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Research Article

Some New Hilbert's Type Inequalities

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Some new inequalities similar to Hilbert's type inequality involving series of nonnegative terms are established.

1. Introduction

In recent years, several authors [1–10] have given considerable attention to Hilbert's type inequalities and their various generalizations. In particular, in [1], Pachpatte proved some new inequalities similar to Hilbert's inequality [11, page 226] involving series of nonnegative terms. The main purpose of this paper is to establish their general forms.

2. Main Results

In [1], Pachpatte established the following inequality involving series of nonnegative terms.

Theorem A. *Let $p \geq 1, q \geq 1$, and let $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, \ldots, k$, and $n = 1, \ldots, r$, where k, r are natural numbers. Let $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \leq C(p,q,k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{1/2} \times \left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{1/2}, \tag{2.1}$$

where

$$C(p,q,k,r) = \frac{1}{2}pq(kr)^{1/2}. \tag{2.2}$$

We first establish the following general form of inequality (2.1).

Theorem 2.1. *Let $p \geq 1, q \geq 1, t > 0$, and $1/\alpha + 1/\beta = 1, \alpha > 1$. Let $\{a_{m_1,\ldots,m_n}\}$, and $\{b_{n_1,\ldots,n_n}\}$ be positive sequences of real numbers defined for $m_i = 1,2,\ldots,k_i$, and $n_i = 1,2,\ldots,r_i$, where k_i, r_i $(i = 1,\ldots,n)$ are natural numbers. Let $A_{m_1,\ldots,m_n} = \sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} a_{s_1,\ldots,s_n}$, and $B_{n_1,\ldots,n_n} = \sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n} b_{t_1,\ldots,t_n}$. Then*

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\frac{\alpha\beta t^{1/\beta}A^p_{m_1,\ldots,m_n}B^q_{n_1,\ldots,n_n}}{m_1\cdots m_n\beta + n_1\cdots n_n\alpha t}\\
&\quad\leq L(k_1,\ldots,k_n,r_1,\ldots,r_n,p,q,\alpha,\beta)\\
&\qquad\times\left(\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{j=1}^{n}(k_j - m_j + 1)\left(a_{m_1,\ldots,m_n}A^{p-1}_{m_1,\ldots,m_n}\right)^{\beta}\right)^{1/\beta}\\
&\qquad\times\left(\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\prod_{j=1}^{n}(r_j - n_j + 1)\left(b_{n_1,\ldots,n_n}B^{q-1}_{n_1,\ldots,n_n}\right)^{\alpha}\right)^{1/\alpha},
\end{aligned} \tag{2.3}$$

where

$$L(k_1,\ldots,k_n,r_1,\ldots,r_n,p,q,\alpha,\beta) = pq(k_1\cdots k_n)^{1/\alpha}(r_1\cdots r_n)^{1/\beta}. \tag{2.4}$$

Proof. By using the following inequality (see [12]):

$$\left(\sum_{m_1=1}^{n_1}\cdots\sum_{m_n=1}^{n_n} z_{m_1,\ldots,m_n}\right)^p \leq p\sum_{m_1=1}^{n_1},\ldots,\sum_{m_n=1}^{n_n} z_{m_1,\ldots,m_n}\left(\sum_{k_1=1}^{m_1}\cdots\sum_{k_n=1}^{m_n} z_{k_1,\ldots,k_n}\right)^{p-1}, \tag{2.5}$$

where $p \geq 1$ is a constant, and $z_{m_1,\ldots,m_n} \geq 0$, $m_i = 1,2,\ldots,k_i$, $i = 1,2,\ldots,n$, we obtain

$$A^p_{m_1,\ldots,m_n} \leq p\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} a_{s_1,\ldots,s_n}A^{p-1}_{s_1,\ldots,s_n}. \tag{2.6}$$

Similarly, we have

$$B^q_{n_1,\ldots,n_n} \leq q\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n} b_{t_1,\ldots,t_n}B^{q-1}_{t_1,\ldots,t_n}. \tag{2.7}$$

From (2.6) and (2.7), using Hölder's inequality [13] and the elementary inequality:

$$a^{1/\alpha} b^{1/\beta} \le \frac{a}{\alpha t^{1/\beta}} + \frac{t^{1/\alpha} b}{\beta}, \tag{2.8}$$

where $1/\alpha + 1/\beta = 1, \alpha > 1, b > 0, a > 0$, and $t > 0$, we have

$$\begin{aligned}
A^{p}_{m_i,\ldots,m_n} B^{q}_{n_i,\ldots,n_n} &\le pq\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} a_{s_1,\ldots,s_n} A^{p-1}_{s_1,\ldots,s_n}\right)\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n} b_{t_1,\ldots,t_n} B^{q-1}_{t_1,\ldots,t_n}\right) \\
&\le pq(m_1,\ldots,m_n)^{1/\alpha}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(a_{s_1,\ldots,s_n} A^{p-1}_{s_1,\ldots,s_n}\right)^{\beta}\right)^{1/\beta} \\
&\quad \times (n_1,\ldots,n_n)^{1/\beta}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(b_{t_1,\ldots,t_n} B^{q-1}_{t_1,\ldots,t_n}\right)^{\alpha}\right)^{1/\alpha} \\
&\le pq\left(\frac{m_1\cdots m_n}{\alpha t^{1/\beta}} + \frac{n_1\cdots n_n t^{1/\alpha}}{\beta}\right)\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(a_{s_1,\ldots,s_n} A^{p-1}_{s_1,\ldots,s_n}\right)^{\beta}\right)^{1/\beta} \\
&\quad \times \left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(b_{t_1,\ldots,t_n} B^{q-1}_{t_1,\ldots,t_n}\right)^{\alpha}\right)^{1/\alpha}.
\end{aligned} \tag{2.9}$$

Dividing both sides of (2.9) by $((m_1\cdots m_n\beta + n_1\cdots n_n\alpha t)/\alpha\beta t^{1/\beta})$, summing up over n_i from 1 to r_i $(i = 1, 2, \ldots, n)$ first, then summing up over m_i from 1 to k_i $(i = 1, 2, \ldots, n)$, using again Hölder's inequality, then interchanging the order of summation, we obtain

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\frac{\alpha\beta t^{1/\beta} A^{p}_{m_1,\ldots,m_n} B^{q}_{n_1,\ldots,n_n}}{m_1\cdots m_n\beta + n_1\cdots n_n\alpha t} \\
&\quad \le pq\left\{\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(a_{s_1,\ldots,s_n} A^{p-1}_{s_1,\ldots,s_n}\right)^{\beta}\right)^{1/\beta}\right\} \\
&\qquad \times \left\{\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(b_{t_1,\ldots,t_n} B^{q-1}_{t_1,\ldots,t_n}\right)^{\alpha}\right)^{1/\alpha}\right\} \\
&\quad \le pq(k_1\cdots k_n)^{1/\alpha}\left\{\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(a_{s_1,\ldots,s_n} A^{p-1}_{s_1,\ldots,s_n}\right)^{\beta}\right)\right\}^{1/\beta} \\
&\qquad \times (r_1\cdots r_n)^{1/\beta}\left\{\sum_{n_1=1}^{k_1}\cdots\sum_{n_n=1}^{r_n}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(b_{t_1,\ldots,t_n} B^{q-1}_{t_1,\ldots,t_n}\right)^{\alpha}\right)\right\}^{1/\alpha}
\end{aligned}$$

$$
\begin{aligned}
&= L(k_1,\ldots,k_n,r_1,\ldots,r_n,p,q,\alpha,\beta)\\
&\quad\times\left\{\sum_{s_1=1}^{k_1}\cdots\sum_{s_n=1}^{k_n}\left(a_{s_1,\ldots,s_n}A_{s_1,\ldots,s_n}^{p-1}\right)^{\beta}\left(\sum_{m_1=s_1}^{k_1}\cdots\sum_{m_n=s_n}^{k_n}1\right)\right\}^{1/\beta}\\
&\quad\times\left\{\sum_{t_1=1}^{r_1}\cdots\sum_{t_n=1}^{r_n}\left(b_{t_1,\ldots,t_n}B_{t_1,\ldots,t_n}^{q-1}\right)^{\alpha}\left(\sum_{n_1=t_1}^{r_1}\cdots\sum_{n_n=t_n}^{r_n}1\right)\right\}^{1/\alpha}\\
&= L(k_1,\ldots,k_n,r_1,\ldots,r_n,p,q,\alpha,\beta)\\
&\quad\times\left\{\sum_{s_1=1}^{k_1}\cdots\sum_{s_n=1}^{k_n}\prod_{j=1}^{n}(k_j-s_j+1)\left(a_{s_1,\ldots,s_n}A_{s_1,\ldots,s_n}^{p-1}\right)^{\beta}\right\}^{1/\beta}\\
&\quad\times\left\{\sum_{t_1=1}^{r_1}\cdots\sum_{t_n=1}^{r_n}\prod_{j=1}^{n}(r_j-t_j+1)\left(b_{t_1,\ldots,t_n}B_{t_1,\ldots,t_n}^{q-1}\right)^{\alpha}\right\}^{1/\alpha}\\
&= L(k_1,\ldots,k_n,r_1,\ldots,r_n,p,q,\alpha,\beta)\\
&\quad\times\left\{\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{j=1}^{n}(k_j-m_j+1)\left(a_{m_1,\ldots,m_n}A_{m_1,\ldots,m_n}^{p-1}\right)^{\beta}\right\}^{1/\beta}\\
&\quad\times\left\{\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\prod_{j=1}^{n}(r_j-n_j+1)\left(b_{n_1,\ldots,n_n}B_{n_1,\ldots,n_n}^{q-1}\right)^{\alpha}\right\}^{1/\alpha}.
\end{aligned}
\tag{2.10}
$$

This completes the proof. □

Remark 2.2. Taking $\alpha=\beta=n=j=2$, (2.3) becomes

$$
\begin{aligned}
&\sum_{m=1}^{k_1}\sum_{m_2=1}^{k_2}\left(\sum_{n_1=1}^{r_1}\sum_{n_2=1}^{r_2}\frac{A_{m_1,m_2}^{p}B_{n_1,n_2}^{q}}{m_1m_2t^{-1/2}+n_1n_2t^{1/2}}\right)\\
&\quad\le\frac{1}{2}pq\sqrt{k_1k_2r_1r_2}\left(\sum_{m=1}^{k_1}\sum_{m_2=1}^{k_2}(k_1-m_1+1)(k_2-m_2+1)\left(a_{m_1,m_2}A_{m_1,m_2}^{p-1}\right)^{2}\right)^{1/2}\\
&\quad\times\left(\sum_{n=1}^{r_1}\sum_{n_2=1}^{r_2}(r_1-n_1+1)(r_2-n_2+1)\left(b_{n_1,n_2}B_{n_1,n_2}^{p-1}\right)^{2}\right)^{1/2}.
\end{aligned}
\tag{2.11}
$$

Taking $t=1$, and changing $\{a_{m_1,m_2}\}$, $\{b_{n_1,n_2}\}$, $\{A_{m_1,m_2}\}$, and $\{B_{n_1,n_2}\}$ into $\{a_m\}$, $\{b_n\}$, $\{A_m\}$, and $\{B_n\}$, respectively, and with suitable changes, (2.11) reduces to Pachpatte [1, inequality (1)].

In [1], Pachpatte also established the following inequality involving series of nonnegative terms.

Theorem B. *Let $\{a_m\}$, $\{b_n\}, A_m, B_n$ be as defined in Theorem A. Let $\{p_m\}$ and $\{q_n\}$ be positive sequences for $m = 1, \ldots, k$, and $n = 1, \ldots, r$, where k, r are natural numbers. Define $P_m = \sum_{s=1}^{m} p_s$, and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be real-valued, nonnegative, convex, submultiplicative functions defined on $\mathbb{R}_+ = [0, +\infty)$. Then*

$$\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{\phi(A_m)\psi(B_n)}{m+n} \leq M(k,r) \left(\sum_{m=1}^{k} (k-m+1) \left(p_m \phi\left(\frac{a_m}{p_m} \right) \right)^2 \right)^{1/2} \times \left(\sum_{n=1}^{r} (r-n+1) \left(q_n \phi\left(\frac{b_n}{q_n} \right) \right)^2 \right)^{1/2}, \tag{2.12}$$

where

$$M(k,r) = \frac{1}{2} \left(\sum_{m=1}^{k} \left(\frac{\phi(P_m)}{P_m} \right)^2 \right)^{1/2} \left(\sum_{n=1}^{r} \left(\frac{\phi(Q_n)}{Q_n} \right)^2 \right)^{1/2}. \tag{2.13}$$

Inequality (2.12) can also be generalized to the following general form.

Theorem 2.3. *Let $\{a_{m_1,\ldots,m_n}\}$, $\{b_{n_1,\ldots,n_n}\}$, α, β, t, $A_{m_1,\ldots,m_n}$, and $B_{n_1,\ldots,n_n}$ be as defined in Theorem 2.1. Let $\{p_{m_1,\ldots,m_n}\}$ and $\{q_{n_1,\ldots,n_n}\}$ be positive sequences for $m_i = 1, 2, \ldots, k_i$, and $n_i = 1, 2, \ldots, r_i$ $(i = 1, 2, \ldots, n)$. Define $P_{m_1,\ldots,m_n} = \sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} p_{s_1,\ldots,s_n}$, and $Q_{n_1,\ldots,n_n} = \sum_{t_1=1}^{n_1} \cdots \sum_{t_n=1}^{n_n} q_{t_1,\ldots,t_n}$. Let ϕ and ψ be real-valued, nonnegative, convex, submultiplicative functions defined on $\mathbb{R}_+ = [0, +\infty)$. Then*

$$\begin{aligned} &\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \sum_{n_1=1}^{r_1} \cdots \sum_{n_n=1}^{r_n} \frac{\alpha\beta t^{1/\beta} \phi(A_{m_1,\ldots,m_n}) \psi(B_{n_1,\ldots,n_n})}{m_1 \cdots m_n \beta + n_1 \cdots n_n \alpha t} \\ &\quad \leq M(k_1, \ldots, k_n, r_1, \ldots, r_n, \alpha, \beta) \\ &\qquad \times \left\{ \sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \prod_{j=1}^{n} (k_j - m_j + 1) \left(p_{m_1,\ldots,m_n} \phi\left(\frac{a_{m_1,\ldots,m_n}}{p_{m_1,\ldots,m_n}} \right) \right)^{\beta} \right\}^{1/\beta} \\ &\qquad \times \left\{ \sum_{n_1=1}^{r_1} \cdots \sum_{n_n=1}^{r_n} \prod_{j=1}^{n} (r_j - n_j + 1) \left(q_{n_1,\ldots,n_n} \psi a\left(\frac{b_{n_1,\ldots,n_n}}{q_{n_1,\ldots,m_n}} \right) \right)^{\alpha} \right\}^{1/\alpha}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} &M(k_1, \ldots, k_n, r_1, \ldots, r_n, \alpha, \beta) \\ &\quad = \left(\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \left(\frac{\phi(P_{m_1,\ldots,m_n})}{P_{m_1,\ldots,m_n}} \right)^{\alpha} \right)^{1/\alpha} \left(\sum_{n_1=1}^{r_1} \cdots \sum_{n_n=1}^{r_n} \left(\frac{\psi(Q_{n_1,\ldots,n_n})}{Q_{n_1,\ldots,n_n}} \right)^{\beta} \right)^{1/\beta}. \end{aligned} \tag{2.15}$$

Proof. By the hypotheses, Jensen's inequality, and Hölder's inequality, we obtain

$$\begin{aligned}
\phi(A_{m_1,\dots,m_n}) &= \phi\left(\frac{P_{m_1,\dots,m_n}\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} p_{s_1,\dots,s_n}\left(a_{s_1,\dots,s_n}/p_{s_1,\dots,s_n}\right)}{\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} p_{s_1,\dots,s_n}}\right)\\
&\le \phi(P_{m_1,\dots,m_n})\phi\left(\frac{\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} p_{s_1,\cdots,s_n}\left(a_{s_1,\dots,s_n}/p_{s_1,\dots,s_n}\right)}{\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} p_{s_1,\dots,s_n}}\right)\\
&\le \frac{\phi(P_{m_1,\dots,m_n})}{P_{m_1,\dots,m_n}}\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n} p_{s_1,\dots,s_n}\phi\left(\frac{a_{s_1,\dots,s_n}}{p_{s_1,\dots,s_n}}\right)\\
&\le \frac{\phi(P_{m_1,\dots,m_n})}{P_{m_1,\dots,m_n}}(m_1\cdots m_n)^{1/\alpha}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(p_{s_1,\dots,s_n}\phi\left(\frac{a_{s_1,\dots,s_n}}{p_{s_1,\dots,s_n}}\right)\right)^{\beta}\right)^{1/\beta}.
\end{aligned} \tag{2.16}$$

Similarly,

$$\phi(B_{n_1,\dots,n_n}) \le \frac{\phi(Q_{n_1,\dots,n_n})}{Q_{n_1,\dots,n_n}}(n_1\cdots n_n)^{1/\beta}\left(\sum_{n_1=1}^{n_1}\cdots\sum_{n_n=1}^{n_n}\left(q_{t_1,\dots,t_n}\psi\left(\frac{b_{t_1,\dots,t_n}}{q_{t_1,\dots,t_n}}\right)\right)^{\alpha}\right)^{1/\alpha}. \tag{2.17}$$

By (2.16) and (2.17), and using the elementary inequality:

$$a^{1/\alpha}b^{1/\beta} \le \frac{a}{\alpha t^{1/\beta}} + \frac{t^{1/\alpha}b}{\beta}, \tag{2.18}$$

where $1/\alpha + 1/\beta = 1, \alpha > 1, b > 0, a > 0$, and $t > 0$, we have

$$\begin{aligned}
\phi(A_{m_1,\dots,m_n})\phi(B_{n_1,\dots,n_n}) \le{}& \left(\frac{m_1\cdots m_n}{\alpha t^{1/\beta}} + \frac{n_1\cdots n_n t^{1/\alpha}}{\beta}\right)\\
&\times\frac{\phi(P_{m_1,\dots,m_n})}{P_{m_1,\dots,m_n}}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(p_{s_1,\dots,s_n}\phi\left(\frac{a_{s_1,\dots,s_n}}{p_{s_1,\dots,s_n}}\right)\right)^{\beta}\right)^{1/\beta}\\
&\times\frac{\phi(Q_{n_1,\dots,n_n})}{Q_{n_1,\dots,n_n}}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(q_{t_1,\dots,t_n}\psi\left(\frac{b_{t_1,\dots,t_n}}{q_{t_1,\dots,t_n}}\right)\right)^{\alpha}\right)^{1/\alpha}.
\end{aligned} \tag{2.19}$$

Dividing both sides of (2.19) by $(m_1\cdots m_n\beta + n_1\cdots n_n\alpha t/\alpha\beta t^{1/\beta})$, and summing up over n_i from 1 to r_i $(i = 1, 2, \ldots, n)$ first, then summing up over m_i from 1 to k_i $(i = 1, 2, \ldots, n)$, using again inverse Hölder's inequality, and then interchanging the order of summation, we obtain

$$
\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\frac{\alpha\beta t^{1/\beta}\phi(A_{m_1,\ldots,m_n})\psi(B_{n_1,\ldots,n_n})}{m_1\ldots m_n\beta + n_1\ldots n_n\alpha t}\\
&\quad\le \sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\left(\frac{\phi(P_{m_1,\ldots,m_n})}{P_{m_1,\ldots,m_n}}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(p_{s_1,\ldots,s_n}\phi\left(\frac{a_{s_1,\ldots,s_n}}{p_{s_1,\ldots,s_n}}\right)\right)^{\beta}\right)^{1/\beta}\right)\\
&\qquad\times\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\left(\frac{\phi(Q_{n_1,\ldots,n_n})}{Q_{n_1,\ldots,n_n}}\left(\sum_{n_1=1}^{n_1}\cdots\sum_{n_n=1}^{n_n}\left(q_{t_1,\ldots,t_n}\psi\left(\frac{b_{t_1,\ldots,t_n}}{q_{t_1,\ldots,t_n}}\right)\right)^{\alpha}\right)^{1/\alpha}\right)\\
&\quad\le\left(\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\left(\frac{\phi(P_{m_1,\ldots,m_n})}{P_{m_1,\ldots,m_n}}\right)^{\alpha}\right)^{1/\alpha}\\
&\qquad\times\left(\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(p_{s_1,\ldots,s_n}\phi\left(\frac{a_{s_1,\ldots,s_n}}{p_{s_1,\ldots,s_n}}\right)\right)^{\beta}\right)\right)^{1/\beta}\\
&\qquad\times\left(\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\left(\frac{\psi(Q_{n_1,\ldots,n_n})}{Q_{n_1,\ldots,n_n}}\right)^{\beta}\right)^{1/\beta}\\
&\qquad\times\left(\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(q_{t_1,\ldots,t_n}\psi\left(\frac{b_{t_1,\ldots,t_n}}{q_{t_1,\ldots,t_n}}\right)\right)^{\alpha}\right)\right)^{1/\alpha}\\
&\quad= M(k_1,\ldots,k_n,r_1,\ldots,r_n,\alpha,\beta)\\
&\qquad\times\left\{\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{j=1}^{n}(k_j-m_j+1)\left(p_{m_1,\ldots,m_n}\phi\left(\frac{a_{m_1,\ldots,m_n}}{p_{m_1,\ldots,m_n}}\right)\right)^{\beta}\right\}^{1/\beta}\\
&\qquad\times\left\{\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\prod_{j=1}^{n}(r_j-n_j+1)\left(q_{n_1,\ldots,n_n}\psi\left(\frac{b_{n_1,\ldots,n_n}}{q_{n_1,\ldots,m_n}}\right)\right)^{\alpha}\right\}^{1/\alpha}.
\end{aligned}
\tag{2.20}
$$

The proof is complete. □

Remark 2.4. Taking $\alpha = \beta = n = j = 2$, (2.14) becomes

$$\begin{aligned}
&\sum_{m=1}^{k_1}\sum_{m_2=1}^{k_2}\left(\sum_{n_1=1}^{r_1}\sum_{n_2=1}^{r_2}\frac{\phi(A_{m_1,m_2})\psi(B_{n_1,n_2})}{m_1m_2t^{-1/2}+n_1n_2t^{1/2}}\right)\\
&\quad\le \overline{M}(k_1,k_2,r_1,r_2)\\
&\qquad\times\left(\sum_{m=1}^{k_1}\sum_{m_2=1}^{k_2}(k_1-m_1+1)(k_2-m_2+1)\left(p_{m_1,m_2}\phi\left(\frac{a_{m_1,\dots,m_n}}{p_{m_1,\dots,m_n}}\right)\right)^2\right)^{1/2}\\
&\qquad\times\left(\sum_{n=1}^{r_1}\sum_{n_2=1}^{r_2}(r_1-n_1+1)(r_2-n_2+1)\left(q_{n_1,n_2}\psi\left(\frac{b_{n_1,\dots,n_n}}{q_{n_1,\dots,n_n}}\right)\right)^2\right)^{1/2},
\end{aligned}\tag{2.21}$$

where

$$\overline{M}(k_1,k_2,r_1,r_2) = \left(\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\left(\frac{\phi(P_{m_1,m_2})}{P_{m_1,m_2}}\right)^2\right)^{1/2}\left(\sum_{n_1=1}^{r_1}\sum_{n_2=1}^{r_2}\left(\frac{\psi(Q_{n_1,n_2})}{Q_{n_1,n_2}}\right)^2\right)^{1/2}.\tag{2.22}$$

Taking $t = 1$, and changing $\{a_{m_1,m_2}\}$, $\{b_{n_1,n_2}\}$, $\{A_{m_1,m_2}\}$, and $\{B_{n_1,n_2}\}$ into $\{a_m\}$, $\{b_n\}$, $\{A_m\}$, and $\{B_n\}$, respectively, and with suitable changes, (2.21) reduces to Pachpatte [1, Inequality (7)].

Theorem 2.5. *Let $\{a_{m_1,\dots,m_n}\}$, $\{b_{n_1,\dots,n_n}\}$, $\{p_{m_1,\dots,m_n}\}$, $\{q_{n_1,\dots,n_n}\}$, $P_{m_1,\dots,m_n}$, and $Q_{n_1,\dots,n_n}$, α,β,t, be as defined in Theorem 2.3. Define*

$$\begin{aligned}
A_{m_1,\dots,m_n} &= \frac{1}{P_{m_1,\dots,m_n}}\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}p_{s_1,\dots,s_n}a_{s_1,\dots,s_n},\\
B_{n1,\dots,n2} &= \frac{1}{Q_{n_1,\dots,n_n}}\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}q_{t_1,\dots,t_n}b_{t_1,\dots,t_n},
\end{aligned}\tag{2.23}$$

for $m_i = 1,2,\dots,k_i$, and $n_i = 1,2,\dots,r_i$ $(i = 1,2,\dots,n)$, where k_i, r_i $(i = 1,\dots,n)$ are natural numbers. Let ϕ and ψ be real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0,+\infty)$. Then

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\frac{\alpha\beta t^{1/\beta}P_{m_1,\dots,m_n}Q_{n_1,\dots,n_n}\phi(A_{m_1,\dots,m_n})\psi(B_{n_1,\dots,n_n})}{m_1\cdots m_n\beta+n_1\cdots n_n\alpha t}\\
&\quad= (k_1\cdots k_n)^{1/\alpha}(r_1\cdots r_n)^{1/\beta}\\
&\qquad\times\left\{\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{j=1}^{n}(k_j-m_j+1)\left(p_{m_1,\dots,m_n}\phi(a_{m_1,\dots,m_n})\right)^{\beta}\right\}^{1/\beta}\\
&\qquad\times\left\{\sum_{n_1=1}^{r_1}\cdots\sum_{n_n=1}^{r_n}\prod_{j=1}^{n}(r_j-n_j+1)\left(q_{n_1,\dots,n_n}\psi(b_{n_1,\dots,n_n})\right)^{\alpha}\right\}^{1/\alpha}.
\end{aligned}\tag{2.24}$$

Proof. By the hypotheses, Jensen's inequality, and Hölder's inequality, it is easy to observe that

$$\begin{aligned}
\phi(A_{m_1,\ldots,m_n}) &= \phi\left(\frac{1}{P_{m_1,\ldots,m_n}}\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}p_{s_1,\ldots,s_n}a_{s_1,\ldots,s_n}\right)\\
&\le \frac{1}{P_{m_1,\ldots,m_n}}\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}p_{s_1,\ldots,s_n}\phi(a_{s_1,\ldots,s_n})\\
&\le \frac{1}{P_{m_1,\ldots,m_n}}(m_1\cdots m_n)^{1/\alpha}\left(\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\left(p_{s_1,\ldots,s_n}\phi(a_{s_1,\ldots,s_n})\right)^{\beta}\right)^{1/\beta},
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
\psi(B_{n_1,\ldots,n_n}) &= \psi\left(\frac{1}{Q_{n_1,\ldots,n_n}}\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}q_{t_1,\ldots,t_n}b_{t_1,\ldots,t_n}\right)\\
&\le \frac{1}{Q_{n_1,\ldots,n_n}}\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}q_{t_1,\ldots,t_n}\psi(b_{t_1,\ldots,t_n})\\
&\le \frac{1}{Q_{n_1,\ldots,n_n}}(n_1\cdots n_n)^{1/\beta}\left(\sum_{t_1=1}^{n_1}\cdots\sum_{t_n=1}^{n_n}\left(q_{t_1,\ldots,t_n}\psi(b_{t_1,\ldots,t_n})\right)^{\alpha}\right)^{1/\alpha}.
\end{aligned} \tag{2.26}$$

Proceeding now much as in the proof of Theorems 2.1 and 2.3, and with suitable modifications, it is not hard to arrive at the desired inequality. The details are omitted here. □

Remark 2.6. In the special case where $j = 1, t = 1, \alpha = \beta = 2$, and $n = 1$, Theorem 2.5 reduces to the following result.

Theorem C. *Let $\{a_m\}, \{b_n\}, \{p_m\}, \{q_n\}, P_m, Q_n$ be as defined in Theorem B. Define $A_m = (1/P_m)\sum_{s=1}^{m} p_s a_s$, and $B_n = (1/Q_n)\sum_{t=1}^{n} q_t b_t$ for $m = 1,\ldots,k$, and $n = 1,\ldots,r$, where k, r are natural numbers. Let ϕ and ψ be real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0,+\infty)$. Then*

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n} \le \frac{1}{2}(kr)^{1/2}&\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi(a_m)\right)^2\right)^{1/2}\\
&\times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi(b_n)\right)^2\right)^{1/2}.
\end{aligned} \tag{2.27}$$

This is the new inequality of Pachpatte in [1, Theorem 4].

Remark 2.7. Taking $j = 1, t = 1, p_m = 1, q_n = 1, \alpha = \beta = 2$, and $n = 1$ in Theorem 2.5, and in view of $P_m = m, Q_n = n$, we obtain the following theorem.

Theorem D. *Let $\{a_m\}$, $\{b_n\}$ be as defined in Theorem A. Define $A_m = (1/m)\sum_{s=1}^{m} a_s$, and $B_n = (1/n)\sum_{t=1}^{n} b_t$, for $m = 1,\ldots,k$, and $n = 1,\ldots,r$, where k, r are natural numbers. Let ϕ and ψ be real-valued, nonnegative, convex functions defined on $R_+ = [0,+\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{m+n}\phi(A_m)\psi(B_n) \le \frac{1}{2}(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)\left(\phi(a_m)\right)^2\right)^{1/2} \times \left(\sum_{n=1}^{r}(r-n+1)\left(\psi(b_n)\right)^2\right)^{1/2}. \tag{2.28}$$

This is the new inequality of Pachpatte in [1, Theorem 3].

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