+y}{2}\right) \leq \frac{h(x)+h(y)}{2} \quad (a < x, y < b), \end{gathered} \tag{1.9}$$

then h is continuous function.

The next two sections are about mean value theorems, positive semidefiniteness, exponential convexity, log-convexity, Cauchy means, and their monotonicity, that are associated with Hermite-Hadamard inequalities for superquadratic functions.

2. Mean Value Theorems

Definition B. Let $\varphi : [0,\infty) \to R$ be an integrable function; for $0 \leq a < b$ one defines a linear functional Λ_φ as

$$\Lambda_\varphi = \int_a^b \varphi(x)dx - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right)dx. \tag{2.1}$$

It is clear from (1.3) Theorem 1.1 of that; if φ is superquadratic function; then $\Lambda_\varphi \geq 0$.

In [7] we have the following Lemma.

Lemma 2.1. *Suppose that $\varphi : [0,\infty) \to R$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or φ'/x is increasing, then φ is superquadratic.*

Lemma 2.2 (see [12, Lemma 2]). *Let $\varphi \in C^2(I), I = (0,\infty)$ such that*

$$m \leq \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2} \leq M, \quad \forall \xi \in I. \tag{2.2}$$

Consider the functions φ_1, φ_2 defined as

$$\varphi_1(x) = \frac{Mx^3}{3} - \varphi(x), \qquad \varphi_2(x) = \varphi(x) - \frac{mx^3}{3}. \tag{2.3}$$

Then φ_1'/x and φ_2'/x are increasing functions. If also $\varphi_i(0) = 0$, $i = 1, 2$, then they are superquadratic functions.

Theorem 2.3. *If $\varphi'/x \in C^1(I)$ and $\varphi(0) = 0$, then the following equality holds:*

$$\Lambda_\varphi = \frac{1}{96}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}(b-a)\Big(a^2(5a-7b) + b^2(3b-a)\Big), \quad \xi \in I. \tag{2.4}$$

Proof. Suppose that φ'/x is bounded, that is, $\min(\varphi'/x) = m$ and $\max(\varphi'/x) = M$. Using φ_1 instead of φ in (1.3) we get

$$\int_a^b \varphi(t)dt - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|t - \frac{a+b}{2}\right|\right)dt \le \frac{M}{96}(b-a)\Big(a^2(5a-7b) + b^2(3b-a)\Big). \tag{2.5}$$

Similarly, using φ_2 instead of φ in (1.3) we get

$$\int_a^b \varphi(t)dt - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|t - \frac{a+b}{2}\right|\right)dt \ge \frac{m}{96}(b-a)\Big(a^2(5a-7b) + b^2(3b-a)\Big). \tag{2.6}$$

By combining the above two inequalities we get that there exists $\xi \in (0, \infty)$ such that (2.4) holds. Moreover if (for example) φ'/x is bounded from above we have that (2.5) is valid. Also (2.4) holds when φ'/x is not bounded. □

We omit the proofs of Theorems 2.4 and 2.6 as they are similar to the proofs in [9, 13–16].

Theorem 2.4. *If $\varphi'/x, \psi'/x \in C^1(I)$, $\varphi(0) = \psi(0) = 0$, and $a^2(5a-7b) + b^2(3b-a) \neq 0$, then one has*

$$\frac{\Lambda_\varphi}{\Lambda_\psi} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in I, \tag{2.7}$$

provided the denominators are not equal to zero. If K is invertible then

$$\xi = K^{-1}\left(\frac{\Lambda_\varphi}{\Lambda_\psi}\right), \quad \Lambda_\psi \neq 0, \tag{2.8}$$

is a new mean.

It is easy to check that the set of functions $\varphi(x) = x^r/(r(r-2))$, $r > 0$, $r \neq 2$, $x \geq 0$, satisfies Lemma 2.1. Therefore if we put $\varphi(x) = x^r/(r(r-2))$ and $\psi(x) = x^t/(t(t-2))$ in (2.8), we have a new mean $N_{r,t}$ defined as follows.

Definition B_1. One defines new mean $N_{r,t}$ for $r,\ t > 0,\ r \neq t$ and $a, b > 0,\ a \neq b$, as follows:

$$N_{r,t} = \left(\frac{2^t t(t+1)(t-2)(2^r(b^{r+1} - a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)(2^t(b^{t+1} - a^{t+1}) - (b-a)(t+1)(a+b)^t - (b-a)^{t+1})} \right)^{1/(r-t)}, \quad r, t \neq 2. \tag{2.9}$$

When t goes to 2, we have

$$N_{r,2} = N_{2,r} = \left(\frac{24(2^r(b^{r+1} - a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)P} \right)^{1/(r-2)}, \quad r \neq 2, \tag{2.10}$$

where

$$P = 4\ln 2\left(b^3 - a^3\right) + 4\left(b^3 \ln b - a^3 \ln a\right) - (b-a)(a+b)^2(1 + 3\ln(a+b)) - (b-a)^3 \ln(b-a). \tag{2.11}$$

When r goes to 2 we have

$$N_{2,2} = \exp\left(\frac{3Q - (6\ln 2 + 5)P}{6P} \right), \tag{2.12}$$

where P is defined above and

$$\begin{aligned} Q &= 2(\ln 2)^2\left(b^3 - a^3\right) + 8\ln 2\left(b^3 \ln b - a^3 \ln a\right) + 4\left(b^3(\ln b)^2 - a^3(\ln a)^2\right) \\ &\quad - (b-a)(a+b)^2(\ln(a+b)(2 + 3\ln(a+b))) - (b-a)^3(\ln(b-a))^2. \end{aligned} \tag{2.13}$$

In $N_{r,t}$ when t goes to r, we have

$$N_{r,r} = \exp\left(\frac{C}{D} - \frac{\ln 2r(r+1)(r-2) + 3r^2 - 2r - 2}{r(r+1)(r-2)} \right), \quad r \neq 2, \tag{2.14}$$

where

$$\begin{aligned} C &= 2^r\left(\ln 2\left(b^{r+1} - a^{r+1}\right) + b^{r+1}\ln b - a^{r+1}\ln a\right) - (b-a)(a+b)^r(1 + (r+1)\ln(a+b)) \\ &\quad - (b-a)^{r+1}\ln(b-a), \\ D &= 2^r\left(b^{r+1} - a^{r+1}\right) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1}. \end{aligned} \tag{2.15}$$

If we put $\varphi(x) = x^{r/s}/((r/s)(r/s-2))$ and $\psi(x) = x^{t/s}/((t/s)(t/s-2))$ in (2.8), then by the substitution, $a = a^s,\ b = b^s$, we have a new mean defined as

Definition B_2. Let $r, s, t \in R_+, r \neq t$ and $a, b > 0$, $a \neq b$ one defines Cauchy mean $N_{r,t}^{[s]}$ as

$$N_{r,t}^{[s]} = \left(\frac{t(t+s)(t-2s)\left(s(b^{r+s} - a^{r+s}) - (r+s)(b^s - a^s)((a^s+b^s)/2)^{r/s} - \mathfrak{A}\right)}{r(r+s)(r-2s)\left(s(b^{t+s} - a^{t+s}) - (t+s)(b^s - a^s)((a^s+b^s)/2)^{t/s} - \mathfrak{B}\right)} \right)^{1/(r-t)}, \quad r, t \neq 2s, \tag{2.16}$$

where $\mathfrak{A}$ denotes $2s((b^s - a^s)/2)^{(r+s)/s}$ and $\mathfrak{B}$ denotes $2s((b^s - a^s)/2)^{(t+s)/s}$. In limiting case when t goes to 2s $N_{r,2s}^{[s]}$ is equal to

$$\left(\frac{6s^2\left(s(b^{r+s} - a^{r+s}) - (r+s)(b^s - a^s)((a^s+b^s)/2)^{r/s} - 2s((b^s - a^s)/2)^{(r+s)/s}\right)}{r(r+s)(r-2s)(s(b^{3s}\ln b - a^{3s}\ln a) - \mathfrak{P} - \mathfrak{C})} \right)^{1/(r-2s)}, \quad r \neq 2s, \tag{2.17}$$

where $\mathfrak{P}$ denotes $(b^s - a^s)((a^s+b^s)/2)^2(1 + 3\ln((a^s + b^s)/2))$ and $\mathfrak{C}$ denotes $2((b^s - a^s)/2)^3 \ln((b^s - a^s)/2)$. When r goes to $2s$ we have,

$$N_{2s,2s}^{[s]} = \exp\left(\frac{G}{2sH} - \frac{5}{6s} \right), \tag{2.18}$$

where

$$\begin{aligned} G &= 4s^2\left(b^{3s}(\ln b)^2 - a^{3s}(\ln a)^2 - 2(b^s - a^s)(a^s + b^s)^2 \ln\left(\frac{a^s+b^s}{2}\right)\left(2 - 3\ln\left(\frac{a^s+b^s}{2}\right)\right)\right) \\ &\quad - 2(b^s - a^s)^3 \left(\frac{\ln(b^s - a^s)}{2}\right)^2, \\ H &= 4s\left(b^{3s}\ln b - a^{3s}\ln a\right) - (b^s - a^s)(a^s + b^s)^2\left(1 + 3\ln\left(\frac{a^s+b^s}{2}\right)\right) - (b^s - a^s)^3 \ln\left(\frac{b^s - a^s}{2}\right). \end{aligned} \tag{2.19}$$

When t goes to r in $N_{r,t}^{[s]}$, we have

$$\begin{aligned} N_{r,r}^{[s]} = \exp\Bigg(&\frac{H}{s(b^{r+s} - a^{r+s}) - (r+s)(b^s - a^s)((a^s+b^s)/2)^{r/s} - 2s((b^s - a^s)/2)^{(r+s)/s}} \\ &- \frac{6r}{(r+s)(r-2s)} \Bigg), \quad r \neq 2s, \end{aligned} \tag{2.20}$$

where

$$H = 4s\left(b^{3s}\ln b - a^{3s}\ln a\right) - (b^s - a^s)(a^s + b^s)^2\left(1 + 3\ln\left(\frac{a^s + b^s}{2}\right)\right) - (b^s - a^s)^3\ln\left(\frac{b^s - a^s}{2}\right). \tag{2.21}$$

Definition C. Let $\varphi : [0,\infty) \to R$ be an integrable function, for $0 \leq a < b$. One defines a linear functional $\widetilde{\Lambda}_\varphi$ as

$$\widetilde{\Lambda}_\varphi = \frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx. \tag{2.22}$$

It is clear from (1.4) Theorem 1.1 of that if φ is superquadratic function, then $\widetilde{\Lambda}_\varphi \geq 0$.

Theorem 2.5. *If $\varphi'/x \in C^1(I)$ and $\varphi(0) = 0$, then the following equality holds,*

$$\widetilde{\Lambda}_\varphi = \frac{1}{60}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(a^2(7a - 11b) + b^2(a + 3b)\right), \quad \xi \in I. \tag{2.23}$$

Proof. Suppose that φ'/x is bounded, that is, $\min(\varphi'/x) = m$ and $\max(\varphi'/x) = M$. Using φ_1 from Lemma 2.2 instead of φ in (1.4), we get

$$\begin{aligned} &\frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx \\ &\leq \frac{M}{60}\left(a^2(7a - 11b) + b^2(a + 3b)\right). \end{aligned} \tag{2.24}$$

Similarly, using φ_2 from Lemma 2.2 instead of φ in (1.4) we get

$$\begin{aligned} &\frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx \\ &\geq \frac{m}{60}\left(a^2(7a - 11b) + b^2(a + 3b)\right). \end{aligned} \tag{2.25}$$

By combining the above two inequalities we get that there exist $\xi \in (0,\infty)$ such that (2.23) holds. Moreover if (for example) φ'/x is bounded from above we have that (2.24) is valid. Also (2.23) holds when φ'/x is not bounded. □

Theorem 2.6. *If $\varphi'/x, \psi'/x \in C^1(I)$, $\varphi(0) = \psi(0) = 0$ and $a^2(7a - 11b) + b^2(a + 3b) \neq 0$ then, one has*

$$\frac{\widetilde{\Lambda}_\varphi}{\widetilde{\Lambda}_\psi} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = T(\xi), \quad \xi \in I, \tag{2.26}$$

provided the denominators are not equal to zero. If T is invertible, then

$$\xi = T^{-1}\left(\frac{\widetilde{\Lambda}_{\varphi}}{\widetilde{\Lambda}_{\psi}}\right), \quad \widetilde{\Lambda}_{\psi} \neq 0, \tag{2.27}$$

is a new mean.

If we put $\varphi(x) = x^r/(r(r-2))$ and $\psi(x) = x^t/(t(t-2))$ in (2.27) we have new mean $\widetilde{N}_{r,t}$ defined as follows.

Definition C_1. We define $\widetilde{N}_{r,t}$ for $r,t > 0$, $r \neq t$, $a,b > 0$, $a \neq b$ as follows:

$$\widetilde{N}_{r,t} = \left(\frac{t(t+1)(t+2)(t-2)((b-a)(r+1)(r+2)(a^r+b^r)-\mathfrak{D})}{r(r+1)(r+2)(r-2)((b-a)(t+1)(t+2)(a^t+b^t)-\mathfrak{E})}\right)^{1/(r-t)}, \quad r,t \neq 2, \tag{2.28}$$

where $\mathfrak{D}$ denotes $2(r+2)(b^{r+1}-a^{r+1})-4(b-a)^{r+1}$ and $\mathfrak{E}$ denotes $2(t+2)(b^{t+1}-a^{t+1})-4(b-a)^{t+1}$. In the limiting case we have $\widetilde{N}_{r,2} = \widetilde{N}_{2,r}$ which is equal to

$$\left(\frac{24\Big((b-a)(r+1)(r+2)(a^r+b^r)-2(r+2)\big(b^{r+1}-a^{r+1}\big)-4(b-a)^{r+1}\Big)}{r(r+1)(r+2)(r-2)((b-a)(7(a^2+b^2)+12(a^2\ln a+b^2\ln b))-\mathfrak{F})}\right)^{1/(r-2)}, \quad r \neq 2, \tag{2.29}$$

where $\mathfrak{F}$ denotes $2(b^3-a^3+4(b^3\ln b-a^3\ln a))-4(b-a)^3\ln(b-a)$,

$$\widetilde{N}_{2,2} = \exp\left(\frac{12A-13B}{12B}\right), \tag{2.30}$$

where

$$\begin{aligned} A &= (b-a)\Big(a^2+b^2+7\Big(a^2\ln a+b^2\ln b\Big)+6\Big(a^2(\ln a)^2+b^2(\ln b)^2\Big)-2(b-a)^2(\ln(b-a))^2\Big) \\ &\quad -2\Big(b^3\ln b-a^3\ln a\Big)-4\Big(b^3(\ln b)^2-a^3(\ln a)^2\Big), \\ B &= (b-a)\Big(7\Big(a^2+b^2\Big)+12\Big(a^2\ln a+b^2\ln b\Big)\Big) \\ &\quad -2\Big(b^3-a^3+4\Big(b^3\ln b-a^3\ln a\Big)\Big)-4(b-a)^3\ln(b-a). \end{aligned} \tag{2.31}$$

In $\widetilde{N}_{r,t}$ when t goes to r, we have

$$\widetilde{N}_{r,r} = \exp\left(\frac{4r^3+3r^2-8r-4}{r(r+1)(r+2)(r-2)} - \frac{R}{S}\right), \tag{2.32}$$

where

$$R = (b-a)(2r+3)(a^r+b^r) + (b-a)(r+1)(r+2)(a^r\ln a + b^r \ln b) - 2\left(b^{r+1}-a^{r+1}\right)$$
$$- 2(r+2)\left(b^{r+1}\ln b - a^{r+1}\ln a\right) - 4(b-a)^{r+1}\ln(b-a), \tag{2.33}$$
$$S = (b-a)(r+1)(r+2)(a^r+b^r) - 2(r+2)\left(b^{r+1}-a^{r+1}\right) - 4(b-a)^{r+1}.$$

If we put $\varphi(x) = x^{r/s}/((r/s)(r/s-2))$ and $\psi(x) = x^{t/s}/((t/s)(t/s-2))$ in (2.27), then by the substitution $a = a^s,\ b = b^s$ we have new mean $\widetilde{N}_{r,t}^{[s]}$ defined as follows.

Definition C_2. Let $r, s, t \in R_+, r \neq t$, and $a, b > 0$, $a \neq b$, one defines Cauchy mean $\widetilde{N}_{r,t}^{[s]}$ as follows:

$$\widetilde{N}_{r,t}^{[s]} = \left(\frac{t(t+s)(t+2s)(t-2s)((b^s-a^s)(r+s)(r+2s)(a^s+b^s)-\mathfrak{G})}{r(r+s)(r+2s)(r-2s)((b^s-a^s)(t+s)(t+2s)(a^s+b^s)-\mathfrak{H})}\right)^{1/(r-t)}, \quad r,t \neq 2s, \tag{2.34}$$

where $\mathfrak{G}$ denotes $2s(r+2s)(b^{r+s}-a^{r+s}) - 4s^2(b^s-a^s)^{(r+s)/s}$ and $\mathfrak{H}$ denotes $2s(t+2s)(b^{t+s}-a^{t+s}) - 4s^2(b^s-a^s)^{(t+s)/s}$. In limiting case we have $\widetilde{N}_{r,2s}^{[s]} = \widetilde{N}_{2s,r}^{[s]}$ which is equal to

$$\left(\frac{24s^3\left((b^s-a^s)(r+s)(r+2s)(a^r+b^r) - 2s(r+2s)(b^{r+s}-a^{r+s}) - 4s^2(b^s-a^s)^{(r+s)/2}\right)}{r(r+1)(r+2s)(r-2s)T}\right)^{1/(r-2s)}, \quad r \neq 2s, \tag{2.35}$$

where

$$T = (b^s-a^s)\left(7s\left(a^{2s}+b^{2s}\right) + 12s^2\left(a^{2s}\ln a + b^{2s}\ln b\right)\right) - 2s\left(b^{3s}-a^{3s}\right)$$
$$- 8s^2\left(b^{3s}\ln b - a^{3s}\ln a\right) - 4s(b^s-a^s)^3\ln(b^s-a^s). \tag{2.36}$$

When r approaches to $2s$,

$$\widetilde{N}_{2s,2s}^{[s]} = \exp\left(\frac{12U-13T}{12T}\right), \tag{2.37}$$

where

$$U = (b^s - a^s)\Big(2\Big(a^{2s} + b^{2s} + 7s\Big(a^{2s}\ln a + b^{2s}\ln b\Big) + 6s^2\Big(a^{2s}(\ln a)^2 + b^{2s}(\ln b)^2\Big)\Big)\Big)$$
$$- 2(b^s - a^s)^3\Big(\ln(b^s - a^s)^2\Big) - 2s\Big(b^{3s}\ln b - a^{3s}\ln a\Big) - 4s^2\Big(b^{3s}(\ln b)^2 - a^{3s}(\ln a)^2\Big),$$
$$\widetilde{N}^{[s]}_{r,r} = \exp\left(\frac{4(r^3 - s^3) + rs(3r - 8s)K - r(r+s)(r+2s)(r-2s)L}{r(r+s)(r+2s)(r-2s)K}\right), \tag{2.38}$$

where

$$K = (b^s - a^s)(r+s)(r+2s)(a^r + b^r) - 2s(r+2s)(b^{r+s} - a^{r+s}) - 4s^2(b^s - a^s)^{(r+s)/2},$$
$$L = (b^s - a^s)(2r+3s)(a^r + b^r) + (r+s)(r+2s)(a^r\ln a + b^r\ln b) - 2s(b^{r+s} - a^{r+s})$$
$$- 2s(r+2s)(b^{r+s}\ln b - a^{r+s}\ln a) - \frac{4}{s}(b^s - a^s)^{(r+s)/2}\ln(b^s - a^s). \tag{2.39}$$

3. Positive Semidefiniteness, Exponential Convexity, and Log-Convexity

Lemma 3.1 (see [12, Lemma 3]). *Consider the function φ_s for $s > 0$ defined as*

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-2)}, & s \neq 2, \\ \dfrac{x^2}{2}\log x, & s = 2. \end{cases} \tag{3.1}$$

Then, with the convention $0\log 0 = 0$, $\varphi_s(x)$ is superquadratic.

Theorem 3.2. *For Λ_{φ_s} defined in (2.1) one has the following.*

(a) *The matrix $A = [\Lambda_{\varphi_{(p_i+p_j)/2}}]$, $1 \le i, j \le n$, is a positive semidefinite matrix, that is,*

$$\det\left(\left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}\right]_{i,j=1}^{k}\right) \ge 0, \quad k = 1, 2, \ldots, n. \tag{3.2}$$

(b) *One has*

$$\Lambda^2_{\varphi_{(s+t)/2}} \le \Lambda_{\varphi_s}\Lambda_{\varphi_t}, \tag{3.3}$$

that is, Λ_{φ_s} is log-convex in the Jensen sense.

(c) *The function $s \mapsto \Lambda_{\varphi_s}$ is exponentially convex.*

(d) Λ_{φ_s} *is log-convex, that is, for* $r < s < t$ *where* $r, s, t \in R_+$ *one has*

$$(\Lambda_{\varphi_s})^{t-r} \le (\Lambda_{\varphi_r})^{t-s} (\Lambda_{\varphi_t})^{s-r}. \tag{3.4}$$

Proof. (a) Define the function $F(x) = \sum_{i,j=1}^{n} u_i u_j \varphi_{p_{ij}}(x)$, where $p_{ij} = (p_i + p_j)/2$. Then,

$$\left(\frac{F'(x)}{x}\right)' = \sum_{i,j=1}^{n} u_i u_j \left(\frac{\varphi'_{p_{ij}}(x)}{x}\right)' = \left(\sum_{i=1}^{n} u_i x^{(p_i-3)/2}\right)^2 \ge 0 \tag{3.5}$$

and $F(0) = 0$. This implies that F is superquadratic, so using this F in the place of φ in (2.1) we have

$$\Lambda_F = \sum_{i,j=1}^{n} u_i u_j A_{\varphi_{p_{ij}}} \ge 0. \tag{3.6}$$

From this we have that the matrix $A = [\Lambda_{\varphi_{(p_i+p_j)/2}}]_{n\times n}$ is positive semidefinite.

(b) It is a simple consequence of (a) for $k = 2$.

(c) Since we have $\lim_{s\to 2}\Lambda_{\varphi_s} = \Lambda_{\varphi_2}$, so Λ_{φ_s} is continuous for all s; then by (3.6) and Proposition 1.2 we have that $s \mapsto \Lambda_{\varphi_s}$ is exponentially convex.

(d) As Λ_{φ_s} is continuous then we have that Λ_{φ_s} is log-convex and we get (3.4). □

Corollary 3.3. *One has the following*

(i) *For* $s > 4$,

$$\Lambda_{\varphi_s} \ge \frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96} \left(\frac{3(a^2-b^2)^2}{2(3b^3 - ab^2 - 7a^2b + 5a^3)}\right)^{s-3}. \tag{3.7}$$

(ii) *For* $1 < s < 2$,

$$\Lambda_{\varphi_s} \le (a-b)^{4-2s} (\Lambda_{\varphi_2})^{s-1}. \tag{3.8}$$

(iii) *For* $2 < s < 3$,

$$\Lambda_{\varphi_s} \le \left(\frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96\Lambda_{\varphi_2}}\right)^{s-2} \Lambda_{\varphi_s}. \tag{3.9}$$

(iv) *For* $3 < s < 4$,

$$\Lambda_{\varphi_s} \le \frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96} \left(\frac{3(a^2-b^2)^2}{2(3b^3 - ab^2 - 7a^2b + 5a^3)}\right)^{s-3}. \tag{3.10}$$

Proof. By applying Theorem 3.2(b) with $3 < 4 < s$ and $1 < s < 2 < 3 < 4$, respectively, we get the result. □

Similar to Theorem 3.2 we get the following.

Theorem 3.4. *For $\widetilde{\Lambda}_{\varphi_s}$ defined in* (2.22) *one has the following.*

(a) *The matrix $A = [\widetilde{\Lambda}_{\varphi_{(p_i+p_j)/2}}]$, $1 \le i, j \le n$, is a positive-semidefinite matrix, that is,*

$$\det\left(\left[\widetilde{\Lambda}_{\varphi_{(p_i+p_j)/2}}\right]_{i,j=1}^{k}\right) \ge 0, \quad k = 1, 2, \dots, n. \tag{3.11}$$

(b) *One has*

$$\widetilde{\Lambda}^2_{\varphi_{(s+t)/2}} \le \widetilde{\Lambda}_{\varphi_s}\widetilde{\Lambda}_{\varphi_t}, \tag{3.12}$$

that is, $\widetilde{\Lambda}_{\varphi_s}$ is log-convex in the Jensen sense.

(c) *The function $s \mapsto \widetilde{\Lambda}_{\varphi_s}$ is exponentially convex.*

(d) *$\widetilde{\Lambda}_{\varphi_s}$ is log-convex, that is, for $r < s < t$ where $r, s, t \in R_+$ one has*

$$\left(\widetilde{\Lambda}_{\varphi_s}\right)^{t-r} \le \left(\widetilde{\Lambda}_{\varphi_r}\right)^{t-s}\left(\widetilde{\Lambda}_{\varphi_t}\right)^{s-r}. \tag{3.13}$$

Proof. The proof is the same as the proof of Theorem 3.2. □

In the next results we use the continuity of Λ_{φ_s} and $\widetilde{\Lambda}_{\varphi_s}$.
When log f is convex we see that (also see [13])

Lemma 3.5. *Let f be log-convex function, and if $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$, then the following inequality is valid,*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \le \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{3.14}$$

Theorem 3.6. *For $p, r, s, t \in R_+$ such that $r \le s$ and $p \le t$, one has for Nr, t as in Definition B_1*

$$N_{p,r} \le N_{t,s}. \tag{3.15}$$

Proof. According to Theorem 3.2, Λ_{φ_s} defined above is log-convex; so Lemma 3.5 implies that for $p, r, s, t \in R_+$ such that $r \le s$ and $p \le t$ we have

$$\left[\frac{\Lambda_{\varphi_p}}{\Lambda_{\varphi_r}}\right]^{1/(p-r)} \le \left[\frac{\Lambda_{\varphi_t}}{\Lambda_{\varphi_s}}\right]^{1/(t-s)}, \quad p \ne r,\ t \ne s. \tag{3.16}$$

From the continuity of Λ_{φ_s} we get our result for $t \neq r$, $v \neq u$, and for $t = r$, $v = u$ we can consider limiting case. □

Theorem 3.7. *For $t, r, u, v \in R_+$ such that $t \leq v$ and $r \leq u$, one has for $N_{r,t}^{[s]}$ as in Definition B_2*

$$N_{t,r}^{[s]} \leq N_{v,u}^{[s]}. \tag{3.17}$$

Proof. As Λ_{φ_s} defined above is log-convex, Lemma 3.5 implies that for $t, r, u, v \in \mathbb{R}$ such that $t \leq v$ and $r \leq u$ we have

$$\left[\frac{\Lambda_{\varphi_t}}{\Lambda_{\varphi_r}}\right]^{1/(t-r)} \leq \left[\frac{\Lambda_{\varphi_v}}{\Lambda_{\varphi_u}}\right]^{1/(v-u)}, \quad t \neq r,\ v \neq u. \tag{3.18}$$

By substituting $t = t/s$, $r = r/s$, $u = u/s, v = v/s$, $a = a^s$, and $b = b^s$, such that $t/s \neq v/s$, $r/s \neq u/s$, $t \neq r$, and $v \neq u$ we get the result, and for $r = t, u = v$ we can consider the limiting case. □

Theorem 3.8. *For $p, r, s, t \in R_+$ such that $r \leq s$ and $p \leq t$, one has*

$$\widetilde{N}_{p,r} \leq \widetilde{N}_{t,s}. \tag{3.19}$$

Proof. According to Theorem 3.4, $\widetilde{\Lambda}_{\varphi_s}$ defined above is log-convex; so Lemma 3.5 implies that for $p, r, s, t \in \mathbb{R}$ such that $r \leq s$ and $p \leq t$ we have

$$\left[\frac{\widetilde{\Lambda}_{\varphi_p}}{\widetilde{\Lambda}_{\varphi_r}}\right]^{1/(p-r)} \leq \left[\frac{\widetilde{\Lambda}_{\varphi_t}}{\widetilde{\Lambda}_{\varphi_s}}\right]^{1/(t-s)}, \quad p \neq r,\ t \neq s. \tag{3.20}$$

From the continuity of $\widetilde{\Lambda}_{\varphi_s}$ we get our result for $t \neq r$, $v \neq u$; and for $t = r$, $v = u$ we can consider limiting case. □

Theorem 3.9. *For $t, r, u, v \in R_+$ such that $t \leq v$ and $r \leq u$, one has*

$$\widetilde{N}_{t,r}^{[s]} \leq \widetilde{N}_{v,u}^{[s]}. \tag{3.21}$$

Proof. As $\widetilde{\Lambda}_{\varphi_s}$ defined above is log-convex, Lemma 3.5 implies that for $t, r, u, v \in \mathbb{R}$ such that $t \leq v$ and $r \leq u$ we have

$$\left[\frac{\widetilde{\Lambda}_{\varphi_t}}{\widetilde{\Lambda}_{\varphi_r}}\right]^{1/(t-r)} \leq \left[\frac{\widetilde{\Lambda}_{\varphi_v}}{\widetilde{\Lambda}_{\varphi_u}}\right]^{1/(v-u)}, \quad t \neq r,\ v \neq u. \tag{3.22}$$

By substituting $t = t/s$, $r = r/s, u = u/s$, $v = v/s, a = a^s$, and $b = b^s$, such that $t/s \neq v/s$, $r/s \neq u/s$, $t \neq r$, and $v \neq u$ we get the result, and for $r = t$, $u = v$ we can consider the limiting case. □

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References

[1] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[2] S. S. Dragomir, "On Hadamard's inequalities for convex functions," *Mathematica Balkanica*, vol. 6, no. 3, pp. 215–222, 1992.

[3] S. S. Dragomir and A. McAndrew, "Refinements of the Hermite-Hadamard inequality for convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 5, article 140, 2005.

[4] S. Banić, "Mappings connected with Hermite-Hadamard inequalities for superquadratic functions," *Journal of Mathematical Inequalities*, vol. 3, no. 4, pp. 577–589, 2009.

[5] S. S. Dragomir, "A mapping in connection to Hadamard's inequalities," *Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse*, vol. 128, pp. 17–20, 1991.

[6] S. S. Dragomir, "Two mappings in connection to Hadamard's inequalities," *Journal of Mathematical Analysis and Applications*, vol. 167, no. 1, pp. 49–56, 1992.

[7] S. Abramovich, G. Jameson, and G. Sinnamon, "Refining Jensen's inequality," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 47, no. 1-2, pp. 3–14, 2004.

[8] S. Banić and S. Varošanec, "Functional inequalities for superquadratic functions," *International Journal of Pure and Applied Mathematics*, vol. 43, no. 4, pp. 537–549, 2008.

[9] M. Anwar, J. Jakšetić, J. Pečarić, and Atiq Ur Rehman, " Exponential convexity, positive semi-definite matrices and fundamental inequalities," accepted in *Journal of Mathematical Inequalities*.

[10] I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, NJ, USA, 1955.

[11] W. Sierpinski, "A family of the Cauchy type mean-value theorems," *Fundamenta Mathematicae*, vol. 1, pp. 125–129, 1920.

[12] S. Abramovich, G. Farid, and J. Pečarić, "More About Jensen's Inequality and Cauchy's Means for Superquadratic Functions," submitted.

[13] M. Anwar, N. Latif, and J. Pečarić, "Cauchy means of the Popoviciu type," *Journal of Inequalities and Applications*, vol. 2009, Article ID 628051, 16 pages, 2009.

[14] M. Anwar and J. Pečarić, "Cauchy's means of Levinson type," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, article 120, p. 8, 2008.

[15] M. Anwar and J. Pečarić, "New means of Cauchy's type," *Journal of Inequalities and Applications*, vol. 2008, Article ID 163202, 10 pages, 2008.

[16] J. E. Pečarić, I. Perić, and H. M. Srivastava, "A family of the Cauchy type mean-value theorems," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 2, pp. 730–739, 2005.