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Research Article

Abstract Convexity and Hermite-Hadamard Type Inequalities

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The deriving Hermite-Hadamard type inequalities for certain classes of abstract convex functions are considered totally, the inequalities derived for some of these classes before are summarized, new inequalities for others are obtained, and for one class of these functions the results on R_+^2 are generalized to R_n^+. By considering a concrete area in R_{++}^n, the derived inequalities are illustrated.

1. Introduction

Studying Hermite-Hadamard type inequalities for some function classes has been very important in recent years. These inequalities which are well known for convex functions have been also found in different function classes (see [1–5]).

Abstract convex function is one of this type of function classes. Hermite-Hadamard type inequalities are studied for some important classes of abstract convex functions, and the concrete results are found [6–9]. For example, increasing convex-along-rays (ICAR) functions, which are defined in R_+^2, are considered in [8]. First, a correct inequality for these functions is given. For each $(x_0, y_0) \in R_+^2/\{0\}$ there exists a number $b(x_0, y_0) > 0$ such that

$$b(x_0, y_0)\left[\min \frac{x}{x_0}, \frac{y}{y_0} - 1\right] \le f(x, y) - f(x_0, y_0) \tag{1.1}$$

for all (x, y).

Then, based on previous inequality the following inequality is proven. If $Q(D) \neq \emptyset$, then for all continuous ICAR function

$$\max_{(\overline{x},\overline{y})\in Q(D)} f(\overline{x},\overline{y}) \leq \frac{1}{A(D)}\int_{\mathrm{D}} f(x,y)\,dx\,dy \tag{1.2}$$

inequality is correct, where

$$Q(D) = \left\{ (\overline{x},\overline{y}) \in D \subset R_{+}^{2} : \frac{1}{A(D)}\int_{D} \min\left(\frac{x}{\overline{x}},\frac{y}{\overline{y}}\right) dx\,dy = 1 \right\}, \tag{1.3}$$

and $A(D)$ is the area of domain D.

Similar inequalities are found for increasing positively homogenous (IPH) functions in [6], for increasing radiant (InR) functions in [9], and for increasing coradiant (ICR) functions in [7].

In this article, first, the theorem which yields inequality (1.1) is proven for ICAR functions defined on R_{+}^{n}, then the other inequalities are generalized, which based on this theorem.

Another generalization is made for $Q(D)$. A more covering set $Q_k(D)$ is considered and all results (for IPH, ICR, InR, ICAR functions) are examined for this set.

2. Abstract Convexity and Hermite-Hadamard Type Inequalities

Let R be a real line and $R_{+\infty} = R \cup \{+\infty\}$. Consider a set X and a set H of functions $h : X \to R$ defined on X. A function $f : X \to R_{+\infty}$ is called abstract convex with respect to H (or H-*convex*) if there exists a set $U \subset H$ such that

$$f(x) = \sup\{h(x) : h \in U\} \quad \forall x \in X. \tag{2.1}$$

Clearly f is H-convex if and only if

$$f(x) = \sup\{h(x) : h \leq f\} \quad \forall x \in X. \tag{2.2}$$

Let Y be a set of functions $f : X \to R_{+\infty}$. A set $H \subset Y$ is called a supremal generator of the set Y if each function $f \in Y$ is abstract convex with respect to H.

2.1. Increasing Positively Homogeneous Functions and Hermite-Hadamard Type Inequalities

A function f defined on $R_{+}^{n} = \{(x_1,x_2,\ldots,x_n) \in R^n : x_1 \geq 0, x_2 \geq 0,\ldots,x_n \geq 0\}$ is called increasing (with respect to the coordinate-wise order relation) if $x \geq y$ implies that $f(x) \geq f(y)$.

The function f is positively homogeneous of degree one if

$$f(\lambda x) = \lambda f(x) \tag{2.3}$$

for all $x \in R^n_+$ and $\lambda > 0$.

Let L be the set of all min-type functions defined on

$$R^n_{++} = \{(x_1, x_2, \ldots, x_n) \in R^n : x_1 > 0, x_2 > 0, \ldots, x_n > 0\}, \tag{2.4}$$

that is, the set L consists of identical zero and all the functions of the form

$$l(x) = \langle l, x\rangle = \min_i \frac{x_i}{l_i}, \quad x \in R^n_{++} \tag{2.5}$$

with all $l \in R^n_{++}$.

One has that a function $f : R^n_{++} \to R$ is L–convex if and only if f is increasing and positively homogeneous of degree one (IPH) functions (see [10]).

The Hermite-Hadamard type inequalities are shown for IPH functions by using the following proposition which is very important for IPH functions.

Proposition 2.1. *Let f be an IPH function defined on R^n_{++}. Then the following inequality holds for all $x, l \in R^n_{++}$:*

$$f(l)\langle l, x\rangle \leq f(x). \tag{2.6}$$

Proposition 2.2 can be easily shown by using the Proposition 2.1 (see [6]).

Proposition 2.2. *Let $D \subset R^n_{++}$, $f : D \to R_{+\infty}$ be an IPH function, and let f be integrable on D. Then*

$$f(u)\int_D \langle u, x\rangle dx \leq \int_D f(x)dx \tag{2.7}$$

for all $u \in D$, and this inequality is sharp.

Unlike the previous work, inequality (2.7) (obtained for IPH functions) and inequalities in the type of (2.7) (will be obtained for different function classes) are going to be inquired for more general the $Q_k(D)$ sets not for the $Q(D)$ set. $Q_k(D)$ will be certainly different for each function class.

Let $D \subset R^n_{++}$ be a closed domain, that is, $\mathrm{cl}(\mathrm{int}\, D) = D$, and let k be positive number. Let $Q_k(D)$ be the set of all points $x^* \in D$ such that

$$\frac{k}{A(D)}\int_D \langle x^*, x\rangle dx = 1, \tag{2.8}$$

where $A(D) = \int_D dx$.

In the case of $k = 1, Q_1(D)$ will be the set $Q(D)$ in [8, 9].

In [6, Proposition 3.2], the proposition has been given for $Q(D)$, the same proposition is defined for $Q_k(D)$ as follows, and its proof is similar.

Proposition 2.3. *Let f be an IPH function defined on D. If the set $Q_k(D)$ is nonempty and f is integrable on D, then*

$$\sup_{x^* \in Q_k(D)} f(x^*) \leq \frac{k}{A(D)} \int_D f(x) dx. \tag{2.9}$$

We had proved a proposition in [6] by using a function $\langle u, x\rangle^+ = \max_{(1 \leq i \leq n)}(x_i/u_i)$, and we get a right-hand side inequality, similar to (2.7).

Proposition 2.4. *Let f be an IPH function, and let f be integrable function on D. Then*

$$\int_D f(x) dx \leq \inf_{u \in D} \left[f(u) \int_D \langle u, x\rangle^+ \, dx \right]. \tag{2.10}$$

For every $u \in D$ the inequality

$$\int_D f(x) dx \leq f(u) \int_D \langle u, x\rangle^+ \, dx \tag{2.11}$$

is sharp.

2.2. Increasing Positively Homogeneous Functions and Hermite-Hadamard Type Inequalities

A function $f : R^n_{++} \to R_{+\infty}$ is called increasing radiant (InR) function if

(1) f is increasing;

(2) f is radiant; that is, $f(\lambda x) \leq \lambda f(x)$ for all $\lambda \in (0, 1)$, and $x \in R^n_{++}$.

Consider the coupling function φ defined on $R^n_{++} \times R^n_{++}$

$$\varphi(u, x) = \begin{cases} 0, & \text{if } \langle u, x\rangle < 1, \\ \langle u, x\rangle, & \text{if} \langle u, x\rangle \geq 1. \end{cases} \tag{2.12}$$

Denote by φ_u the function defined on R^n_{++} by the formula

$$\varphi_u(x) = \varphi(u, x). \tag{2.13}$$

It is known that the set

$$U = \left\{ \frac{1}{c}\varphi_u : u \in R^n_{++},\ c \in (0, +\infty] \right\} \tag{2.14}$$

is supremal generator of all increasing radiant functions defined on R^n_{++} (see[9]).

Note that for $c = +\infty$ we get $c\varphi_u(x) = \sup_{h>0}(h\varphi_u(x))$.

The very important property for InR functions is given here in after. It can be easily proved.

Proposition 2.5. *Let f be an InR function defined on R^n_{++}. Then the following inequality holds for all $x, l \in R^n_{++}$:*

$$f(l)\varphi(l, x) \le f(x). \tag{2.15}$$

By using [9, Proposition 2.5], the following proposition is proved.

Proposition 2.6. *Let $D \subset R^n_{++}$, $f : D \to R_{+\infty}$ be InR functions and integrable on D. Then*

$$f(u)\int_D \varphi(u, x) \le \int_D f(x)dx \tag{2.16}$$

for all $u \in D$. This inequality is sharp for any $u \in D$ since one has the inequality in [9] for $f(x) = \varphi_u(x)$.

We determine the set $Q_k(D)$ for InR functions. Let $Q_k(D)$ be the set of all points $x^* \in D$ such that

$$\frac{k}{A(D)}\int_D \varphi(x^*, x)\, dx = 1, \tag{2.17}$$

which is given in [9, Proposition 3.1] can be generalized for $Q_k(D)$.

Proposition 2.7. *Let f be an InR function defined on R^n_{++}. If the set $Q_k(D)$ is nonempty and f is integrable on D, then*

$$\sup_{\overline{x} \in Q_k(D)} f(\overline{x}) \le \frac{k}{A(D)}\int_D f(x)dx. \tag{2.18}$$

Proof. The proof of the proposition can be made in a similar way to the proof in [9, Proposition 3.1]. □

Now, we will study to achieve right-hand side inequality for InR functions.

First, Let us prove the auxiliary proposition.

Proposition 2.8. *Let f be an InR function on D. Then the following inequalities hold for all $l, x \in D$:*

$$f(l) \le \varphi^+_x(l) f(x), \tag{2.19}$$

where

$$\varphi_x^+(l) = \begin{cases} \langle x, l\rangle^+, & \text{if } \langle x, l\rangle^+ \le 1, \\ +\infty, & \text{if } \langle x, l\rangle^+ > 1. \end{cases} \tag{2.20}$$

Proof. Since f is InR function on D, then

$$f(l)\varphi(x) \le f(x) \tag{2.21}$$

for all $x, l \in D$. From this

$$f(l)\langle l, x\rangle \le f(x)), \quad \text{if } \langle l, x\rangle \ge 1. \tag{2.22}$$

That is,

$$f(l) \le \langle x, l\rangle^+ f(x), \quad \text{if } \langle l, x\rangle^+ \le 1. \tag{2.23}$$

If we consider the definition of $\varphi_x^+(l)$, then

$$f(l) \le \varphi_x^+(l) f(x) \tag{2.24}$$

for all $x, l \in D$. □

Proposition 2.9. *Let f be an InR function and integrable on D, $u \in D$ and*

$$D(u) = \{x \in D \mid \langle u, x\rangle^+ \le 1\}, \tag{2.25}$$

then

$$\int_{D(u)} f(x)dx \le f(u) \int_{D(u)} \varphi_u^+(x)dx \tag{2.26}$$

holds and is sharp since we get equality for $f(x) = \langle u, x\rangle^+$.

Proof. It follows from Proposition 2.8. □

Corollary 2.10. *Let f be an InR function and integrable on D. If $u \in D$ and $u \ge x$ for all $x \in D$, then*

$$\int_D f(x)dx \le f(u) \int_D \langle u, x\rangle^+ dx \tag{2.27}$$

holds and is sharp.

2.3. Increasing Coradiant Functions and Hermit-Hadamard Type Inequalities

A function $f : K \to R_{+\infty}$ defined on a cone $K \subset R^n$ is called coradiant if

$$f(\lambda x) \geq \lambda f(x) \quad \forall x \in K,\ \lambda \in [0,1]. \tag{2.28}$$

It is easy to check that f is coradiant if and only if

$$f(\nu x) \leq \nu f(x) \quad \forall x \in K,\ \nu \geq 1. \tag{2.29}$$

We will consider increasing coradiant (ICR) function defined on the cone R^n_{++}.

Consider the function Ψ_l defined on R^n_{++} :

$$\Psi_l(x) = \begin{cases} \langle l, x\rangle, & \text{if } \langle l, x\rangle \leq 1, \\ 1, & \text{if } \langle l, x\rangle > 1, \end{cases} \tag{2.30}$$

where $l \in R^n_{++}$.

Recall that the set

$$H = \{c\Psi_l : l \in R^n_{++},\ c \in [0,\infty]\} \tag{2.31}$$

is supremal generator of the class ICR functions defined on R^n_{++} (see [10]).

The Hermit-Hadamard type inequalities have been obtained for ICR functions by using the following proposition in [7].

Proposition 2.11. *Let f be an ICR function defined on R^n_{++}. Then the following inequality holds for all $x, l \in R^n_{++}$:*

$$f(l)\Psi_l(x) \leq f(x). \tag{2.32}$$

Proposition 2.12. *Let $D \subset R^n_{++}$, $f : D \to R_{+\infty}$ be ICR function and integrable on D. Then the following inequality holds for all $u \in D$:*

$$f(u)\int_D \Psi_u(x)\,dx \leq \int_D f(x)\,dx, \tag{2.33}$$

and it is sharp.

The set $Q_k(D)$ is defined for ICR function, namely, $Q_k(D)$ denotes the set of all points $x^* \in D$ such that

$$\frac{k}{A(D)}\int_D \Psi_{x^*}(x)dx = 1. \tag{2.34}$$

Proposition 2.13. *Let f be an ICR function on D. If the set $Q_k(D)$ is nonempty and f is integrable on D, then*

$$\sup_{\overline{x}\in Q_k(D)} f(\overline{x}) \leq \frac{k}{A(D)}\int_D f(x)dx. \tag{2.35}$$

Let us define a new function $\Psi_u^+(x)$ such that

$$\Psi_u^+(x) = \begin{cases} \langle u,x\rangle^+, & \text{if } \langle u,x\rangle^+ \geq 1, \\ 1, & \text{if } \langle u,x\rangle^+ < 1, \end{cases} \tag{2.36}$$

where $\langle u,x\rangle^+$ is max-type function. By including the new function $\Psi_u^+(x)$, we can achieve right-hand side inequalities for ICR functions, too.

Proposition 2.14. *Let function f be an ICR function and integrable on D. Then*

$$\int_D f(x)dx \leq \min_{u\in D}\left[f(u)\int_D \Psi_u^+(x)dx\right], \tag{2.37}$$

and for every $u \in D$ the inequality

$$\int_D f(x)dx \leq f(u)\int_D \Psi_u^+(x)dx \tag{2.38}$$

is sharp.

2.4. Increasing Convex Along Rays Functions and Hermit-Hadamard Type Inequalities

The Hermite-Hadamard type inequalities are studied for ICAR functions in [8]. But only the functions which are defined on R_+^2 are considered.

In this article, the functions which are defined on R_+^n are considered, and general results are found.

Let $K \subset R^n$ be a conic set. A function $f : K \to R_{+\infty}$ is called convex-along-rays if its restriction to each ray starting from zero is a convex function of one variable. In other words, it means that the function

$$f_x(t) = f(tx), \quad t \geq 0 \tag{2.39}$$

is convex for each $x \in K$.

In this paper we consider increasing convex-along- rays (ICARs) functions defined on $K = R_+^n$.

It is known that a finite ICAR function is continuous on the R_{++}^n and lower semicontinuous on R_+^n in [10].

Let us give two theorems which had been proved in [10, Theorems 3.2 and 3.4].

Theorem 2.15. *Let H_L be the class of all functions h defined by*

$$h(x) = \langle l, x \rangle - c, \tag{2.40}$$

where $\langle l, x \rangle$ is a min-type function and $c \in R$. A function $f : R^n_+ \to R_{+\infty}$ is H_L-convex if and only if f is lower semicontinuous and ICAR.

Theorem 2.16. *Let f be ICAR function, and let $x \in R^n_+ \setminus \{0\}$ be a point such that $(1+\varepsilon)x \in \operatorname{dom} f$ for some $\varepsilon > 0$. Then the sup differential*

$$\partial_L f(x) \equiv \{ l \in L : \langle l, y \rangle - \langle l, x \rangle \le f(y) - f(x) \} \tag{2.41}$$

is not empty and

$$\left\{ \frac{u}{x} : u \in \partial f_x(1) \right\} \subset \partial_L f(x), \tag{2.42}$$

where $f_x(t) = f(tx)$.

Now, we can define the following theorem which is important to achieve Hermit-Hadamard type inequalities for ICAR functions.

Theorem 2.17. *Let f be a finite ICAR function defined on R^n_+. Then for each $y \in R^n_+ \setminus \{0\}$ there exists a number $b = b(y) > 0$ such that*

$$b(\langle y, x \rangle - 1) \le f(x) - f(y) \tag{2.43}$$

for all x.

Proof. The result follows directly from Theorem 2.16. □

We will apply Theorem 2.17 in the study of Hermit-Hadamard type inequalities for ICAR functions.

Proposition 2.18. *Let $D \subset R^n_+$, $f : D \to R_+$ be ICAR function. Then the following inequality holds for all $u \in D$:*

$$b(u) \int_D [\langle u, x \rangle - 1] \, dx + f(u) A(D) \le \int_D f(x) \, dx. \tag{2.44}$$

Proof. It follows from Theorem 2.17. □

Formula (2.44) can be made simply with the sets $Q(D)$.

Let $D \subset R^n_+$ be a bounded set such that $\operatorname{cl}(\operatorname{int} D) = D$ and

$$Q(D) \equiv \left\{ x^* \in D \mid \frac{1}{\mathrm{A(D)}} \int_D \langle x^*, x \rangle dx = 1 \right\}. \tag{2.45}$$

Proposition 2.19. *Let the set $Q(D)$ be nonempty, and let f be a continuous ICAR function defined on D. Then the following inequality holds:*

$$\max_{u\in Q(D)} f(u) \le \frac{1}{A(D)}\int_D f(x)dx. \tag{2.46}$$

Proof. Let $u \in Q(D)$. It follows from (2.43) and the definition of $Q(D)$ that

$$\begin{aligned} 0 &= b(u)\left[\frac{1}{A(D)}\int_D \langle u, x\rangle\, dx - 1\right] \\ &= \frac{1}{A(D)}\int_D b(u)[\langle u, x\rangle - 1]\, dx \\ &\le \frac{1}{A(D)}\int_D [f(x) - f(u)]\, dx. \end{aligned} \tag{2.47}$$

Thus

$$f(u) \le \frac{1}{A(D)}\int_D f(x)\, dx. \tag{2.48}$$

Since $Q(D)$ is compact (see [8]) and f is continuous (finite ICAR functions is continuous), it follows that the maximum in (2.46) is attained. □

Remark 2.20. Inequalities (2.9), (2.18), (2.35), and (2.46), which are obtained for different convex classes, are actually different, even if they appear to be the same. The reason is that these are determined with the (2.8), (2.17), (2.34), and (2.45) formulas appropriate for the sets of $Q_k(D)$ and also yielding different sets.

3. Examples

The results of different classes of convex functions are defined for same triangle region $D \subset R^2_{++}$:

$$D = \left\{(x_1, x_2) \in R^2_{++} : 0 < x_1 \le a,\ 0 < x_2 \le v x_1\right\}. \tag{3.1}$$

The inequalities (2.7) and (2.10) have been defined for IPH functions. The inequalities are examined for the region D in [6]. If we combine two results, then we get

$$\frac{a^3(2u_1 v - u_2)}{6u_1^2} f(u_1, u_2) \le \int_D f(x_1, x_2)\, dx \le \frac{a^3}{6}\left(\frac{u_2}{u_1^2} + \frac{v^2}{u_2}\right) f(u_1, u_2) \tag{3.2}$$

for all $(u_1, u_2) \in D$.

If we study the set $Q_k(D)$ for IPH functions, a point $(x_1^*, x_2^*) \in D$ belongs to $Q_k(D)$ if and only if

$$x_2^* = -\frac{3v}{ak}\left(x_1^*\right)^2 + 2vx_1^*. \tag{3.3}$$

That is, the set $Q_k(D)$ is intersection with the set D and the parabola by formula (3.3).

Let us consider the InR functions for same region D. The inequality (2.16) has been examined for D, and the following inequality has been obtained in [9]:

$$\left[\left(\frac{va^3}{3u_1} - \frac{vu_1^2}{3}\right) + u_2\left(\frac{2u_1}{3} - \frac{a}{2} - \frac{a^3}{6u_1^2}\right)\right] f(u_1, u_2) \leq \int_D f(x_1, x_2) dx_1 dx_2 \tag{3.4}$$

for all $(u_1, u_2) \in D$.

Let us study on the right-hand side inequality (2.26), which is obtained in this article, for same region D, which has been defined as follows:

$$D(u) = \left\{x \in D : \langle u, x\rangle^+ \leq 1\right\} \tag{3.5}$$

for all $u \in D$.

We will separate two sets:

$$\begin{gathered}
D_1(u) = \left\{x \in D : \frac{x_2}{u_2} \leq \frac{x_1}{u_1} \leq 1\right\} = \left\{x \in D : 0 \leq x_1 \leq u_1,\ 0 \leq x_2 \leq \frac{u_2}{u_1}x_1\right\}, \\
D_2(u) = \left\{x \in D : \frac{x_1}{u_1} \leq \frac{x_2}{u_2} \leq 1\right\} = \left\{x \in D : 0 \leq x_2 \leq u_2,\ \frac{x_2}{v} \leq x_1 \leq \frac{u_1}{u_2}x_2\right\},
\end{gathered} \tag{3.6}$$

such that $D(u) = D_1(u) \cup D_2(u)$.

In this case, we get

$$\begin{aligned}
\int_{D(u)} \langle u, x\rangle^+ dx_1\, dx_2 &= \frac{1}{u_1}\int_{D_1(u)} x_1\, dx_1\, dx_2 + \frac{1}{u_2}\int_{D_2(u)} x_2\, dx_1\, dx_2 \\
&= \frac{1}{u_1}\int_0^{u_1}\int_0^{(u_2/u_1)x_1} x_1\, dx_1\, dx_2 + \frac{1}{u_2}\int_0^{u_1}\int_0^{(u_1/u_2)x_2} x_2\, dx_1\, dx_2 \\
&= \frac{2vu_1u_2 - u_2^2}{3v}.
\end{aligned} \tag{3.7}$$

Thus, the inequality (2.26) becomes

$$\int_{D(u)} f(x_1, x_2)\, dx_1\, dx_2 \leq \frac{2vu_1u_2 - u_2^2}{3v} f(u_1, u_2) \tag{3.8}$$

for all $u \in D$; it is held.

The set $Q_k(D)$ can be defined for InR functions such that, a point x^* belongs to $Q_k(D)$ if and only if

$$x_2^*\left(1+\frac{3}{a^2}(x_1^*)^2-\frac{4}{a^3}(x_1^*)^3\right)=vx_1^*\left(2-\frac{3}{ak}x_1^*-\frac{2}{a^3}(x_1^*)^3\right). \tag{3.9}$$

The inequalities (2.33) and (2.35) had been obtained for ICR functions. If these inequalities are examined for the same triangle region D, then the following inequality is obtained in [7]:

$$\begin{aligned}\frac{1}{6}\left(2u_1u_2+3a^2v-vu_1^2-3au_2\right)f(u_1,u_2) &\le \int_D f(x_1,x_2)\,dx_1\,dx_2 f(u_1,u_2)\\ &\le \frac{6vu_1^2u_2}{a^3vu_2^2+2vu_1^3u_2^2+a^3v^3u_1^2-u_1^2u_2^3}\end{aligned} \tag{3.10}$$

for all $u \in D$.

The set $Q_k(D)$ has been obtained for ICR functions as formula (2.34). Formula (2.34) becomes formula (3.11) for the triangle region D. That is a point x^* belongs to $Q_k(D)$ if and only if

$$2x_1^*x_2^*-3ax_2^*-v(x_1^*)^2=\frac{3va^2}{k}. \tag{3.11}$$

Lastly, formula (2.44) has been defined for ICAR functions. Now, we will define the same formula for the triangle region D:

$$b(u_1,u_2)\left[\frac{a^3(2u_1v-u_2)}{6u_1^2}-\frac{a^2v}{2}\right]+\frac{a^2v}{2}f(u_1,u_2)\le\int_D f(x_1,x_2)\,dx_1\,dx_2, \tag{3.12}$$

and the inequality is held for all $u \in D$, where $b(u_1,u_2)$ is parameter which depends on f (see [10]).

ICAR functions had been studied for the set $Q(D)$ which is determined by formula (2.45). If $k=1$ in $Q_k(D)$, then the set $Q(D)$ is special case of the given formula (2.8). Then a point $x^* \in D$ belongs to $Q(D)$ if and only if

$$x_2=-\frac{3v}{a}x_1^2+2vx_1. \tag{3.13}$$

In other words, $x^* \in D$ belongs to the parabola by formula (3.13).

4. Conclusion

Hermite-Hadamard type inequalities are investigated for specific functions classes. One of these functions classes is abstract convex functions. The deriving Hermite-Hadamard type inequalities for IPH, InR, ICR, and ICAR functions, which are important classes of abstract convex functions, are investigated by different authors [6–10].

In this article, this problem is considered entirely; findings from [6–10] are summarized; new results are found for some classes; results of some classes are generalized. For example, all results are found for more general $Q_k(D)$ case, not all for $Q(D)$. Even though the results, (2.9), (2.18), (2.35), (2.46), are similar in appearance, they represent different inequalities, since the sets, which are defined with formulas (2.8), (2.17), (2.34), and (2.45), for different classes, are different.

Right-hand side inequalities, which are found for InR functions classes in [9], are considered here as well; more general results are found with the support of $\varphi_u^+(x)$ functions and explained as Proposition 2.9.

ICAR functions, which are studied in [8], are investigated on R_+^2 here, and results are explained in Proposition 2.18 . The inequality, which is explained in formula (2.44), is a new inequality for these functions classes.

Finally, all the results are explained for the same region given on R_{++}^2. Formulas (3.2), (3.8), (3.10), and (3.12) are concrete results of Hermit-Hadamard type inequalities of different abstract convex function classes on given triangle region. Formulas (3.3), (3.9), (3.11), and (3.13) are concrete explanations of $Q_k(D)$ sets in this region.

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