Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 892936, 17 pages
doi:10.1155/2009/892936

Research Article

Sufficient and Necessary Conditions for Oscillation of nth-Order Differential Equation with Retarded Argument

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Received 10 July 2009; Revised 11 November 2009; Accepted 9 December 2009

Recommended by Martin Bohner

Necessary and sufficient conditions are found for oscillation of the solutions of a class of strongly superlinear and strongly sublinear differential equations of even order with retarded argument.

1. Introduction

We consider the following nth-order differential equation with retarded argument:

$$x^{(n)}(t) + f(t, x(t), x(\tau(t))) = 0, \quad n \text{ is even.} \tag{1.1}$$

Firstly, we introduce several conditions as follows:

$(H1)$ $f \in C(R_+ \times R^2, R)$, $uf(t, u, v) > 0$ for $uv > 0$ and $t \in R_+$.

$(H2)$ $\tau \in C(R_+, R)$, $\tau(t) \le t$ for $t \in R_+$ and $\lim_{t\to\infty}\tau(t) = \infty$.

As customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

Definition 1.1. The function $f(t, u, v)$ is said to be strongly superlinear if there exists $\alpha > 1$, such that $|f(t, u, v)|/|u|^\alpha$ is a nondecreasing function with respect to $|u|, |v|$ for each fixed $t \in R_+$.

It is easy to see that the function $|f(t,u,v)|/|u|$ is nondecreasing with respect to $|u|,|v|$ for $t \in R_+$ if $f(t,u,v)$ is strongly superlinear. The function $|f(t,u,v)|$ is nondecreasing with respect to $|u|,|v|$ for $t \in R_+$ if $|f(t,u,v)|/|u|$ is nondecreasing with respect to $|u|,|v|$.

Definition 1.2. The function $f(t,u,v)$ is said to be strongly sublinear if there exists $0 < \beta < 1$, such that $|f(t,u,v)|/|u|^{\beta}$ is a nonincreasing function with respect to $|u|,|v|$ for each fixed $t \in R_+$.

We should indicate that there are many ways in which one can define the concept of strongly superlinearity, superlinearity, strongly sublinearity and sublinearity, to characterize functions satisfying different conditions. For example, in [1] the strongly superlinearity is used to specify functions with specific behavior at 0 and ∞; in [2] the superlinearity and sublinearity are defined for multivariable functions. In this paper, we adopt the definitions as in monograph [3].

In particular, if $f(t,x(t),x(\tau(t)) = p(t)x^{\gamma}(t)$, where $p(t) \in C([t_0,\infty),R^{+})$, $t_0 > 0$, and γ is the quotient of odd positive integers, then (1.1) becomes

$$x^{(n)}(t) + p(t)x^{\gamma}(t) = 0. \tag{1.2}$$

It is easy to see that $p(t)x^{\gamma}(t)$ is strongly superlinear for $\gamma > 1$ and $p(t)x^{\gamma}(t)$ is strongly sublinear for $0 < \gamma < 1$. If $n = 2$; then (1.2) reduces to

$$x''(t) + p(t)x^{\gamma}(t) = 0, \tag{1.3}$$

Equation (1.3) is the well-known Emden-Fowler equation [4].

Recently, many remarkable results have been established for the oscillation of solutions of the second- and higher-order functional differential equations. For example, Theorem A is presented in [2].

Theorem A. *If $\gamma = 1$, then every bounded solution of* (1.2) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1}p(s)ds = \infty. \tag{1.4}$$

For (1.3), the well-known Theorems B–D are presented in [5–7].

Theorem B (see [5, 7]). *If $\gamma > 0$, then* (1.3) *has a bounded nonoscillatory solution if and only if*

$$\int_{t_0}^{\infty} sp(s)ds < \infty. \tag{1.5}$$

Theorem C (see [5]). *If $\gamma > 1$, then all solutions of* (1.3) *are oscillatory if and only if*

$$\int_{t_0}^{\infty} sp(s)ds = \infty. \tag{1.6}$$

Theorem D (see [7]). *If* $0 < \gamma < 1$, *then* (1.3) *is oscillatory if and only if*

$$\int_{t_0}^{\infty} s^{\gamma} p(s) ds = \infty. \tag{1.7}$$

In [8], Waltman studied the oscillation of the solutions for the equation

$$x''(t) + p(t) f(x(t)) = 0. \tag{1.8}$$

Equation (1.8) is the prototype of (1.1) and (1.2). Theorems E and F were proved in [8].

Theorem E. *If* $f(x(t))$ *satisfies* (i) $f(0) = 0$ *and* $f(x) \neq 0$ *for* $x \neq 0$ *and* (ii) $f'(x)$ *is continuous and non-negative, then* (1.8) *has a bounded and eventually monotonic solution if and only if*

$$\int^{\infty} tp(t) dt < \infty. \tag{1.9}$$

Theorem F. *Suppose that the conditions* (i) *and* (ii) *in Theorem E are satisfied. If*

$$\liminf_{x \to \infty} \frac{|f(x)|}{|x|^{\alpha}} \neq 0 \tag{1.10}$$

for some $\alpha > 1$, *then all solutions of* (1.8) *are oscillatory if and only if*

$$\int^{\infty} tp(t) dt = \infty. \tag{1.11}$$

Some other related results can be found in [2, 4, 9–12] and the references cited therein. Due to some problems of theoretical and technical character in handling with higher-order nonlinear differential equations, there are only a few results which concern necessary and sufficient conditions for the oscillatory behavior for (1.1). So there are a lot of things worth further consideration for (1.1). The main purpose of this paper is to establish necessary and sufficient conditions for (1.1). The obtained results extend the above theorems.

2. Main Results

In order to establish our main results we need introduce and establish two lemmas.

Lemma 2.1 (see [13–15]). *If* $x(t)$ *is a positive and* n*-times differentiable function on* $[t_0, \infty)$, *and* $x^{(n)}(t)$ *is nonpositive and not identically zero on any subinterval* $[t_1, \infty)$, *then there exist* $T \geq t_0$ *and an integer* $k \in \{0, 1, \ldots, n-1\}$ *such that* $n + k$ *is odd and*

(i) $x^{(i)}(t) \geq 0$ *for* $t \geq T$, $i = 0, 1, \ldots, k-1$;

(ii) $(-1)^{i+k} x^{(i)}(t) > 0$ *for* $i = k, k+1, \ldots, n$;

(iii) $(t - T)|x^{(k-i)}(t)| \leq (1 + i)|x^{(k-i-1)}(t)|$ *for* $t \geq T$, $i = 0, 1, \ldots, k-1$, $k = 1, \ldots, n-1$.

Lemma 2.2. *If $f(t,u,v)$ is a strongly sublinear function, then*

$$\frac{f(t,u_1,v_1)}{u_1} \geq \frac{f(t,u_2,v_2)}{u_2} \tag{2.1}$$

for $0 < u_1 \leq u_2$ and $0 < v_1 \leq v_2$.

Proof. From $0 < u_1 \leq u_2$ and $0 < v_1 \leq v_2$ together with Definition 1.2 we clearly see that

$$\frac{f(t,u_1,v_1)}{u_1^{\gamma}} \geq \frac{f(t,u_2,v_2)}{u_2^{\gamma}}, \tag{2.2}$$

where $0 < \gamma < 1$. From $0 < u_1/u_2 \leq 1$ we know that $(u_1/u_2)^{\gamma-1} \geq 1$, and therefore

$$\begin{aligned} \frac{f(t,u_1,v_1)}{u_1} &\geq \frac{f(t,u_2,v_2)}{u_2}\left(\frac{u_1}{u_2}\right)^{\gamma-1} \\ &\geq \frac{f(t,u_2,v_2)}{u_2}. \end{aligned} \tag{2.3}$$

Our main result is Theorem 2.3. □

Theorem 2.3. *The following statements are true.*

(a) *Suppose that $|f(t,u,v)|$ is a nondecreasing function with respect to $|u|$ and $|v|$ for $t \in R_+$. If*

$$\int_{t_0}^{\infty} s^{n-1} f(s,c,c)ds < \infty \tag{2.4}$$

for some constants $c > 0$, then (1.1) *has a bounded nonoscillatory solution.*

(b) *If $f(t,u,v)$ is a strongly superlinear function, then every solution of* (1.1) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1} f(s,c,c)ds = \infty \tag{2.5}$$

for any $c > 0$.

(c) *If* $|f(t,u,v)|/|u|$ *is a nondecreasing function with respect to* $|u|$ *and* $v|$, *then every bounded solution of* (1.1) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1} f(s,c,c)ds = \infty \tag{2.6}$$

for each $c > 0$.

(d) *If* $f(t,u,v)$ *is a strongly sublinear function, then every solution of* (1.1) *oscillates if and only if*

$$\int_{t_0}^{\infty} f\left(s, cs^{n-1}, c\tau(s)^{n-1}\right)ds = \infty \tag{2.7}$$

for any $c > 0$.

Proof. (a) Assume that (2.4) holds. Choose $T_0 \geq t_0$ sufficiently large such that

$$\int_{t}^{\infty} s^{n-1} f(s,c,c)ds < \frac{c}{2} \tag{2.8}$$

for $t \geq T_0$ and some $c > 0$.

Observing that if $x(t)$ satisfies the equation

$$x(t) = \int_{T_0}^{t} ds \int_{s}^{\infty} (u-s)^{n-2} f(u, x(u), x(\tau(u)))du + \frac{c}{2}, \tag{2.9}$$

then $x(t)$ is a solution of (1.1). Therefore it suffices to show that (2.9) has a bounded nonoscillatory solution.

Consider the functional set

$$M = \left\{x \in C([T_0,\infty), R) : \frac{c}{2} \leq x(t) \leq c\right\}. \tag{2.10}$$

Define the operator $S : M \to C([T_0,\infty), R)$ as follows:

$$S(x(t)) = \int_{T_0}^{t} ds \int_{s}^{\infty} (u-s)^{n-2} f(u, x(u), x(\tau(u)))du + \frac{c}{2}. \tag{2.11}$$

Then we have

$$
\begin{aligned}
S(x(t)) &= \int_{T_0}^{t} f(u,x(u),x(\tau(u)))du \int_{T_0}^{u} (u-s)^{n-2} ds \\
&\quad + \int_{t}^{\infty} f(u,x(u),x(\tau(u)))du \int_{T_0}^{t} (u-s)^{n-2} ds + \frac{c}{2} \\
&= \int_{T_0}^{t} \frac{1}{n-1}(u-T_0)^{n-1} f(u,x(u),x(\tau(u)))du \\
&\quad + \int_{t}^{\infty} \frac{1}{n-1}\left[(u-T_0)^{n-1} - (u-t)^{n-1}\right] f(u,x(u),x(\tau(u)))du + \frac{c}{2} \\
&\le \int_{T_0}^{t} \frac{1}{n-1}(u-T_0)^{n-1} f(u,x(u),x(\tau(u)))du \\
&\quad + \int_{t}^{\infty} \frac{n-1}{n-1}(t-T_0)(u-T_0)^{n-2} f(u,x(u),x(\tau(u)))du + \frac{c}{2} \\
&\le \int_{T_0}^{t} \frac{1}{n-1}(u-T_0)^{n-1} f(u,x(u),x(\tau(u)))du \\
&\quad + \frac{n-1}{n-1}\int_{t}^{\infty} (u-T_0)^{n-1} f(u,x(u),x(\tau(u)))du + \frac{c}{2} \\
&\le \frac{n-1}{n-1}\int_{T_0}^{\infty} (u-T_0)^{n-1} f(u,x(u),x(\tau(u)))du + \frac{c}{2} \\
&\le \int_{T_0}^{\infty} (u-T_0)^{n-1} f(u,c,c)du + \frac{c}{2} \le c.
\end{aligned}
\tag{2.12}
$$

Clearly, we have $S(x(t)) \ge c/2$, and therefore $SM \subseteq M$.

Now, we define the functions $u_n : [T_0, \infty) \to R$ as follows:

$$
u_n = S(u_{n-1}(t)), \quad n \in N, \tag{2.13}
$$

where

$$
u_0 = \frac{c}{2}, \quad t > T_0. \tag{2.14}
$$

Since the function $f(t,u,v)$ is nondecreasing with respect to $u > 0$ and $v > 0$, a straightforward verification shows the validity of the inequalities

$$
\frac{c}{2} \le u_{n-1} \le u_n \le c, \quad t \ge T_0. \tag{2.15}
$$

Therefore $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \ge T_0$. It follows from the Lebesgue convergence theorem that $u \in M$ and $u(t) = S(u(t))$.

It is easy to see that $u(t)$ is the desired bounded and nonoscillatory solution of (2.9)

(b) Sufficiency. Assume that $\int_{t_0}^{\infty} s^{n-1} f(s,c,c)ds = \infty$ for each $c > 0$. We will prove that every solution of (1.1) oscillates. Otherwise, assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0$. Then according to Lemma 2.1, there exists an odd integer $k \in \{1,\dots,n-1\}$ and $T \geq t_0$ such that

$$\begin{aligned} x^{(i)}(t) > 0 \quad & \text{for } t \geq T,\ 0 \leq i \leq k, \\ (-1)^{i+k} x^{(i)}(t) > 0 \quad & \text{for } t \geq T,\ k \leq i \leq n. \end{aligned} \tag{2.16}$$

There are two possible cases.

Case 1 ($k = 1$). In this case we see that

$$x'(t) > 0, \quad x''(t) < 0, \quad x^{(3)}(t) > 0, \dots, x^{(n)}(t) < 0. \tag{2.17}$$

Since $x(t)$ is an increasing function, hence for $t \geq T > 0$ and some constants $c > 0$, one has

$$x(t) \geq c > 0, \qquad x(\tau(t)) \geq c > 0. \tag{2.18}$$

Making use of the Taylor expansion we get

$$x'(t) = \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} x^{(1+j)}(\delta)(\delta - t)^j + \frac{(-1)^{n-1}}{(n-2)!} \int_t^{\delta} (s-t)^{n-2} x^{(n)}(s)ds, \quad T \leq t \leq \delta. \tag{2.19}$$

From (1.1) and (2.17) together with (2.19) we get

$$x'(t) > \int_t^{\delta} \frac{(s-t)^{n-2}}{(n-2)!} f(s, x(s), x(\tau(s)))ds. \tag{2.20}$$

The strong superlinearity of f leads to

$$f(s, x(s), x(\tau(s))) = \frac{f(s, x(s), x(\tau(s)))}{x^{\alpha}(s)} x^{\alpha}(s) \geq \frac{f(s,c,c)}{c^{\alpha}} x^{\alpha}(s), \tag{2.21}$$

which implies

$$x'(t) > \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \frac{f(s,c,c)}{c^{\alpha}} x^{\alpha}(s)ds > \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \frac{f(s,c,c)}{c^{\alpha}} x^{\alpha}(t)ds. \tag{2.22}$$

From (2.22) we have

$$\begin{aligned}\int_T^t \frac{x'(s)}{x^\alpha(s)}ds &> \int_T^t ds \int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!}\frac{f(u,c,c)}{c^\alpha}du \\ &= \int_T^t \frac{f(u,c,c)}{c^\alpha}du \int_T^u \frac{(u-s)^{n-2}}{(n-2)!}ds + \int_t^\infty \frac{f(u,c,c)}{c^\alpha}du \int_T^t \frac{(u-s)^{n-2}}{(n-2)!}ds.\end{aligned} \tag{2.23}$$

By using the elementary inequality $a^{n-1} - b^{n-1} \geq (a-b)a^{n-2}$ for $0 < b \leq a$, we have

$$\begin{aligned}\int_T^t (u-s)^{n-2}ds &= -\frac{(u-s)^{n-1}}{n-1}\Bigg|_T^t \\ &= \frac{1}{n-1}\left[(u-T)^{n-1} - (u-t)^{n-1}\right] \geq \frac{1}{n-1}(t-T)(u-T)^{n-2}.\end{aligned} \tag{2.24}$$

Therefore, we get

$$\int_T^t \frac{x'(s)}{x^\alpha(s)}ds \geq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!}\frac{f(u,c,c)}{c^\alpha}du + (t-T)\int_t^\infty \frac{(u-T)^{n-2}}{(n-1)!}\frac{f(u,c,c)}{c^\alpha}du, \tag{2.25}$$

$$\int_T^t \frac{(u-T)^{n-1}}{(n-1)!}\frac{f(u,c,c)}{c^\alpha}du < \int_T^t \frac{x'(s)}{x^\alpha(s)}ds, \tag{2.26}$$

or

$$\int_T^t \frac{(u-T)^{n-1}}{(n-1)!}\frac{f(u,c,c)}{c^\alpha}du < \frac{x^{1-\alpha}(t)}{\alpha-1} < \infty, \tag{2.27}$$

which contradicts with (2.5).

Case 2 ($k > 1$). Making use of (2.21) we have

$$x^{(n)}(t) + \frac{f(t,c,c)}{c^\alpha}x^\alpha(t) < 0. \tag{2.28}$$

For $t \geq T$, it follows from (iii) of Lemma 2.1 that

$$x(t) \geq \frac{(t-T)^{k-1}}{k!}x^{(k-1)}(t). \tag{2.29}$$

For sufficiently large t, one has

$$x^\alpha(t) \geq \frac{(t-T)^{(k-1)\alpha}}{(k!)^\alpha}\left(x^{(k-1)}(t)\right)^\alpha > \frac{(t-T)^{k-1}}{(k!)^\alpha}\left(x^{(k-1)}(t)\right)^\alpha, \quad \alpha > 1. \tag{2.30}$$

Let $z(t) = x^{(k-1)}(t)$, then

$$z(t) > 0, \quad z'(t) > 0, \quad z''(t) < 0, \dots, \tag{2.31}$$

and therefore

$$z^{(n-k+1)}(t) + \frac{f(t,c,c)}{c^{\alpha}} \frac{(t-T)^{k-1}}{(k!)^{\alpha}} z^{\alpha}(t) < 0. \tag{2.32}$$

Using the same method as in the proof of Case 1, we get

$$\int_T^{\infty} s^{n-k} \frac{f(s,c,c)}{c^{\alpha}} \frac{(s-T)^{k-1}}{(k!)^{\alpha}} ds < \infty, \tag{2.33}$$

that is

$$\int_T^{\infty} s^{n-1} f(s,c,c) ds < \infty, \tag{2.34}$$

which contradicts with (2.5).

Conversely, if every solution of (1.1) oscillates, then (2.5) holds. Otherwise (2.4) holds. Theorem 2.3(a) implies that (1.1) has a nonoscillatory solution.

(c) Sufficiency. Without loss of generality, we assume that $x(t)$ is a bounded positive solution. We divided the proof into two cases.

Case 1 ($k = 1$). The same argument as in the proof of Theorem 2.3(b) implies that inequality (2.26) holds for $\alpha = 1$, that is,

$$\int_T^{t} \frac{(u-T)^{n-1}}{(n-1)!} \frac{f(u,c,c)}{c} du < \int_T^{t} \frac{x'(s)}{x(s)} ds = \ln x(t) - \ln x(T) < \infty, \tag{2.35}$$

which contradicts with (2.6).

Case 2 ($k > 1$). From the proof of Theorem 2.3(b) we also clearly see that

$$\int_T^{\infty} u^{n-1} f(u,c,c) du < \infty, \tag{2.36}$$

which contradicts with (2.6).

Conversely, if every bounded solution of (1.1) oscillates, and then (2.6) holds. Otherwise (2.4) holds, then Theorem 2.3(a) implies that (1.1) has a nonoscillatory bounded solution.

(d) Sufficiency. Without loss of generality, we assume that $x(t)$ is a finally positive solution, that is, $x(t) > 0$ for $t \geq T > 0$. We consider the following two cases.

Case 1 ($k = 1$). In this case we see that

$$x(t) > 0, \quad x'(t) > 0, \quad x''(t) < 0, \dots, x^{n}(t) < 0, \tag{2.37}$$

then we know that

$$\lim_{t\to\infty} x'(t) = L \in [0, \infty), \tag{2.38}$$

and there exist constants $T_1 > T$ and $c > 0$ such that $x(t) \le c(t-T)$ and $x(\tau(t)) \le c(\tau(t) - T)$ for $t \ge T_1 > T$. The strong sublinearity of f implies that

$$\begin{aligned} f(t, x(t), x(\tau(t))) &= \frac{f(t, x(t), x(\tau(t)))}{x^{\beta}(t)} x^{\beta}(t) \\ &\ge \frac{f(t, c(t-T), c(\tau(t)-T))}{c^{\beta}} \left(\frac{x(t)}{t-T}\right)^{\beta}. \end{aligned} \tag{2.39}$$

The same argument as in the proof of Case 1 of Theorem 2.3(b) yields

$$x'(t) > \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} f(s, x(s), x(\tau(s))) ds. \tag{2.40}$$

Integrating from T to t leads to

$$\begin{aligned} x(t) &> x(t) - x(T) \\ &> \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} f(u, x(u), x(\tau(u))) du + (t-T) \int_t^{\infty} \frac{(u-T)^{n-2}}{(n-1)!} f(u, x(u), x(\tau(u))) du \\ &> (t-T) \int_t^{\infty} \frac{(u-T)^{n-2}}{(n-1)!} f(u, x(u), x(\tau(u))) du. \end{aligned} \tag{2.41}$$

That is

$$\frac{x(t)}{t-T} > \int_t^{\infty} \frac{(u-T)^{n-2}}{(n-1)!} \frac{f(u, c(u-T), c(\tau(u)-T))}{c^{\beta}} \left(\frac{x(u)}{u-T}\right)^{\beta} du. \tag{2.42}$$

Let

$$z(t) = \int_t^{\infty} \frac{(u-T)^{n-2}}{(n-1)!} \frac{f(u, c(u-T), c(\tau(u)-T))}{c^{\beta}} \left(\frac{x(u)}{u-T}\right)^{\beta} du, \tag{2.43}$$

then $z'(t) < 0$, $0 < z(t) < x(t)/(t-T)$ and

$$\begin{aligned} z'(t) &= -\frac{(t-T)^{n-2}}{(n-1)!}\frac{f(t,c(t-T),c(\tau(t)-T))}{c^{\beta}}\left(\frac{x(t)}{t-T}\right)^{\beta} \\ &< -\frac{(t-T)^{n-2}}{(n-1)!}\frac{f(t,c(t-T),c(\tau(t)-T))}{c^{\beta}}z^{\beta}(t), \\ \frac{z'(t)}{z^{\beta}(t)} &\leq -\frac{(t-T)^{n-2}}{(n-1)!}\frac{f(t,c(t-T),c(\tau(t)-T))}{c^{\beta}}, \end{aligned} \tag{2.44}$$

and for $T_1 > T$, one has

$$\begin{aligned} \int_{T_1}^{t}\frac{z'(u)}{z^{\beta}(u)}du &\leq -\int_{T_1}^{t}\frac{(u-T)^{n-2}}{(n-1)!}\frac{f(u,c(u-T),c(\tau(u)-T))}{c^{\beta}}du, \\ \frac{1}{1-\beta}\left[z^{1-\beta}(t)-z^{1-\beta}(T_1)\right] &\leq -\frac{1}{(n-1)!}\int_{T_1}^{t}(u-T)^{n-2+\beta}\frac{f(u,c(u-T),c(\tau(u)-T))}{c^{\beta}}du. \end{aligned} \tag{2.45}$$

Therefore

$$\int_{T_1}^{t}(u-T)^{n-2}\frac{f(u,c(u-T),c(\tau(u)-T))}{c^{\beta}}du < \infty \tag{2.46}$$

or

$$\int_{T_1}^{\infty}(u-T)^{n-2}f(u,c(u-T),c(\tau(u)-T))du < \infty. \tag{2.47}$$

By condition (H2), we can choose $\widetilde{T}_1 > T_1$ such that $\tau(u) - T > (1/2)\tau(u) > 1$ and $u - T > (1/2)u > 1$ for $u > \widetilde{T}_1$. Then making use of Lemma 2.2, we have

$$\begin{aligned} \frac{f(u,c(u-T),c(\tau(u)-T))}{c(u-T)} &\geq \frac{f\left(u,c(u-T)^{n-1},c(\tau(u)-T)^{n-1}\right)}{c(u-T)^{n-1}} \\ &\geq \frac{f\left(u,cu^{n-1},c(\tau(u))^{n-1}\right)}{cu^{n-1}}, \quad u > \widetilde{T}_1. \end{aligned} \tag{2.48}$$

From (2.47) and (2.48) together with $u - T > (1/2)u$ we get

$$\int_{\widetilde{T}_1}^{\infty}f\left(u,cu^{n-1},c(\tau(u))^{n-1}\right)dt < \infty, \tag{2.49}$$

which contradicts with (2.7).

Case 2 $(k > 1)$. That is,

$$x(t) > 0, \quad x'(t) > 0, \ldots, x^{(k-1)}(t) > 0, \quad x^{(k)}(t) > 0, \quad x^{(k+1)}(t) < 0, \ldots, x^{(n)}(t) < 0. \tag{2.50}$$

From $x^{(k)}(t) > 0$ and $x^{(k+1)}(t) < 0$ for $t \geq T > 0$ we know that

$$\lim_{t\to\infty} x^{(k)}(t) = L \in [0, \infty), \tag{2.51}$$

and there exist constants $T_2 > T$ and $c_1 > 0$ such that $x(t) < c_1(t-T)^k$ and $x(\tau(t)) < c_1(\tau(t)-T)^k$ for $t \geq T_2 > T > 0$. The strong sublinearity of f leads to

$$\begin{aligned} f(t, x(t), x(\tau(t))) &= \frac{f(t, x(t), x(\tau(t)))}{x^{\beta}(t)} x^{\beta}(t) \\ &\geq \frac{f\left(t, c(t-T)^k, c_1(\tau(t)-T)^k\right)}{\left[c_1(t-T)^k\right]^{\beta}} x^{\beta}(t). \end{aligned} \tag{2.52}$$

It follows from (iii) of Lemma 2.1, that

$$x(t) \geq \frac{(t-T)^{k-1}}{k!} x^{(k-1)}(t), \tag{2.53}$$

and thus

$$x^{\beta}(t) \geq \frac{(t-T)^{(k-1)\beta}}{(k!)^{\beta}} \left[x^{(k-1)}(t)\right]^{\beta}. \tag{2.54}$$

Let $z(t) = x^{(k-1)}(t)$, then $z(t) > 0$, $z'(t) > 0$, $z''(t) < 0, \ldots, z^{(n-k+1)} < 0$ and

$$z^{(n-k+1)}(t) + \frac{f\left(t, c_1(t-T)^k, c_1(\tau(t)-T)^k\right)}{\left[c_1(t-T)^k\right]^{\beta}} \frac{(t-T)^{(k-1)\beta}}{(k!)^{\beta}} z^{\beta}(t) < 0, \tag{2.55}$$

where $n-k+1$ is also even. According to the same process as the one used in the proof of Case 1 of Theorem 2.3(d) we conclude that

$$\int_{T_2}^{\infty} s^{(n-k)\beta} \frac{f\left(s, c_1(s-T)^k, c_1(\tau(s)-T)^k\right)}{\left[c_1(s-T)^k\right]^{\beta}} (s-T)^{(k-1)\beta} ds < \infty. \tag{2.56}$$

By condition (H2), we can choose $\widetilde{T}_2 > T_2$ such that $\tau(s) - T > (1/2)\tau(s) > 1$ and $s - T > (1/2)s > 1$ for $s > \widetilde{T}_2$. Now making use of Lemma 2.2, we have

$$\begin{aligned}\frac{f\left(s, c_1(s-T)^k, c_1(\tau(s)-T)^k\right)}{\left[c_1(s-T)^k\right]^\beta} &\geq \frac{f\left(s, c_1(s-T)^{n-1}, c_1(\tau(s)-T)^{n-1}\right)}{\left[c_1(s-T)^{n-1}\right]^\beta} \\ &\geq \frac{f\left(s, c_1 s^{n-1}, c_1(\tau(s))^{n-1}\right)}{\left(c_1 s^{n-1}\right)^\beta}.\end{aligned} \tag{2.57}$$

From (2.56) and (2.57) together with $s - T > (1/2)s$ we clearly see that

$$\int_{\widetilde{T}_2}^{\infty} f\left(s, c_1 s^{n-1}, c_1(\tau(s))^{n-1}\right) dt < \infty, \tag{2.58}$$

which contradicts with (2.7).

Necessity

If every solution of (1.1) oscillates, then (2.7) holds. Otherwise, assuming that

$$\int_{t_0}^{\infty} f\left(s, cs^{n-1}, c\tau(s)^{n-1}\right) ds < \infty \tag{2.59}$$

for some constants $c > 0$, we should prove that (1.1) has a nonoscillatory solution. From (2.59) we know that there exist $t \geq T$ and some $c > 0$ such that

$$\int_t^{\infty} f\left(u, \frac{c}{2(n-1)!}(u-T)^{n-1}, \frac{c}{2(n-1)!}(\tau(u)-T)^{n-1}\right) du \leq \frac{c}{4}. \tag{2.60}$$

Let X be the Banach space of all real-valued continuous functions $x(t)$ endowed with the norm

$$\|x\| = \sup_{t \geq T} \left| \frac{x(t)}{(t-T)^{n-1}} \right|, \tag{2.61}$$

and let M be the subset of X defined by

$$M = \left\{ x \in X : \frac{c}{2(n-1)!}(t-T)^{n-1} \leq x(t) \leq \frac{c}{(n-1)!}(t-T)^{n-1}, t \geq T \right\}. \tag{2.62}$$

Define the mapping S on M by

$$S(x(t)) = \int_T^t ds_1 \int_T^{s_1} ds_2 \cdots \int_T^{s_{n-2}} \left[\frac{c}{2} + \int_{s_{n-1}}^{\infty} f(u, x(u), x(\tau(u))) du \right] ds_{n-1}, \tag{2.63}$$

where the integration is $n - 1$ times.

By Lemma 2.2, for $x(t) \in M$ one has

$$\frac{f(t,x(t),x(\tau(t)))}{x(t)} \leq \frac{f\left(t,(c/2(n-1)!)(t-T)^{n-1},(c/2(n-1)!)(\tau(t)-T)^{n-1}\right)}{(c/2(n-1)!)(t-T)^{n-1}} \tag{2.64}$$

for sufficient large $t \geq \tau(t) > T$, that is,

$$\begin{aligned} f(t,x(t),x(\tau(t))) &\leq f\left(t,\frac{c}{2(n-1)!}(t-T)^{n-1},\frac{c}{2(n-1)!}(\tau(t)-T)^{n-1}\right)\frac{x(t)}{(c/2(n-1)!)(t-T)^{n-1}} \\ &\leq 2f\left(t,\frac{c}{2(n-1)!}(t-T)^{n-1},\frac{c}{2(n-1)!}(\tau(t)-T)^{n-1}\right). \end{aligned} \tag{2.65}$$

From (2.60) and (2.65) we get

$$\int_T^\infty f(u,x(u),x(\tau(u)))du \leq 2\int_T^\infty f\left(u,\frac{c}{2(n-1)!}(u-T)^{n-1},\frac{c}{2(n-1)!}(\tau(u)-T)^{n-1}\right)du \leq \frac{c}{2}. \tag{2.66}$$

Equation (2.66) and the definition of the operator S imply that $S(x(t)) \leq (c/(n-1)!)(t-T)^{n-1}$. On the other hand, we clearly see that $S(x(t)) \geq (c/2(n-1)!)(t-T)^{n-1}$ for $t \geq T$. Therefore, $SM \subseteq M$.

It is routine to prove that S is continuous and Sx is relatively compact in the topology of the Frechet space $C[T,\infty)$. Therefore, there exists $x(t) \in M$ such that $x(t) = S(x(t))$ follows from the well-known Schauder's fixed point Theorem. It is easy to see that $x(t)$ is the solution of (1.1).

The proof of Theorem 2.3 is completed. □

Remark 2.4. If $f(s,x(s),x(\tau(s))) = p(s)x^\gamma(s)$, then $f(s,c,c) = c^\gamma p(s) \equiv \tilde{c}p(s)$ and $f(s, cs^{n-1}, c\tau(s)^{n-1}) = c^\gamma s^{(n-1)\gamma}p(s) \equiv \tilde{c}s^{(n-1)\gamma}p(s)$. For (1.2) we can derive Corollary 2.5 from Theorem 2.3.

Corollary 2.5. *If n is even, then the following statements are true.*

(a) *If*

$$\int_{t_0}^\infty s^{n-1}p(s)ds < \infty, \tag{2.67}$$

then (1.2) *has a bounded nonoscillatory solution.*

(b) *If $\gamma > 1$, then every solution of* (1.2) *oscillates if and only if*

$$\int_{t_0}^\infty s^{n-1}p(s)ds = \infty. \tag{2.68}$$

(c) *If* $\gamma = 1$, *then every bounded solution of* (1.2) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1} p(s) ds = \infty. \tag{2.69}$$

(d) *If* $0 < \gamma < 1$, *then every solution of* (1.2) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{(n-1)\gamma} p(s) ds = \infty. \tag{2.70}$$

It is easy to see that Theorem A can be obtained directly from our Corollary 2.5(c).

For $n = 2$, we have Corollary 2.6 for (1.3).

Corollary 2.6. *If* $n = 2$, *then the following statements are true.*
(a) *If*

$$\int_{t_0}^{\infty} sp(s) ds < \infty. \tag{2.71}$$

then (1.3) *has a bounded nonoscillatory solution.*
(b) *If* $\gamma > 1$, *then every solution of* (1.3) *oscillates if and only if*

$$\int_{t_0}^{\infty} sp(s) ds = \infty. \tag{2.72}$$

(c) *If* $\gamma = 1$, *then every bounded solution of* (1.3) *oscillates if and only if*

$$\int_{t_0}^{\infty} sp(s) ds = \infty. \tag{2.73}$$

(d) *If* $0 < \gamma < 1$, *then every solution of* (1.3) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{\gamma} p(s) ds = \infty. \tag{2.74}$$

We clearly see that our results in Corollary 2.6(a), (b), and (d) are exactly corresponding to the results in Theorems B, C, and D, respectively.

Remark 2.7. If $f(t,x(t),x(\tau(t))) = p(t)f(x(t))$, then (1.1) becomes

$$x^{(n)}(t) + p(t)f(x(t)) = 0. \tag{2.75}$$

From the proof of Theorem 2.3(b) we indicate that the strongly superlinearity of $f(x(t))$ can be replaced by the condition

$$\liminf_{x\to\infty} \frac{|f(x(t))|}{|x|^{\alpha}} \neq 0. \tag{2.76}$$

In fact, if $x(t)$ is a nonoscillatory solution of (2.75), then from Theorem 2.3(a) we may assume that $x(t)$ is unbounded, and (2.76) implies that $\lim_{x\to\infty}\inf(|f(x(t)|/|x|^{\alpha}) \geq K > 0$, and there exists T such that $x(t) > 0$ and $f(x(t))/x^{\alpha}(t) \geq K > 0$ for $t > T$. Then we get

$$p(t)f(x(t)) \geq Kp(t)x^{\alpha}(t), \quad t > T. \tag{2.77}$$

We notice that if (2.21) is replaced by (2.77), then Corollary 2.8 follows from the proof of Theorem 2.3(b).

Corollary 2.8. *If* $\lim_{x\to\infty}\inf(|f(x(t)|/|x|^{\alpha}) \neq 0$, *then all solutions of* (2.75) *oscillate if and only if*

$$\int_{t_0}^{\infty} t^{n-1}p(t) = \infty. \tag{2.78}$$

If $n = 2$, *then one clearly sees that Theorem F is the special case of* Corollary 2.8.

Example 2.9. The equation

$$x^{(4)}(t) + \frac{24}{t^3(t-2)(t-1)}x^2(t)x(t-1) = 0, \quad t \geq 3 \tag{2.79}$$

satisfies the assumptions of Theorem 2.3(a) but does not satisfy the assumptions of Theorem 2.3(b) and (c); hence there exists a bounded nonoscillatory solution. In fact $x(t) = 1 - 1/t$ is one such solution.

Example 2.10. The equation

$$x''(t) + (2 - \sin t)\sin^{2/3}t\frac{x^{1/3}(t)}{2 + x(t-\pi)} = 0 \tag{2.80}$$

satisfies the assumptions of Theorem 2.3(d). Hence every solution of (1.1) is oscillatory. In fact $x(t) = \sin t$ is one such solution.

Acknowledgments

The authors wish to thank the anonymous referees for the very careful reading of the manuscript and fruitful comments and suggestions. This work was supported by the National Natural Science Foundation of China (Grant no. 60850005) and the Natural Science Foundation of Zhejiang Province (Grant nos. D7080080 and Y607128).

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