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Research Article

Some Subclasses of Meromorphic Functions Associated with a Family of Integral Operators

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Making use of the principle of subordination between analytic functions and a family of integral operators defined on the space of meromorphic functions, we introduce and investigate some new subclasses of meromorphic functions. Such results as inclusion relationships and integral-preserving properties associated with these subclasses are proved. Several subordination and superordination results involving this family of integral operators are also derived.

1. Introduction and Preliminaries

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C},\ 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \tag{1.2}$$

Let $f, g \in \Sigma$, where f is given by (1.1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k. \tag{1.3}$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.4}$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{1.5}$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z), \tag{1.7}$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}) \tag{1.8}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \tag{1.9}$$

Indeed, it is known that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.10}$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.11}$$

Analogous to the integral operator defined by Jung et al. [1], Lashin [2] introduced and investigated the following integral operator:

$$Q_{\alpha,\beta} : \Sigma \longrightarrow \Sigma \tag{1.12}$$

defined, in terms of the familiar Gamma function, by

$$\begin{aligned} Q_{\alpha,\beta}f(z) &= \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\frac{1}{z^{\beta+1}}\int_0^z t^\beta\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt \\ &= \frac{1}{z}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)}a_k z^k \quad (\alpha>0;\ \beta>0;\ z\in\mathbb{U}^*). \end{aligned} \tag{1.13}$$

By setting

$$f_{\alpha,\beta}(z) := \frac{1}{z}+\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+\alpha+1)}{\Gamma(k+\beta+1)}z^k \quad (\alpha>0;\ \beta>0;\ z\in\mathbb{U}^*), \tag{1.14}$$

we define a new function $f_{\alpha,\beta}^{\lambda}(z)$ in terms of the Hadamard product (or convolution):

$$f_{\alpha,\beta}(z) * f_{\alpha,\beta}^{\lambda}(z) = \frac{1}{z(1-z)^{\lambda}} \quad (\alpha>0;\ \beta>0;\ \lambda>0;\ z\in\mathbb{U}^*). \tag{1.15}$$

Then, motivated essentially by the operator $Q_{\alpha,\beta}$, we now introduce the operator

$$Q_{\alpha,\beta}^{\lambda} : \Sigma \longrightarrow \Sigma, \tag{1.16}$$

which is defined as

$$Q_{\alpha,\beta}^{\lambda}f(z) := f_{\alpha,\beta}^{\lambda}(z) * f(z) \quad (z\in\mathbb{U}^*;\ f\in\Sigma), \tag{1.17}$$

where (and throughout this paper unless otherwise mentioned) the parameters α, β, and λ are constrained as follows:

$$\alpha>0; \qquad \beta>0; \qquad \lambda>0. \tag{1.18}$$

We can easily find from (1.14), (1.15), and (1.17) that

$$Q_{\alpha,\beta}^{\lambda}f(z) = \frac{1}{z}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{(\lambda)_{k+1}}{(k+1)!}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)}a_k z^k \quad (z\in\mathbb{U}^*), \tag{1.19}$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \begin{cases} 1, & (k=0), \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & (k\in\mathbb{N}:=\{1,2,\ldots\}). \end{cases} \tag{1.20}$$

Clearly, we know that $Q_{\alpha,\beta}^{1} = Q_{\alpha,\beta}$.

It is readily verified from (1.19) that

$$z\left(Q_{\alpha,\beta}^{\lambda}f\right)'(z) = \lambda Q_{\alpha,\beta}^{\lambda+1}f(z) - (\lambda+1)Q_{\alpha,\beta}^{\lambda}f(z), \tag{1.21}$$

$$z\left(Q_{\alpha+1,\beta}^{\lambda}f\right)'(z) = (\beta+\alpha)Q_{\alpha,\beta}^{\lambda}f(z) - (\beta+\alpha+1)Q_{\alpha+1,\beta}^{\lambda}f(z). \tag{1.22}$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}^*(\eta;\phi)$, $\mathcal{MK}(\eta;\phi)$, $\mathcal{MC}(\eta,\delta;\phi,\psi)$, and $\mathcal{MQC}(\eta,\delta;\phi,\psi)$ of the class Σ which are defined by

$$\begin{aligned}
\mathcal{MS}^*(\eta;\phi) &:= \left\{f\in\Sigma : \frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)}-\eta\right) \prec \phi(z)\ (\phi\in\mathcal{P};\ 0\leqq\eta<1;\ z\in\mathbb{U})\right\},\\
\mathcal{MK}(\eta;\phi) &:= \left\{f\in\Sigma : \frac{1}{1-\eta}\left(-1-\frac{zf''(z)}{f'(z)}-\eta\right) \prec \phi(z)\ (\phi\in\mathcal{P};\ 0\leqq\eta<1;\ z\in\mathbb{U})\right\},\\
\mathcal{MC}(\eta,\delta;\phi,\psi) &:= \left\{f\in\Sigma : \exists\, g\in\mathcal{MS}^*(\eta;\phi) \text{ such that } \frac{1}{1-\delta}\left(-\frac{zf'(z)}{g(z)}-\delta\right) \prec \psi(z)\right.\\
&\qquad \left.(\phi,\psi\in\mathcal{P};\ 0\leqq\eta,\ \delta<1;\ z\in\mathbb{U})\right\},\\
\mathcal{MQC}(\eta,\delta;\phi,\psi) &:= \left\{f\in\Sigma :\ \exists\, g\in\mathcal{MK}(\eta;\phi) \text{ such that } \frac{1}{1-\delta}\left(-\frac{(zf'(z))'}{g'(z)}-\delta\right) \prec \psi(z)\right.\\
&\qquad \left.(\phi,\psi\in\mathcal{P};\ 0\leqq\eta,\ \delta<1;\ z\in\mathbb{U})\right\}.
\end{aligned} \tag{1.23}$$

Indeed, the above mentioned function classes are generalizations of the general meromorphic starlike, meromorphic convex, meromorphic close-to-convex and meromorphic quasi-convex functions in analytic function theory (see, for details, [3–12]).

Next, by using the operator defined by (1.19), we define the following subclasses $\mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi)$, $\mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi)$, $\mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi)$, and $\mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi)$ of the class Σ:

$$\begin{aligned}
\mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi) &:= \left\{f\in\Sigma : Q_{\alpha,\beta}^{\lambda}f\in\mathcal{MS}^*(\eta;\phi)\right\},\\
\mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi) &:= \left\{f\in\Sigma : Q_{\alpha,\beta}^{\lambda}f\in\mathcal{MK}(\eta;\phi)\right\},\\
\mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi) &:= \left\{f\in\Sigma : Q_{\alpha,\beta}^{\lambda}f\in\mathcal{MC}(\eta,\delta;\phi,\psi)\right\},\\
\mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi) &:= \left\{f\in\Sigma : Q_{\alpha,\beta}^{\lambda}f\in\mathcal{MQC}(\eta,\delta;\phi,\psi)\right\}.
\end{aligned} \tag{1.24}$$

Obviously, we know that

$$f \in \mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi) \Longleftrightarrow -zf' \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi), \tag{1.25}$$

$$f \in \mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi) \Longleftrightarrow -zf' \in \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi). \tag{1.26}$$

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (see [13]). *Let $\kappa, \vartheta \in \mathbb{C}$. Suppose also that $\mathfrak{m}$ is convex and univalent in $\mathbb{U}$ with*

$$\mathfrak{m}(0) = 1, \qquad \Re(\kappa\mathfrak{m}(z) + \vartheta) > 0 \quad (z \in \mathbb{U}). \tag{1.27}$$

If $\mathfrak{u}$ is analytic in $\mathbb{U}$ with $\mathfrak{u}(0) = 1$, then the subordination

$$\mathfrak{u}(z) + \frac{z\mathfrak{u}'(z)}{\kappa\mathfrak{u}(z) + \vartheta} \prec \mathfrak{m}(z) \tag{1.28}$$

implies that

$$\mathfrak{u}(z) \prec \mathfrak{m}(z). \tag{1.29}$$

Lemma 1.2 (see [14]). *Let h be convex univalent in $\mathbb{U}$ and let ζ be analytic in $\mathbb{U}$ with*

$$\Re(\zeta(z)) \geqq 0 \quad (z \in \mathbb{U}). \tag{1.30}$$

If q is analytic in $\mathbb{U}$ and $q(0) = h(0)$, then the subordination

$$q(z) + \zeta(z)zq'(z) \prec h(z) \tag{1.31}$$

implies that

$$q(z) \prec h(z). \tag{1.32}$$

The main purpose of the present paper is to investigate some inclusion relationships and integral-preserving properties of the subclasses

$$\mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi), \qquad \mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi), \qquad \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi), \qquad \mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi) \tag{1.33}$$

of meromorphic functions involving the operator $\mathcal{Q}_{\alpha,\beta}^{\lambda}$. Several subordination and superordination results involving this operator are also derived.

2. The Main Inclusion Relationships

We begin by presenting our first inclusion relationship given by Theorem 2.1.

Theorem 2.1. *Let* $0 \leqq \eta < 1$ *and* $\phi \in \mathcal{P}$ *with*

$$\max_{z\in\mathbb{U}}\{\Re(\phi(z))\} < \min\left\{\frac{\lambda-\eta+1}{1-\eta}, \frac{\beta+\alpha-\eta+1}{1-\eta}\right\} \quad (z \in \mathbb{U}). \tag{2.1}$$

Then

$$\mathcal{MS}_{\alpha,\beta}^{\lambda+1}(\eta;\phi) \subset \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi) \subset \mathcal{MS}_{\alpha+1,\beta}^{\lambda}(\eta;\phi). \tag{2.2}$$

Proof. We first prove that

$$\mathcal{MS}_{\alpha,\beta}^{\lambda+1}(\eta;\phi) \subset \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi). \tag{2.3}$$

Let $f \in \mathcal{MS}_{\alpha,\beta}^{\lambda+1}(\eta;\phi)$ and suppose that

$$\mathfrak{h}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda}f(z)} - \eta\right), \tag{2.4}$$

where $\mathfrak{h}$ is analytic in $\mathbb{U}$ with $\mathfrak{h}(0) = 1$. Combining (1.21) and (2.4), we find that

$$\lambda\frac{Q_{\alpha,\beta}^{\lambda+1}f(z)}{Q_{\alpha,\beta}^{\lambda}f(z)} = -(1-\eta)\mathfrak{h}(z) - \eta + \lambda + 1. \tag{2.5}$$

Taking the logarithmical differentiation on both sides of (2.5) and multiplying the resulting equation by z, we get

$$\frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda+1}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda+1}f(z)} - \eta\right) = \mathfrak{h}(z) + \frac{z\mathfrak{h}'(z)}{-(1-\eta)\mathfrak{h}(z) - \eta + \lambda + 1} \prec \phi(z). \tag{2.6}$$

By virtue of (2.1), an application of Lemma 1.1 to (2.6) yields $\mathfrak{h} \prec \phi$, that is $f \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi)$. Thus, the assertion (2.3) of Theorem 2.1 holds.

To prove the second part of Theorem 2.1, we assume that $f \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi)$ and set

$$\mathfrak{g}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha+1,\beta}^{\lambda}f\right)'(z)}{Q_{\alpha+1,\beta}^{\lambda}f(z)} - \eta\right), \tag{2.7}$$

where $\mathfrak{g}$ is analytic in $\mathbb{U}$ with $\mathfrak{g}(0) = 1$. Combining (1.22), (2.1), and (2.7) and applying the similar method of proof of the first part, we get $\mathfrak{g} \prec \phi$, that is $f \in \mathcal{MS}^{\lambda}_{\alpha+1,\beta}(\eta;\phi)$. Therefore, the second part of Theorem 2.1 also holds. The proof of Theorem 2.1 is evidently completed. □

Theorem 2.2. *Let* $0 \leqq \eta < 1$ *and* $\phi \in \mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MK}^{\lambda+1}_{\alpha,\beta}(\eta;\phi) \subset \mathcal{MK}^{\lambda}_{\alpha,\beta}(\eta;\phi) \subset \mathcal{MK}^{\lambda}_{\alpha+1,\beta}(\eta;\phi). \tag{2.8}$$

Proof. In view of (1.25) and Theorem 2.1, we find that

$$\begin{aligned}
f \in \mathcal{MK}^{\lambda+1}_{\alpha,\beta}(\eta;\phi) &\Longleftrightarrow \mathcal{Q}^{\lambda+1}_{\alpha,\beta} f \in \mathcal{MK}(\eta;\phi) \\
&\Longleftrightarrow -z\left(\mathcal{Q}^{\lambda+1}_{\alpha,\beta} f\right)' \in \mathcal{MS}^{*}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{Q}^{\lambda+1}_{\alpha,\beta}(-zf') \in \mathcal{MS}^{*}(\eta;\phi) \\
&\Longleftrightarrow -zf' \in \mathcal{MS}^{\lambda+1}_{\alpha,\beta}(\eta;\phi) \\
&\Longrightarrow -zf' \in \mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{Q}^{\lambda}_{\alpha,\beta}(-zf') \in \mathcal{MS}^{*}(\eta;\phi) \\
&\Longleftrightarrow -z\left(\mathcal{Q}^{\lambda}_{\alpha,\beta} f\right) \in \mathcal{MS}^{*}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{Q}^{\lambda}_{\alpha,\beta} f \in \mathcal{MK}(\eta;\phi) \\
&\Longleftrightarrow f \in \mathcal{MK}^{\lambda}_{\alpha,\beta}(\eta;\phi),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
f \in \mathcal{MK}^{\lambda}_{\alpha,\beta}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi) \\
&\Longrightarrow -zf' \in \mathcal{MS}^{\lambda}_{\alpha+1,\beta}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{Q}^{\lambda}_{\alpha+1,\beta}(-zf') \in \mathcal{MS}^{*}(\eta;\phi) \\
&\Longleftrightarrow \mathcal{Q}^{\lambda}_{\alpha+1,\beta} f \in \mathcal{MK}(\eta;\phi) \\
&\Longleftrightarrow f \in \mathcal{MK}^{\lambda}_{\alpha+1,\beta}(\eta;\phi).
\end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), we deduce that the assertion of Theorem 2.2 holds. □

Theorem 2.3. *Let* $0 \leqq \eta < 1$, $0 \leqq \delta < 1$ *and* $\phi, \psi \in \mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MC}^{\lambda+1}_{\alpha,\beta}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}^{\lambda}_{\alpha+1,\beta}(\eta,\delta;\phi,\psi). \tag{2.11}$$

Proof. We begin by proving that

$$\mathcal{MC}_{\alpha,\beta}^{\lambda+1}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi). \tag{2.12}$$

Let $f \in \mathcal{MC}_{\alpha,\beta}^{\lambda+1}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that

$$\frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda+1}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda+1}g(z)}-\delta\right) \prec \psi(z) \tag{2.13}$$

with $g \in \mathcal{MS}_{\alpha,\beta}^{\lambda+1}(\eta;\phi)$, Moreover, by Theorem 2.1, we know that $g \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi)$, which implies that

$$\mathfrak{q}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}g\right)'(z)}{Q_{\alpha,\beta}^{\lambda}g(z)}-\eta\right) \prec \phi(z). \tag{2.14}$$

We now suppose that

$$\mathfrak{p}(z) := \frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda}g(z)}-\delta\right), \tag{2.15}$$

where $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$. Combining (1.21) and (2.15), we find that

$$-[(1-\delta)\mathfrak{p}(z)+\delta]Q_{\alpha,\beta}^{\lambda}g(z) = \lambda Q_{\alpha,\beta}^{\lambda+1}f(z) - (\lambda+1)Q_{\alpha,\beta}^{\lambda}f(z). \tag{2.16}$$

Differentiating both sides of (2.16) with respect to z and multiplying the resulting equation by z, we get

$$-(1-\delta)z\mathfrak{p}'(z) - [(1-\delta)\mathfrak{p}(z)+\delta]\left[-(1-\eta)\mathfrak{q}(z)-\eta+\lambda+1\right] = \lambda\frac{z\left(Q_{\alpha,\beta}^{\lambda+1}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda}g(z)}. \tag{2.17}$$

In view of (1.21), (2.14), and (2.17), we conclude that

$$\frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda+1}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda+1}g(z)}-\delta\right) = \mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{-(1-\eta)\mathfrak{q}(z)-\eta+\lambda+1} \prec \psi(z). \tag{2.18}$$

By noting that (2.1) holds and

$$\mathfrak{q}(z) \prec \phi(z), \tag{2.19}$$

we know that

$$\Re(-(1-\eta)\mathfrak{q}(z)-\eta+\lambda+1)>0. \tag{2.20}$$

Thus, an application of Lemma 1.2 to (2.18) yields

$$\mathfrak{p}(z)\prec\psi(z), \tag{2.21}$$

that is $f\in\mathcal{MC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)$, which implies that the assertion (2.12) of Theorem 2.3 holds.

By virtue of (1.22) and (2.1), making use of the similar arguments of the details above, we deduce that

$$\mathcal{MC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)\subset\mathcal{MC}^{\lambda}_{\alpha+1,\beta}(\eta,\delta;\phi,\psi). \tag{2.22}$$

The proof of Theorem 2.3 is thus completed. □

Theorem 2.4. *Let* $0\leqq\eta<1$, $0\leqq\delta<1$ *and* $\phi,\psi\in\mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MQC}^{\lambda+1}_{\alpha,\beta}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{\lambda}_{\alpha+1,\beta}(\eta,\delta;\phi,\psi). \tag{2.23}$$

Proof. In view of (1.26) and Theorem 2.3, and by similarly applying the method of proof of Theorem 2.2, we conclude that the assertion of Theorem 2.4 holds. □

3. A Set of Integral-Preserving Properties

In this section, we derive some integral-preserving properties involving two families of integral operators.

Theorem 3.1. *Let* $f\in\mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$ *with* $\phi\in\mathcal{P}$ *and*

$$\Re(\phi(z))<\frac{\Re(\nu)-\eta}{1-\eta}\quad(z\in\mathbb{U};\ \Re(\nu)>1). \tag{3.1}$$

Then the integral operator $F_\nu(f)$ *defined by*

$$F_\nu(f):=F_\nu(f)(z)=\frac{\nu-1}{z^\nu}\int_0^z t^{\nu-1}f(t)dt\quad(z\in\mathbb{U};\ \Re(\nu)>1) \tag{3.2}$$

belongs to the class $\mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$.

Proof. Let $f\in\mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$. Then, from (3.2), we find that

$$z\left(\mathcal{Q}^{\lambda}_{\alpha,\beta}F_\nu(f)\right)'(z)+\nu\mathcal{Q}^{\lambda}_{\alpha,\beta}F_\nu(f)(z)=(\nu-1)\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z). \tag{3.3}$$

By setting

$$\mathbb{P}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}F_{\nu}(f)\right)'(z)}{Q_{\alpha,\beta}^{\lambda}F_{\nu}(f)(z)} - \eta\right), \tag{3.4}$$

we observe that $\mathbb{P}$ is analytic in $\mathbb{U}$ with $\mathbb{P}(0) = 0$. It follows from (3.3) and (3.4) that

$$-(1-\eta)\mathbb{P}(z) - \eta + \nu = (\nu - 1)\frac{Q_{\alpha,\beta}^{\lambda}f(z)}{Q_{\alpha,\beta}^{\lambda}F_{\nu}(f)(z)}. \tag{3.5}$$

Differentiating both sides of (3.5) with respect to z logarithmically and multiplying the resulting equation by z, we get

$$\mathbb{P}(z) + \frac{z\mathbb{P}'(z)}{-(1-\eta)\mathbb{P}(z) - \eta + \nu} = \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}f\right)'(z)}{Q_{\alpha,\beta}^{\lambda}f(z)} - \eta\right) \prec \phi(z). \tag{3.6}$$

Since (3.1) holds, an application of Lemma 1.1 to (3.6) yields

$$\frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda}F_{\nu}(f)\right)'(z)}{Q_{\alpha,\beta}^{\lambda}F_{\nu}(f)(z)} - \eta\right) \prec \phi(z), \tag{3.7}$$

which implies that the assertion of Theorem 3.1 holds. □

Theorem 3.2. *Let $f \in \mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and* (3.1) *holds. Then the integral operator $F_{\nu}(f)$ defined by* (3.2) *belongs to the class $\mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi)$.*

Proof. By virtue of (1.25) and Theorem 3.1, we easily find that

$$\begin{aligned} f \in \mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi) \\ &\Longrightarrow F_{\nu}(-zf') \in \mathcal{MS}_{\alpha,\beta}^{\lambda}(\eta;\phi) \\ &\Longleftrightarrow -z(F_{\nu}(f))' \in \mathcal{MS}^{*}(\eta;\phi) \\ &\Longleftrightarrow F_{\nu}(f) \in \mathcal{MK}_{\alpha,\beta}^{\lambda}(\eta;\phi). \end{aligned} \tag{3.8}$$

The proof of Theorem 3.2is evidently completed. □

Theorem 3.3. *Let* $f \in \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta, \delta; \phi, \psi)$ *with* $\phi \in \mathcal{P}$ *and* (3.1) *holds. Then the integral operator* $F_\nu(f)$ *defined by* (3.2) *belongs to the class* $\mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta, \delta; \phi, \psi)$.

Proof. Let $f \in \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta, \delta; \phi, \psi)$. Then, by definition, we know that there exists a function $g \in \mathcal{MS}^*(\eta; \phi)$ such that

$$\frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda} f\right)'(z)}{Q_{\alpha,\beta}^{\lambda} g(z)} - \eta\right) \prec \psi(z). \tag{3.9}$$

Since $g \in \mathcal{MS}^*(\eta; \phi)$, by Theorem 3.1, we easily find that $F_\nu(g) \in \mathcal{MS}^*(\eta; \phi)$, which implies that

$$\mathbb{H}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda} F_\nu(g)\right)'(z)}{Q_{\alpha,\beta}^{\lambda} F_\nu(g)(z)} - \eta\right) \prec \phi(z). \tag{3.10}$$

We now set

$$\mathbb{Q}(z) := \frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda} F_\nu(f)\right)'(z)}{Q_{\alpha,\beta}^{\lambda} F_\nu(g)(z)} - \delta\right), \tag{3.11}$$

where $\mathbb{Q}$ is analytic in $\mathbb{U}$ with $\mathbb{Q}(0) = 1$. From (3.3), and (3.11), we get

$$-[(1-\delta)\mathbb{Q}(z) + \delta] Q_{\alpha,\beta}^{\lambda} F_\nu(g)(z) + \nu Q_{\alpha,\beta}^{\lambda} F_\nu(f)(z) = (\nu - 1) Q_{\alpha,\beta}^{\lambda} f(z). \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), we find that

$$-(1-\delta) z\mathbb{Q}'(z) - [(1-\delta)\mathbb{Q}(z) + \delta]\left[-(1-\eta)\mathbb{H}(z) - \eta + \nu\right] = (\nu - 1)\frac{z\left(Q_{\alpha,\beta}^{\lambda} f\right)'(z)}{Q_{\alpha,\beta}^{\lambda} F_\nu(g)(z)}. \tag{3.13}$$

By virtue of (1.21), (3.10), and (3.13), we deduce that

$$\frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^{\lambda} f\right)'(z)}{Q_{\alpha,\beta}^{\lambda} g(z)} - \delta\right) = \mathbb{Q}(z) + \frac{z\mathbb{Q}'(z)}{-(1-\eta)\mathbb{H}(z) - \eta + \nu} \prec \psi(z). \tag{3.14}$$

The remainder of the proof of Theorem 3.3 is much akin to that of Theorem 2.3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{Q}(z) \prec \psi(z), \tag{3.15}$$

which implies that $F_\nu(f) \in \mathcal{MC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)$. The proof of Theorem 3.3 is thus completed. □

Theorem 3.4. *Let $f \in \mathcal{MQC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and* (3.1) *holds. Then the integral operator $F_\nu(f)$ defined by* (3.2) *belongs to the class $\mathcal{MQC}^{\lambda}_{\alpha,\beta}(\eta,\delta;\phi,\psi)$.*

Proof. In view of (1.26) and Theorem 3.3, and by similarly applying the method of proof of Theorem 3.2, we deduce that the assertion of Theorem 3.4 holds. □

Theorem 3.5. *Let $f \in \mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and*

$$\Re\left(\sigma - \eta\xi - (1-\eta)\xi\phi(z)\right) > 0 \quad (z \in \mathbb{U};\ \xi \neq 0). \tag{3.16}$$

Then the function $K^{\sigma}_{\xi}(f) \in \Sigma$ defined by

$$\begin{aligned} Q^{\lambda}_{\alpha,\beta}K^{\sigma}_{\xi}(f) &:= Q^{\lambda}_{\alpha,\beta}K^{\sigma}_{\xi}(f)(z) \\ &= \left(\frac{\sigma-\xi}{z^{\sigma}}\int_0^z t^{\sigma-1}\left(Q^{\lambda}_{\alpha,\beta}f(t)\right)^{\xi}dt\right)^{1/\xi} \quad (z \in \mathbb{U}^*;\ \xi \neq 0) \end{aligned} \tag{3.17}$$

belongs to the class $\mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$.

Proof. Let $f \in \mathcal{MS}^{\lambda}_{\alpha,\beta}(\eta;\phi)$ and suppose that

$$\mathbb{M}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q^{\lambda}_{\alpha,\beta}K^{\sigma}_{\xi}(f)\right)'(z)}{Q^{\lambda}_{\alpha,\beta}K^{\sigma}_{\xi}(f)(z)} - \eta\right). \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\sigma - \eta\xi - (1-\eta)\xi\mathbb{M}(z) = (\sigma-\xi)\left(\frac{Q^{\lambda}_{\alpha,\beta}f(z)}{Q^{\lambda}_{\alpha,\beta}K^{\sigma}_{\xi}(f)(z)}\right)^{\xi}. \tag{3.19}$$

Now, in view of (3.17), (3.18), and (3.19), we get

$$\mathbb{M}(z) + \frac{z\mathbb{M}'(z)}{\sigma - \eta\xi - (1-\eta)\xi\mathbb{M}(z)} = \frac{1}{1-\eta}\left(-\frac{z\left(Q^{\lambda}_{\alpha,\beta}f\right)'(z)}{Q^{\lambda}_{\alpha,\beta}f(z)} - \eta\right) \prec \phi(z). \tag{3.20}$$

Since (3.16) holds, an application of Lemma 1.1 to (3.20) yields

$$\mathbb{M}(z) \prec \phi(z), \tag{3.21}$$

that is, $K_\xi^\sigma(f) \in \mathcal{MS}_{\alpha,\beta}^\lambda(\eta;\phi)$. We thus complete the proof of Theorem 3.5. □

Theorem 3.6. *Let* $f \in \mathcal{MK}_{\alpha,\beta}^\lambda(\eta;\phi)$ *with* $\phi \in \mathcal{P}$ *and* (3.16) *holds. Then the function* $K_\xi^\sigma(f) \in \Sigma$ *defined by* (3.17) *belongs to the class* $\mathcal{MK}_{\alpha,\beta}^\lambda(\eta;\phi)$.

Proof. By virtue of (1.25) and Theorem 3.5, and by similarly applying the method of proof of Theorem 3.2, we conclude that the assertion of Theorem 3.6 holds. □

Theorem 3.7. *Let* $f \in \mathcal{MC}_{\alpha,\beta}^\lambda(\eta,\delta;\phi,\psi)$ *with* $\phi \in \mathcal{P}$ *and* (3.16) *holds. Then the function* $K_\xi^\sigma(f) \in \Sigma$ *defined by* (3.17) *belongs to the class* $\mathcal{MC}_{\alpha,\beta}^\lambda(\eta,\delta;\phi,\psi)$.

Proof. Let $f \in \mathcal{MC}_{\alpha,\beta}^\lambda(\eta,\delta;\phi,\psi)$. Then, by definition, we know that there exists a function $g \in \mathcal{MS}^*(\eta;\phi)$ such that (3.9) holds. Since $g \in \mathcal{MS}^*(\eta;\phi)$, by Theorem 3.5, we easily find that $K_\xi^\sigma(g) \in \mathcal{MS}^*(\eta;\phi)$, which implies that

$$\mathbb{R}(z) := \frac{1}{1-\eta}\left(-\frac{z\left(Q_{\alpha,\beta}^\lambda K_\xi^\sigma(g)\right)'(z)}{Q_{\alpha,\beta}^\lambda K_\xi^\sigma(g)(z)} - \eta\right) \prec \phi(z). \tag{3.22}$$

We now set

$$\mathbb{D}(z) := \frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^\lambda K_\xi^\sigma(f)\right)'(z)}{Q_{\alpha,\beta}^\lambda K_\xi^\sigma(g)(z)} - \delta\right), \tag{3.23}$$

where $\mathbb{D}$ is analytic in $\mathbb{U}$ with $\mathbb{D}(0) = 1$. From (3.17) and (3.23), we get

$$-\xi[(1-\delta)\mathbb{D}(z)+\delta]Q_{\alpha,\beta}^\lambda K_\xi^\sigma(g)(z) + \delta Q_{\alpha,\beta}^\lambda K_\xi^\sigma(f)(z) = (\delta-\xi)Q_{\alpha,\beta}^\lambda f(z). \tag{3.24}$$

Combining (3.22), (3.23), and (3.24), we find that

$$-\xi(1-\delta)z\mathbb{D}'(z) - [(1-\delta)\mathbb{D}(z)+\delta]\left[-(1-\eta)\xi\mathbb{R}(z) - \eta\xi + \delta\right] = (\delta-\xi)\frac{z\left(Q_{\alpha,\beta}^\lambda f\right)'(z)}{Q_{\alpha,\beta}^\lambda K_\xi^\sigma(g)(z)}. \tag{3.25}$$

Furthermore, by virtue of (1.22), (3.22), and (3.25), we deduce that

$$\frac{1}{1-\delta}\left(-\frac{z\left(Q_{\alpha,\beta}^\lambda f\right)'(z)}{Q_{\alpha,\beta}^\lambda g(z)} - \delta\right) = \mathbb{D}(z) + \frac{z\mathbb{D}'(z)}{-(1-\eta)\xi\mathbb{R}(z) - \eta\xi + \delta} \prec \psi(z). \tag{3.26}$$

The remainder of the proof of Theorem 3.7 is similar to that of Theorem 2.3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{D}(z) \prec \psi(z), \tag{3.27}$$

which implies that $K_{\xi}^{\sigma}(f) \in \mathcal{MC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi)$. The proof of Theorem 3.7 is thus completed. □

Theorem 3.8. *Let $f \in \mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and* (3.16) *holds. Then the function $K_{\xi}^{\sigma}(f) \in \Sigma$ defined by* (3.17) *belongs to the class $\mathcal{MQC}_{\alpha,\beta}^{\lambda}(\eta,\delta;\phi,\psi)$.*

Proof. By virtue of (1.26) and Theorem 3.7, and by similarly applying the method of proof of Theorem 3.2, we deduce that the assertion of Theorem 3.8 holds. □

4. Subordination and Superordination Results

In this section, we derive some subordination and superordination results associated with the operator $Q_{\alpha,\beta}^{\lambda}$. By similarly applying the methods of proof of the results obtained by Cho et al. [15], we get the following subordination and superordination results. Here, we choose to omit the details involved. For some other recent sandwich-type results in analytic function theory, one can find in [16–30] and the references cited therein.

Corollary 4.1. *Let $f, g \in \Sigma$. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho \quad \left(z \in \mathbb{U};\ \varphi(z) := zQ_{\alpha,\beta}^{\lambda}g(z)\right), \tag{4.1}$$

where

$$\varrho := \frac{1+(\beta+\alpha)^2-\left|1-(\beta+\alpha)^2\right|}{4(\beta+\alpha)}, \tag{4.2}$$

then the subordination relationship

$$zQ_{\alpha,\beta}^{\lambda}f(z) \prec zQ_{\alpha,\beta}^{\lambda}g(z) \tag{4.3}$$

implies that

$$zQ_{\alpha+1,\beta}^{\lambda}f(z) \prec zQ_{\alpha+1,\beta}^{\lambda}g(z). \tag{4.4}$$

Furthermore, the function $zQ_{\alpha+1,\beta}^{\lambda}g$ is the best dominant.

Corollary 4.2. *Let $f, g \in \Sigma$. If*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi \quad \left(z \in \mathbb{U};\ \chi(z) := zQ_{\alpha,\beta}^{\lambda+1}g(z)\right), \tag{4.5}$$

where

$$\varpi := \frac{1+\lambda^2-\left|1-\lambda^2\right|}{4\lambda}, \tag{4.6}$$

then the subordination relationship

$$zQ_{\alpha,\beta}^{\lambda+1}f(z) \prec zQ_{\alpha,\beta}^{\lambda+1}g(z) \tag{4.7}$$

implies that

$$zQ_{\alpha,\beta}^{\lambda}f(z) \prec zQ_{\alpha,\beta}^{\lambda}g(z). \tag{4.8}$$

Furthermore, the function $zQ_{\alpha,\beta}^{\lambda}g$ is the best dominant.

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} : \lim_{z\to\varepsilon} f(z) = \infty\right\}, \tag{4.9}$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$. If f is subordinate to $\digamma$, then $\digamma$ is superordinate to f. We now derive the following superordination results.

Corollary 4.3. *Let $f, g \in \Sigma$. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho \quad \left(z \in \mathbb{U};\ \varphi(z) := zQ_{\alpha,\beta}^{\lambda}g(z)\right), \tag{4.10}$$

where ϱ is given by (4.2)*, also let the function $zQ_{\alpha,\beta}^{\lambda}f$ be univalent in $\mathbb{U}$ and $zQ_{\alpha+1,\beta}^{\lambda}f \in Q$, then the subordination relationship*

$$zQ_{\alpha,\beta}^{\lambda}g(z) \prec zQ_{\alpha,\beta}^{\lambda}f(z) \tag{4.11}$$

implies that

$$zQ_{\alpha+1,\beta}^{\lambda}g(z) \prec zQ_{\alpha+1,\beta}^{\lambda}f(z). \tag{4.12}$$

Furthermore, the function $zQ_{\alpha+1,\beta}^{\lambda}g$ is the best subordinant.

Corollary 4.4. *Let $f, g \in \Sigma$. If*

$$\Re\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi \quad \left(z \in \mathbb{U};\ \chi(z) := zQ_{\alpha,\beta}^{\lambda+1}g(z)\right), \tag{4.13}$$

where ϖ is given by (4.6), *also let the function $zQ_{\alpha,\beta}^{\lambda+1}f$ be univalent in $\mathbb{U}$ and $zQ_{\alpha,\beta}^{\lambda}f \in Q$, then the subordination relationship*

$$zQ_{\alpha,\beta}^{\lambda+1}g(z) \prec zQ_{\alpha,\beta}^{\lambda+1}f(z) \tag{4.14}$$

implies that

$$zQ_{\alpha,\beta}^{\lambda}g(z) \prec zQ_{\alpha,\beta}^{\lambda}f(z). \tag{4.15}$$

Furthermore, the function $zQ_{\alpha,\beta}^{\lambda}g$ is the best subordinant.

Combining the above mentioned subordination and superordination results involving the operator $Q_{\alpha,\beta}^{\lambda}$, we get the following "sandwich-type results".

Corollary 4.5. *Let $f, g_k \in \Sigma$ $(k = 1, 2)$. If*

$$\Re\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\varrho \quad \left(z \in \mathbb{U};\ \varphi_k(z) := zQ_{\alpha,\beta}^{\lambda}g_k(z)\ (k = 1, 2)\right), \tag{4.16}$$

where ϱ is given by (4.2), *also let the function $zQ_{\alpha,\beta}^{\lambda}f$ be univalent in $\mathbb{U}$ and $zQ_{\alpha+1,\beta}^{\lambda}f \in Q$, then the subordination chain*

$$zQ_{\alpha,\beta}^{\lambda}g_1(z) \prec zQ_{\alpha,\beta}^{\lambda}f(z) \prec zQ_{\alpha,\beta}^{\lambda}g_2(z) \tag{4.17}$$

implies that

$$zQ_{\alpha+1,\beta}^{\lambda}g_1(z) \prec zQ_{\alpha+1,\beta}^{\lambda}f(z) \prec zQ_{\alpha+1,\beta}^{\lambda}g_2(z). \tag{4.18}$$

Furthermore, the functions $zQ_{\alpha+1,\beta}^{\lambda}g_1$ and $zQ_{\alpha+1,\beta}^{\lambda}g_2$ are, respectively, the best subordinant and the best dominant.

Corollary 4.6. *Let $f, g_k \in \Sigma$ $(k = 1, 2)$. If*

$$\Re\left(1 + \frac{z\chi_{k''}(z)}{\chi_{k'}(z)}\right) > -\varpi \quad \left(z \in \mathbb{U};\ \chi_k(z) := zQ_{\alpha,\beta}^{\lambda+1}g_k(z)\ (k = 1, 2)\right), \tag{4.19}$$

where ϖ *is given by* (4.6), *also let the function* $zQ_{\alpha,\beta}^{\lambda+1}f$ *be univalent in* $\mathbb{U}$ *and* $zQ_{\alpha,\beta}^{\lambda}f \in Q$, *then the subordination chain*

$$zQ_{\alpha,\beta}^{\lambda+1}g_1(z) \prec zQ_{\alpha,\beta}^{\lambda+1}f(z) \prec zQ_{\alpha,\beta}^{\lambda+1}g_2(z) \tag{4.20}$$

implies that

$$zQ_{\alpha,\beta}^{\lambda}g_1(z) \prec zQ_{\alpha,\beta}^{\lambda}f(z) \prec zQ_{\alpha,\beta}^{\lambda}g_2(z). \tag{4.21}$$

Furthermore, the functions $zQ_{\alpha,\beta}^{\lambda}g_1$ *and* $zQ_{\alpha,\beta}^{\lambda}g_2$ *are, respectively, the best subordinant and the best dominant.*

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