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Research Article

Markov Inequalities for Polynomials with Restricted Coefficients

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Essentially sharp Markov-type inequalities are known for various classes of polynomials with constraints including constraints of the coefficients of the polynomials. For $\mathbb{N}$ and $\delta > 0$ we introduce the class $\mathcal{F}_{n,\delta}$ as the collection of all polynomials of the form $P(x) = \sum_{k=h}^{n} a_k x^k$, $a_k \in \mathbb{Z}$, $|a_k| \leq n^{\delta}$, $|a_h| = \max_{h \leq k \leq n} |a_k|$. In this paper, we prove essentially sharp Markov-type inequalities for polynomials from the classes $\mathcal{F}_{n,\delta}$ on $[0,1]$. Our main result shows that the Markov factor $2n^2$ valid for all polynomials of degree at most n on $[0,1]$ improves to $c_\delta n \log(n+1)$ for polynomials in the classes $\mathcal{F}_{n,\delta}$ on $[0,1]$.

1. Introduction

In this paper, n always denotes a nonnegative integer; c and c_i always denote absolute positive constants. In this paper c_δ will always denote a positive constant depending only on δ the value of which may vary from place to place. We use the usual notation $L^p = L^p[a,b]$ $(0 < p \leq \infty, -\infty \leq a < b \leq \infty)$ to denote the Banach space of functions defined on $[a,b]$ with the norms

$$\|f\|_p = \|f\|_{L^p[a,b]} = \left\{ \int_a^b |f(x)|^p dx \right\}^{1/p} < \infty, \quad 0 < p < \infty,$$
$$\|f\|_{[a,b]} = \|f\|_{L^\infty[a,b]} = \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)|. \tag{1.1}$$

We introduce the following classes of polynomials. Let

$$P_n = \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{R} \right\} \tag{1.2}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients. Let

$$P_n^c = \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{C} \right\} \tag{1.3}$$

denote the set of all algebraic polynomials of degree at most n with complex coefficients. For $\delta > 0$ we introduce the class $\mathcal{F}_{n,\delta}$ as the collection of all polynomials of the form

$$P(x) = \sum_{k=h}^{n} a_k x^k, \quad a_k \in \mathbb{Z}, \quad |a_k| \leq n^{\delta}, \quad |a_h| = \max_{h \leq k \leq n} |a_k|. \tag{1.4}$$

So obviously

$$\mathcal{F}_{n,\delta} \subset P_n \subset P_n^c. \tag{1.5}$$

The following so-called Markov inequality is an important tool to prove inverse theorems in approximation theory. See, for example, Duffin and Schaeffer [1], Devore and Lorentz [2], and Borwein and Erdelyi [3].

Markov inequality. The inequality

$$\|P'\|_p \leq n^2 \|P\|_p, \quad 1 \leq p \leq \infty \tag{1.6}$$

holds for every $P \in P_n$.

It is well known that there have been some improvements of Markov-type inequality when the coefficients of polynomial are restricted; see, for example, [3–7]. In [5], Borwein and Erdélyi restricted the coefficients of polynomials and improved the Markov inequality as in following form.

Theorem 1.1. *There is an absolute constant $c > 0$ such that*

$$\|P'\|_{[0,1]} \leq \text{cn } \log(n+1) \|P\|_{[0,1]} \tag{1.7}$$

for every $P \in L_n = \{f : f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \{-1, 0, 1\}\}$.

We notice that the coefficients of polynomials in L_n only take three integers: $-1, 0$, and 1. So, it is natural to raise the question: can we take the coefficients of polynomials as more general integers, and the conclusion of the theorem still holds? This question was not posed by Borwein and Erdélyi in [5, 6]. Also, we have not found the study for the question by now. This paper addresses the question. We shall give an affirmative answer. Indeed, we will prove the following results.

Theorem 1.2. *There are an absolute constant $c_1 > 0$ and a positive constant c_δ depending only on δ such that*

$$c_1 n \log(n+1) \leq \max_{0 \neq P_n \in \mathcal{F}_{n,\delta}} \frac{|P_n'(1)|}{\|P_n\|_{[0,1]}} \leq \max_{0 \neq P_n \in \mathcal{F}_{n,\delta}} \frac{\|P_n'\|_{[0,1]}}{\|P_n\|_{[0,1]}} \leq c_\delta n \log(n+1). \tag{1.8}$$

Our proof follows [6] closely.

Remark 1.3. Theorem 1.2 does not contradict [6, Theorem 2.4] since the coefficients of polynomials in $\mathcal{F}_{n,\delta}$ are assumed to be integers, in which case there is a room for improvement.

2. The Proof of Theorem

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let $M \in \mathbb{R}$ and $n, m \in \mathbb{N}$. Suppose $m \leq M \leq 2n$, f is analytical inside and on the ellipse $A_{n,M}$, which has focal points $(0,0)$ and $(1,0)$, and major axis*

$$\left[-\frac{M}{n}, 1+\frac{M}{n}\right]. \tag{2.1}$$

Let $B_{n,m,M}$ be the ellipse with focal points $(0,1)$ and $(1,0)$, and major axis

$$\left[-\frac{m^2}{nM}, 1+\frac{m^2}{nM}\right]. \tag{2.2}$$

Then there is an absolute constant $c_3 > 0$ such that

$$\max_{z \in B_{n,m,M}} \log\left|f(z)\right| \leq \max_{z \in [0,1]} \log\left|f(z)\right| + \frac{c_3 m}{M}\left(\max_{z \in A_{n,m}} \log\left|f(z)\right| - \max_{z \in [0,1]} \log\left|f(z)\right|\right). \tag{2.3}$$

Proof. The proof of Lemma 2.1 is mainly based on the famous Hadamard's Three Circles Theorem and the proof [6, Corollary 3.2]. In fact, if one uses it with n replaced by n/m and α replaced by M/m, Lemma 2.1 follows immediately from [6, Corollary 3.2]. □

Lemma 2.2. *Let $P \in \mathcal{F}_{n,\delta}$ with $\|P\|_{[0,1]} = \exp(-M)$, $M \geq \log(n+1)$. Suppose $m \in \mathbb{N}$ and $1 \leq m \leq M$. Then there is a constant $c_\delta \geq 2$ such that*

$$\left\|P^{(m)}\right\|_{[0,1]} \leq m!\left(\frac{c_\delta n M}{m^2}\right)^m \|P\|_{[0,1]}. \tag{2.4}$$

Proof. By Chebyshev's inequality, there is an $s_{n-1} \in P_{n-1}$ such that

$$\begin{aligned}
\|P(x)\|_{[0,1]} &= \left\|P\left(\frac{y+1}{2}\right)\right\|_{[-1,1]} \\
&= 2^{-n}\left\|\sum_{j=0}^{n} 2^{n-j} a_j (y+1)^j\right\|_{[-1,1]} \\
&= 2^{-n}|a_n|\left\|y^n - s_{n-1}\right\|_{[-1,1]} \geq 2^{-n} \times 2^{1-n} = 2 \times 4^{-n},
\end{aligned} \tag{2.5}$$

for every $P \in \mathcal{F}_{n,\delta}$ with $a_n \neq 0$. Therefore, $M \leq n \log 4$. Because of the assumption on $P \in \mathcal{F}_{n,\delta}$, we can write

$$\max_{z \in [0,1]} \log|P(z)| = -M. \tag{2.6}$$

Recalling the facts that

$$\max_{z \in A_{n,M}} |z| \leq 1 + \frac{M}{n}, \tag{2.7}$$

$P \in \mathcal{F}_{n,\delta}$, and $z \in A_{n,M}$ we obtain

$$\begin{aligned}
\log|P(z)| = \log \sum_{k=0}^{n} \left|a_k z^k\right| &\leq \log\left(n^{\delta}(n+1)\left(1 + \frac{M}{n}\right)^{n+1}\right) \\
&\leq \log\left(n^{\delta}\right) + \log(n+1) + (n+1)\frac{M}{n} \leq c_{\delta} M.
\end{aligned} \tag{2.8}$$

Now by Lemma 2.1 we have

$$\begin{aligned}
\max_{z \in B_{n,m,M}} |P(z)| &= \max_{z \in B_{n,m,M}} \exp\left(\log|P(z)|\right) \\
&\leq \max_{z \in [0,1]} \exp\left(\log|P(z)|\right) \exp\left(\frac{c_3 m}{M}\left(\max_{z \in A_{n,M}} \log|P(z)| - \max_{z \in [0,1]} \log|P(z)|\right)\right) \\
&\leq \max_{z \in [0,1]} |P(z)| \exp\left(\frac{c_3 m}{M}(c_{\delta}+1)M\right) \leq (c_{\delta})^m \max_{z \in [0,1]} |P(z)|.
\end{aligned} \tag{2.9}$$

Let $y \in [0,1]$, then there is an absolute constant $c_4 \geq 2$ such that

$$B_{\rho} := \left\{w : \left|w - y\right| = \rho := \frac{m^2}{c_4 n M}\right\} \subseteq B_{n,m,M}. \tag{2.10}$$

By Cauchy's integral formula and the above inequality, we obtain

$$\begin{aligned}
\left|P^{(m)}(y)\right| &= \left|\frac{m!}{2\pi i}\int_{B_{n,m,M}} \frac{P(z)}{(z-y)^{m+1}}dz\right| \\
&\leq \frac{m!}{2\pi}(c_\delta)^m \|P\|_{[0,1]}\int_{B_\rho} \frac{dz}{(z-y)^{m+1}} \leq \frac{m!}{2\pi}(c_\delta)^m \|P\|_{[0,1]}\int_{B_\rho} \frac{\rho d e^{i\theta}}{\rho^{m+1}} \\
&\leq m!\left(\frac{c_\delta n M}{m^2}\right)^m \|P\|_{[0,1]}.
\end{aligned} \tag{2.11}$$

The proof of Lemma 2.2 is complete. □

Proof of Theorem 1.2. Noting $\mathcal{F}_{n,\delta} \supseteq L_n$ and the fact

$$c_1 n \log(n+1) \leq \max_{0 \neq P_n \in L_n} \frac{|P_n'(1)|}{\|P_n\|_{[0,1]}} \tag{2.12}$$

proved by [6], we only need to prove the upper bound. To obtain

$$\left|P'(y)\right| \leq c_\delta n \log(n+1)\|P\|_{[0,1]}, \tag{2.13}$$

we distinguish four cases.

Case 1. $y \in [0,1/4]$. Let y be an arbitrary number in $[0,1/4]$, then

$$\begin{aligned}
\left|P'(y)\right| &\leq |a_h| n y^h \left(1+y+y^2+\cdots\right) \\
&\leq 2|a_h| n y^h \left(1-y-y^2-\cdots\right) \\
&= 2ny^h\left(|a_h| - |a_h|y - |a_h|y^2 - \cdots\right) \\
&\leq 2n\left|P(y)\right| \\
&\leq 2n\|P\|_{[0,1]}.
\end{aligned} \tag{2.14}$$

Case 2. $y \in [1-\mu^2/c_\delta n M, 1]$ and $\|P\|_{[0,1]} = \exp(-M) \leq (2n+2)^{-4}$, where $\mu = \min\{[M], k\}$ and k denotes the number of zeros of P at 1. Let n be a positive integer. If $P \in \mathcal{F}_{n,\delta}$ satisfies the assumptions, then $|P^{(k)}(1)| \neq 0$, and $P^{(r)}(1) = 0$ $(0 \leq r < k)$. Therefore, Markov inequality implies

$$1 \leq \left|P^{(k)}(1)\right| \leq n^2 \cdots (n-k+1)^2 \|P\|_{[0,1]} \leq (2n)^{2k}\exp(-M). \tag{2.15}$$

Hence

$$k \geq \frac{M}{2\log(2n)}. \tag{2.16}$$

So, the last inequality and $M \geq 4\log(2n+2)$ imply

$$\begin{gathered}\mu \geq \min\left\{M-1, \frac{M}{2\log(2n)}\right\} \geq \frac{M}{2\log(2n+2)} \geq 2, \\ \frac{M}{\mu} \leq 2\log(2n+2).\end{gathered} \tag{2.17}$$

Now using Taylor's theorem, Lemma 2.2 with $m = \mu - 1$, the above inequality, and the fact $P^{(r)}(1) = 0$ $(0 \leq r < k)$, we obtain

$$\begin{aligned}\left|P'(y)\right| &\leq \frac{1}{(\mu-1)!}\left\|\left(P'\right)^{(\mu-1)}\right\|_{[1-y,1]}(1-y)^{\mu-1} \\ &\leq \frac{\mu!}{(\mu-1)!}\left(\frac{c_\delta nM}{\mu^2}\right)^{\mu}\|P\|_{[0,1]}(1-y)^{\mu-1} \\ &\leq \frac{\mu!}{(\mu-1)!}\left(\frac{c_\delta nM}{\mu^2}\right)^{\mu}\|P\|_{[0,1]}\left(\frac{\mu^2}{c_\delta nM}\right)^{\mu-1} \\ &\leq 2^{1-\mu}c_\delta n\frac{M}{\mu}\|P\|_{[0,1]} \leq c_\delta n\log(2n+2)\|P\|_{[0,1]}.\end{aligned} \tag{2.18}$$

Case 3. $y \in [1/4, 1-\mu^2/c_\delta nM]$ and $\|P\|_{[0,1]} = \exp(-M) \leq (2n+2)^{-4}$. Let $(u,v) \in B_{n,m,M}$. We have $u = 1/2 + a\cos\theta$, $v = b\sin\theta$, where $2a$ and $2b$ are the major axis and minor axis of $B_{n,m,M}$, respectively, and $0 \leq \theta < 2\pi$. Let $m = 1$, we see

$$a = \frac{1}{2} + \frac{1}{nM}, \quad b = \sqrt{\frac{1}{nM}\left(1+\frac{1}{nM}\right)}. \tag{2.19}$$

Denote

$$h(\theta) = \left(\frac{1}{2} - y + a\cos\theta\right)^2 + b^2\sin^2\theta. \tag{2.20}$$

The solution of equation $h'(\theta) = 0$ is

$$\cos\theta_1 = 4a\left(y - \frac{1}{2}\right), \quad \sin\theta_2 = 0. \tag{2.21}$$

It is obvious that

$$\min_{\theta\in[0,2\pi)} h(\theta) = h(\theta_1). \tag{2.22}$$

So, $a^2 = b^2 + 1/4$ and the assumption of Lemma 2.2 imply

$$\begin{aligned} h(\theta_1) &= \left(y-\frac{1}{2}\right)^2\left(4a^2-1\right)^2 + b^2\left(1-16a^2\left(y-\frac{1}{2}\right)^2\right) \\ &= b^2 + \left(y-\frac{1}{2}\right)^2\left(16a^4-8a^2+1-16a^2b^2\right) \\ &= b^2 + \left(y-\frac{1}{2}\right)^2\left(1-4a^2\right) = b^2\left(1-(2y-1)^2\right) \\ &= 4b^2 y(1-y) \ge \frac{\mu^2}{c_\delta (nM)^2}. \end{aligned} \tag{2.23}$$

And from (2.17) and Cauchy's integral formula, it follows that for every $y \in [1/4, 1-\mu^2/c_\delta nM]$,

$$B_{\rho'} := \left\{ w : |w-y| \le \rho' = \sqrt{\frac{\mu^2}{c_\delta nM}} \right\} \subseteq B_{n,1,M}, \tag{2.24}$$

and there holds

$$\begin{aligned} |P'(y)| &= \left|\frac{1}{2\pi i}\int_{B_{n,1,M}} \frac{P(z)}{(z-y)^2}dz\right| \\ &\le c_\delta \|P\|_{[0,1]}\left|\int_{B_{\rho'}} \frac{\rho'}{(\rho')^2} de^{i\theta}\right| \\ &\le c_\delta \frac{nM}{\mu^2}\|P\|_{[0,1]} \\ &\le c_\delta n \log(n+1)\|P\|_{[0,1]}. \end{aligned} \tag{2.25}$$

Case 4. $\|P\|_{[0,1]} \ge (2n+2)^{-4}$. Applying Lemma 2.1 with $m=1$ and $M=\log(n+2)$, we obtain that there is constant $c_\delta > 0$ such that

$$\max_{z\in B_{n,1,\log(n+2)}} |P(z)| \le c_\delta \|P\|_{[0,1]}. \tag{2.26}$$

Indeed, noting that

$$\max_{z\in[0,1]} \log|P(z)| \geq -4\log(2n+2),$$
$$\max_{z\in A_{n,\log(n+2)}} \log|P(z)| \leq \log\left(n^{\delta}\left(1+\frac{\log(n+2)}{n}\right)^{n+1}\right) \leq c_{\delta}\log(n+2), \tag{2.27}$$

we get the result want to be proved by a simple modification of the proof of Lemma 2.2. We omit the details. The proof of Theorem 1.2 is complete. □

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