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Research Article

The Schur Harmonic Convexity of the Hamy Symmetric Function and Its Applications

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We prove that the Hamy symmetric function $F_n(x,r) = \sum_{1\le i_1<i_2<\cdots<i_r\le n}(\prod_{j=1}^{r}x_{i_j})^{1/r}$ is Schur harmonic convex for $x \in R_+^n$. As its applications, some analytic inequalities including the well-known Weierstrass inequalities are obtained.

1. Introduction

Throughout this paper we use R^n to denote the n-dimensional Euclidean space over the field of real numbers, and $R_+^n = \{x = (x_1, x_2, \ldots, x_n) \in R^n : x_i > 0, i = 1, 2, \ldots, n\}$.

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R_+^n$ and $\alpha > 0$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ xy &= (x_1y_1, x_2y_2, \ldots, x_ny_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \\ \frac{1}{x} &= \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right). \end{aligned} \tag{1.1}$$

For $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, the Hamy symmetric function [1–3] was defined as

$$\begin{aligned} F_n(x,r) &= F_n(x_1, x_2, \ldots, x_n; r) \\ &= \sum_{1\le i_1<i_2<\cdots<i_r\le n}\left(\prod_{j=1}^{r}x_{i_j}\right)^{1/r}, \quad r = 1, 2, \ldots, n. \end{aligned} \tag{1.2}$$

Corresponding to this is the rth order Hamy mean

$$\sigma_n(x,r) = \sigma_n(x_1, x_2, \ldots, x_n; r) = \frac{1}{\binom{n}{r}} F_n(x,r), \tag{1.3}$$

where $\binom{n}{r} = n!/(n-r)!r!$. Hara et al. [1] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x,n) \leq \sigma_n(x,n-1) \leq \cdots \leq \sigma_n(x,2) \leq \sigma_n(x,1) = A_n(x). \tag{1.4}$$

Here $A_n(x) = 1/n \sum_{i=1}^{n} x_i$ and $G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ denote the classical arithmetic and geometric means, respectively.

The paper [4] by Ku et al. contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave, the more results can also be found in the book [5] by Bullen. In [2], the Schur convexity of Hamy's symmetric function and its generalization were discussed. In [3] , Jiang defined the dual form of the Hamy symmetric function as follows:

$$H_n^*(x,r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\sum_{j=1}^{r} x_{i_j}{}^{1/r} \right), \quad r = 1, 2, \ldots, n, \tag{1.5}$$

discussed the Schur concavity Schur convexity of $H_n^*(x,r)$, and established some analytic inequalities.

The main purpose of this paper is to investigate the Schur harmonic convexity of the Hamy symmetric function $F_n(x,r)$. Some analytic inequalities including Weierstrass inequalities are established.

2. Definitions and Lemmas

Schur convexity was introduced by Schur in 1923 [6], and it has many important applications in analytic inequalities [7–12], linear regression [13], graphs and matrices [14], combinatorial optimization [15], information-theoretic topics [16], Gamma functions [17], stochastic orderings [18], reliability [19], and other related fields.

For convenience of readers, we recall some definitions as follows.

Definition 2.1. A set $E_1 \subseteq R^n$ is called a convex set if $(x+y)/2 \in E_1$ whenever $x, y \in E_1$. A set $E_2 \subseteq R_+^n$ is called a harmonic convex set if $2xy/(x+y) \in E_2$ whenever $x, y \in E_2$.

It is easy to see that $E \subseteq R_+^n$ is a harmonic convex set if and only if $1/E = \{1/x : x \in E\}$ is a convex set.

Definition 2.2. Let $E \subseteq R^n$ be a convex set a function $f : E \to R^1$ is said to be convex on E if $f((x+y)/2) \leq (f(x) + f(y))/2$ for all $x, y \in E$. Moreover, f is called a concave function if $-f$ is a convex function.

Definition 2.3. Let $E \subseteq R_+^n$ be a harmonic convex set a function $f : E \longrightarrow R_+^1$ is called a harmonic convex (or concave, resp.) function on E if $f(2xy/(x+y)) \leq$ (or $\geq$ resp.) $2f(x)f(y)/(f(x)+f(y))$ for all $x, y \in E$.

Definitions 2.2 and 2.3 have the following consequences.

Fact A. If $E_1 \subseteq R_+^n$ is a harmonic convex set and $f : E_1 \longrightarrow R_+^1$ is a harmonic convex function, then

$$F(x) = \frac{1}{f(1/x)} : \frac{1}{E_1} \longrightarrow R_+^1 \tag{2.1}$$

is a concave function. Conversely, if $E_2 \subseteq R_+^n$ is a convex set and $F : E_2 \longrightarrow R_+^1$ is a convex function, then

$$f(x) = \frac{1}{F(1/x)} : \frac{1}{E_2} \longrightarrow R_+^1 \tag{2.2}$$

is a harmonic concave function.

Definition 2.4. Let $E \subseteq R^n$ be a set a function $F : E \longrightarrow R^1$ is called a Schur convex function on E if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \tag{2.3}$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E, such that $x \prec y$, that is,

$$\begin{gathered} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{gathered} \tag{2.4}$$

where $x_{[i]}$ denotes the ith largest component in x. F is called a Schur concave function on E if $-F$ is a Schur convex function on E .

Definition 2.5. Let $E \subseteq R_+^n$ be a set a function $F : E \longrightarrow R_+^1$ is called a Schur harmonic convex (or concave, resp.) function on E if

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) \leq (\text{or} \geq \ \text{resp.})\ F\left(\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n}\right) \tag{2.5}$$

for each pair of $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E, such that $x \prec y$.

Definitions 2.4 and 2.5 have the following consequences.

Fact B. Let $E \subseteq R_+^n$ be a set, and $H = 1/E = \{1/x : x \in E\}$, then $f : E \to R_+^1$ is a Schur harmonic convex (or concave, resp.) function on E if and only if $1/f(1/x)$ is a Schur concave (or convex, resp.) function on H.

The notion of generalized convex function was first introduced by Aczél in [20]. Later, many authors established inequalities by using harmonic convex function theory [21–28]. Recently, Anderson et al. [29] discussed an attractive class of inequalities, which arise from the notation of harmonic convex functions.

The following well-known result was proved by Marshall and Olkin [6].

Theorem A. *Let $E \subseteq R^n$ be a symmetric convex set with nonempty interior intE, and let $\varphi : E \to R^1$ be a continuous symmetric function on E. If φ is differentiable on intE, then φ is Schur convex (or concave, resp.) on E if and only if*

$$(x_i - x_j)\left(\frac{\partial\varphi}{\partial x_i} - \frac{\partial\varphi}{\partial x_j}\right) \geq (\text{or} \leq \ \text{resp.})\, 0 \tag{2.6}$$

for all $i, j = 1, 2, \ldots, n$ and $(x_1, x_2, \ldots, x_n) \in intE$. Here, E is a symmetric set means that $x \in E$ implies $Px \in E$ for any $n \times n$ permutation matrix P.

Remark 2.6. Since φ is symmetric, the Schur's condition in Theorem A, that is, (2.6) can be reduced to

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq \quad (\text{or} \leq \ \text{resp.}) \ \ 0. \tag{2.7}$$

The following Lemma 2.7 can easily be derived from Fact B, Theorem A and Remark 2.6 together with elementary computation.

Lemma 2.7. *Let $E \subseteq R_+^n$ be a symmetric harmonic convex set with nonempty interior intE, and let $\varphi : E \to R_+^1$ be a continuous symmetry function on E. If φ is differentiable on intE, then φ is Schur harmonic convex (or concave, resp.) on E if and only if*

$$(x_1 - x_2)\left({x_1}^2\frac{\partial\varphi}{\partial x_1} - {x_2}^2\frac{\partial\varphi}{\partial x_2}\right) \geq (\text{or} \leq \ \text{resp.}) \quad 0 \tag{2.8}$$

for all $(x_1, x_2, \ldots, x_n) \in intE$.

Next we introduce two lemmas, which are used in Sections 3 and 4.

Lemma 2.8 (see [5, page 234]). *For $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, if th rth order symmetric function is defined as*

$$E_n(x,r) = E_n(x_1, x_2, \ldots, x_n; r) = \begin{cases} 0, & r < 0 \text{ or } r > n, \\ 1, & r = 0, \\ \displaystyle\sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\prod_{j=1}^{r} x_{i_j} \right), & r = 1, 2, \ldots, n, \end{cases} \tag{2.9}$$

then

$$\begin{aligned} E_n(x_1, x_2, \ldots, x_n; r) = x_1 x_2 E_{n-2}(x_3, x_4, \ldots, x_n; r-2) \\ + (x_1 + x_2) E_{n-2}(x_3, x_4, \ldots, x_n; r-1) \\ + E_{n-2}(x_3, x_4, \ldots, x_n; r). \end{aligned} \tag{2.10}$$

Lemma 2.9 (see [2, Lemma 2.2]). *Suppose that $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $\sum_{i=1}^{n} x_i = s$. If $c \ge s$, then*

$$\begin{aligned} &\text{(i)} \quad \frac{c-x}{nc/s-1} = \left(\frac{c-x_1}{nc/s-1}, \frac{c-x_2}{nc/s-1}, \ldots, \frac{c-x_n}{nc/s-1} \right) \prec (x_1, x_2, \ldots, x_n) = x; \\ &\text{(ii)} \quad \frac{c+x}{nc/s+1} = \left(\frac{c+x_1}{nc/s+1}, \frac{c+x_2}{nc/s+1}, \ldots, \frac{c+x_n}{nc/s+1} \right) \prec (x_1, x_2, \ldots, x_n) = x. \end{aligned} \tag{2.11}$$

3. Main Result

In this section, we give and prove the main result of this paper.

Theorem 3.1. *The Hamy symmetric function $F_n(x, r)$, $r = 1, 2, \ldots, n$, is Schur harmonic convex in R_+^n.*

Proof. By Lemma 2.7, we only need to prove that

$$(x_1 - x_2) \left({x_1}^2 \ \frac{\partial F_n(x,r)}{\partial x_1} - {x_2}^2 \ \frac{\partial F_n(x,r)}{\partial x_2} \right) \ge 0. \tag{3.1}$$

To prove (3.1), we consider the following possible cases for r.

Case 1 ($r = 1$). Then (1.2) leads to $F_n(x, 1) = \sum_{i=1}^{n} x_i$, and (3.1) is clearly true.

Case 2 ($r = n$). Then (1.2) leads to the following identity:

$$(x_1 - x_2) \left({x_1}^2 \ \frac{\partial F_n(x,n)}{\partial x_1} - {x_2}^2 \ \frac{\partial F_n(x,n)}{\partial x_2} \right) = \frac{F_n(x,n)}{n} (x_1 - x_2)^2, \tag{3.2}$$

and therefore, (3.1) follows from (3.2).

Case 3 ($r = n-1$). Then (1.2) leads to

$$F_n(x, n-1) = \sum_{i=1}^{n} \left(\frac{\prod_{j=1}^{n} x_j}{x_i} \right)^{1/(n-1)}. \tag{3.3}$$

Simple computation yields

$$\begin{aligned} {x_1}^2 \frac{\partial F_n(x, n-1)}{\partial x_1} &= \frac{x_1}{n-1} \left[x_2^{-1/(n-1)} \left(\prod_{j=1}^{n} x_j \right)^{1/(n-1)} + \sum_{i=3}^{n} \left(\frac{\prod_{j=1}^{n} x_j}{x_i} \right)^{1/(n-1)} \right] \\ {x_2}^2 \frac{\partial F_n(x, n-1)}{\partial x_2} &= \frac{x_2}{n-1} \left[x_1^{-1/(n-1)} \left(\prod_{j=1}^{n} x_j \right)^{1/(n-1)} + \sum_{i=3}^{n} \left(\frac{\prod_{j=1}^{n} x_j}{x_i} \right)^{1/(n-1)} \right]. \end{aligned} \tag{3.4}$$

From (3.4) we get

$$\begin{aligned} &(x_1 - x_2) \left({x_1}^2 \frac{\partial F_n(x, n-1)}{\partial x_1} - {x_2}^2 \frac{\partial F_n(x, n-1)}{\partial x_2} \right) \\ &\quad = \frac{1}{n-1} (x_1 - x_2) \left(x_1^{1+1/(n-1)} - x_2^{1+1/n} \right) \left(\prod_{j=3}^{n} x_j \right)^{1/(n-1)} \\ &\qquad + \frac{(x_1 - x_2)^2}{n-1} \sum_{i=3}^{n} \left(\frac{\prod_{j=1}^{n} x_j}{x_i} \right)^{1/(n-1)}. \end{aligned} \tag{3.5}$$

Therefore, (3.1) follows from (3.5) and the fact that $x^{1+1/(n-1)}$ is increasing in R_+^1.

Case 4 ($r = 2, 3, \ldots, n-2$). Fix r and let $u = (u_1, u_2, \ldots, u_n)$ and $u_i = x_i^{1/r}, i = 1, 2, \ldots, n$. We have the following identity:

$$F_n(x_1, x_2, \ldots, x_n; r) = E_n(u_1, u_2, \ldots, u_n; r). \tag{3.6}$$

Differentiating (3.6) with respect to x_1 and x_2, respectively, and using Lemma 2.8, we get

$$\begin{aligned}
\frac{\partial F_n(x,r)}{\partial x_1} &= \sum_{i=1}^{n} \frac{\partial E_n(u,r)}{\partial u_i} \cdot \frac{\partial u_i}{\partial x_1} = \frac{\partial E_n(u,r)}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} \\
&= \frac{1}{r x_1} \sqrt[r]{x_1 x_2} E_{n-2}(u_3, u_4, \ldots, u_n; r-2) \\
&\quad + \frac{\sqrt[r]{x_1}}{r x_1} E_{n-2}(u_3, u_4, \ldots, u_n; r-1), \\
\frac{\partial F_n(x,r)}{\partial x_2} &= \frac{1}{r x_2} \sqrt[r]{x_1 x_2} E_{n-2}(u_3, u_4, \ldots, u_n; r-2) \\
&\quad + \frac{\sqrt[r]{x_2}}{r x_2} E_{n-2}(u_3, u_4, \ldots, u_n; r-1).
\end{aligned} \tag{3.7}$$

From (3.7) we obtain

$$\begin{aligned}
&(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,r)}{\partial x_2}\right) \\
&\quad = \frac{\sqrt[r]{x_1 x_2}}{r}(x_1 - x_2)^2 E_{n-2}(u_3, u_4, \ldots, u_n; r-2) \\
&\qquad + \frac{1}{r}(x_1 - x_2)\left(x_1^{1+1/r} - x_2^{1+1/r}\right) E_{n-2}(u_3, u_4, \ldots, u_n; r-1).
\end{aligned} \tag{3.8}$$

Therefore, (3.1) follows from (3.8) and the fact that $x^{1+1/r}$ is increasing in R_+^1.

□

4. Applications

In this section, making use of our main result, we give some inequalities.

Theorem 4.1. *Suppose that $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ with $\sum_{i=1}^n x_i = s$. If $c \geq s$ and $r = 1, 2, \ldots, n$, then*

$$\begin{aligned}
&\text{(i)} \quad \left(\frac{nc}{s} - 1\right) F_n\left(\frac{1}{c - x_1}, \frac{1}{c - x_2}, \ldots, \frac{1}{c - x_n}; r\right) \leq F_n\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}; r\right); \\
&\text{(ii)} \quad \left(\frac{nc}{s} + 1\right) F_n\left(\frac{1}{c + x_1}, \frac{1}{c + x_2}, \ldots, \frac{1}{c + x_n}; r\right) \leq F_n\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}; r\right).
\end{aligned} \tag{4.1}$$

Proof. The proof follows from Theorem 3.1 and Lemma 2.9 together with (1.2). □

If taking $r = 1$ and $r = n$ in Theorem 4.1, respectively, then we have the following corollaries.

Corollary 4.2. *Suppose that* $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ *with* $\sum_{i=1}^n x_i = s$. *If* $c \geq s$, *then*

$$\text{(i)} \quad \frac{\sum_{i=1}^n 1/x_i}{\sum_{i=1}^n 1/(c - x_i)} \geq \frac{nc}{s} - 1;$$
$$\text{(ii)} \quad \frac{\sum_{i=1}^n 1/x_i}{\sum_{i=1}^n 1/(c + x_i)} \geq \frac{nc}{s} + 1. \tag{4.2}$$

Corollary 4.3. *Suppose that* $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ *with* $\sum_{i=1}^n x_i = s$. *If* $c \geq s$, *then*

$$\text{(i)} \quad \prod_{i=1}^n \frac{c - x_i}{x_i} \geq \left(\frac{nc}{s} - 1\right)^n;$$
$$\text{(ii)} \quad \prod_{i=1}^n \frac{c + x_i}{x_i} \geq \left(\frac{nc}{s} + 1\right)^n. \tag{4.3}$$

Taking $c = s = 1$ in Corollaries 4.2 and 4.3, respectively, we get the following.

Corollary 4.4. *If* $x_i > 0, i = 1, 2, \ldots, n$, *and* $\sum_{i=1}^n x_i = 1$, *then*

$$\text{(i)} \quad \frac{\sum_{i=1}^n 1/x_i}{\sum_{i=1}^n 1/(1 - x_i)} \geq n - 1;$$
$$\text{(ii)} \quad \frac{\sum_{i=1}^n 1/x_i}{\sum_{i=1}^n 1/(1 + x_i)} \geq n + 1. \tag{4.4}$$

Corollary 4.5 (Weierstrass inequalities [30, Page 260]). *If* $x_i > 0, i = 1, 2, \ldots, n$, *and* $\sum_{i=1}^n x_i = 1$, *then*

$$\text{(i)} \quad \prod_{i=1}^n \left(x_i^{-1} - 1\right) \geq (n-1)^n;$$
$$\text{(ii)} \quad \prod_{i=1}^n \left(x_i^{-1} + 1\right) \geq (n+1)^n. \tag{4.5}$$

Theorem 4.6. *If* $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ *and* $r \in \{1, 2, \ldots, n\}$, *then*

$$F_n(x, r) = F_n(x_1, x_2, \ldots, x_n; r) \geq \frac{n(n!)}{r!(n-r)! \sum_{i=1}^n 1/x_i}. \tag{4.6}$$

Proof. Let $t = (1/n) \sum_{i=1}^n 1/x_i$, and $T = (t, t, \ldots, t)$ be the n-tuple, then obviously

$$T = (t, t, \ldots, t) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) = \frac{1}{x}. \tag{4.7}$$

□

Therefore, Theorem 4.6 follows from Theorem 3.1, (4.7) ,and (1.2).

Theorem 4.7. *Let A be an n-dimensional simplex in n-dimensional Euclidean space $R^n (n \geq 3)$, and $\{A_1, A_2, \ldots, A_{n+1}\}$ be the set of vertices. Let P be an arbitrary point in the interior of A. If B_i is the intersection point of the extension line of A_iP and the $(n-1)$-dimensional hyperplane opposite to the point A, and $r \in \{1, 2, \ldots, n+1\}$, then one has*

$$F_{n+1}\left(\frac{A_1B_1}{PB_1}, \frac{A_2B_2}{PB_2}, \ldots, \frac{A_{n+1}B_{n+1}}{PB_{n+1}}; r\right) \geq \frac{(n+1)\,[(n+1)!]}{r!\,(n-r+1)!},$$
$$F_{n+1}\left(\frac{A_1B_1}{PA_1}, \frac{A_2B_2}{PA_2}, \ldots, \frac{A_{n+1}B_{n+1}}{PA_{n+1}}; r\right) \geq \frac{(n+1)\,[(n+1)!]}{n \cdot r!\,(n-r+1)!}. \tag{4.8}$$

Proof. It is easy to see that

$$\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1,$$
$$\sum_{i=1}^{n+1} \frac{PA_i}{A_iB_i} = n. \tag{4.9}$$

(4.9) implies that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1}, \frac{PB_2}{A_2B_2}, \ldots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right),$$
$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \ldots, \frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1}, \frac{PA_2}{A_2B_2}, \ldots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.10}$$

Therefore, Theorem 4.7 follows from Theorem 3.1, (4.10), and (1.2). □

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