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Research Article

Monotonic and Logarithmically Convex Properties of a Function Involving Gamma Functions

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Using the series-expansion of digamma functions and other techniques, some monotonicity and logarithmical concavity involving the ratio of gamma function are obtained, which is to give a partially affirmative answer to an open problem posed by B.-N. Guo and F. Qi. Several inequalities for the geometric means of natural numbers are established.

1. Introduction

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

For extension of these functions to complex variables and for basic properties see [1].

In recent years, many monotonicity results and inequalities involving the Gamma and incomplete Gamma functions have been established. This article is stimulated by an open problem posed by Guo and Qi in [2]. The extensions and generalizations of this problem can be found in [3–5] and some references therein.

Using Stirling formula, for all nonnegative integers k, natural numbers n and m, an upper bound of the quotient of two geometrical means of natural numbers was established in [4] as follows:

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)}} \leq \sqrt{\frac{n+k}{n+m+k}}, \tag{1.2}$$

and the following lower bound was appeared in [2, 5]:

$$\frac{n+k+1}{n+m+k+1} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}, \tag{1.3}$$

Since $\Gamma(n+1) = n!$, as a generalization of inequality (1.3), the following monotonicity result was obtained by Guo and Qi in [2]. The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{1.4}$$

is decreasing with respect to x on $[1,\infty)$ for fixed $y \geq 0$. Hence, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{1.5}$$

Recently, in [6], Qi and Sun proved that the function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{\sqrt{x+y}} \tag{1.6}$$

is strictly increasing with respect to $x \in [y+1,\infty)$ for all $y \geq y_0$.

Now, we generalize the function in (1.4) as follows: for positive real numbers x and y, $\alpha \geq 0$, let

$$F_\alpha(x,y) = \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{(x+y+1)^\alpha}. \tag{1.7}$$

The aim of this paper is to discuss the monotonicity and logarithmical convexity of the function $F_\alpha(x,y)$ with respect to parameter α.

For convenience of the readers, we recall the definitions and basic knowledge of convex function and logarithmically convex function.

Definition 1.1. Let $D \subset R^2$ be a convex set, $f : D \to R$ is called a convex function on D if

$$f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \leq \frac{f(\mathbf{x})+f(\mathbf{y})}{2} \tag{1.8}$$

for all $\mathbf{x}, \mathbf{y} \in D$, and f is called concave if $-f$ is convex.

Definition 1.2. Let $D \subset R^2$ be a convex set, $f : D \to (0, \infty)$ is called a logarithmically convex function on D if $\ln f$ is convex on D, and f is called logarithmically concave if $\ln f$ is concave.

The following criterion for convexity of function was established by Fichtenholz in [7].

Proposition 1.3. *Let $D \subset R^2$ be a convex set, if $f : D \to R$ have continuous second partial derivatives, then f is a convex (or concave) function on D if and only if $L(\mathbf{x})$ is a positive (or negative) semidefinite matrix for all $\mathbf{x} \in D$, where*

$$L(\mathbf{x}) = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix} \tag{1.9}$$

and $f''_{ij} = \partial^2 f(x_1, x_2)/\partial x_i \partial x_j$ for $\mathbf{x} = (x_1, x_2)$, $i, j = 1, 2$.

Notation 1. In Definitions 1.1, 1.2 and Proposition 1.3, we denote $\mathbf{x}, \mathbf{y}$ by the points (or vectors) of R^2, and denote x, y by the real variables in the later.

Our main results are Theorems 1.4 and 1.5.

Theorem 1.4. (1) *For any fixed $y \geq 0$, $F_\alpha(x, y)$ is strictly increasing (or decreasing, resp.) with respect to x on $(0, \infty)$ if and only if $0 \leq \alpha \leq 1/2$ (or $\alpha \geq 1$, resp.);*

(2) *For any fixed $x > 0$, $F_\alpha(x, y)$ is strictly increasing with respect to y on $[0, \infty)$ if and only if $0 \leq \alpha \leq 1$.*

Theorem 1.5. (1) *If $0 \leq \alpha \leq 1/4$, then $F_\alpha(x, y)$ is logarithmically concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$;*

(2) *If $E \subset (0, \infty) \times (0, \infty)$ is a convex set with nonempty interior and $\alpha \geq 1$, then $F_\alpha(x, y)$ is neither logarithmically convex nor logarithmically concave with respect to (x, y) on E.*

The following two corollaries can be derived from Theorems 1.4 and 1.5 immediately.

Corollary 1.6. *If $(x, y) \in (0, \infty) \times (0, \infty)$, then*

$$\frac{x+y+1}{x+y+2} < \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} < \sqrt{\frac{x+y+1}{x+y+2}}. \tag{1.10}$$

Remark 1.7. Inequality (1.3) can be derived from Corollary 1.6 if we take $x, y \in \mathbb{N}$. Although we cannot get the inequality (1.2) exactly from Corollary 1.6, but we can get the following inequality which is close to inequality (1.2):

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)}} \le \sqrt{\frac{n+k+1}{n+m+k+1}}. \tag{1.11}$$

Corollary 1.8. *If* $(x_1, y_1), (x_2, y_2) \in (0, \infty) \times (0, \infty)$, *then*

$$\begin{aligned} &\frac{\left[\Gamma(x_1+y_1+1)/\Gamma(y_1+1)\right]^{1/x_1} \cdot \left[\Gamma(x_2+y_2+1)/\Gamma(y_2+1)\right]^{1/x_2}}{\left[\Gamma((x_1+x_2+y_1+y_2)/2+1)/\Gamma((y_1+y_2)/2+1)\right]^{4/(x_1+x_2)}} \\ &\quad \le \frac{\sqrt{2}\left[(x_1+y_1+1)(x_2+y_2+1)\right]^{1/4}}{\sqrt{x_1+y_1+x_2+y_2+2}}. \end{aligned} \tag{1.12}$$

Remark 1.9. We conjecture that the inequality (1.2) can be improved if we can choose two pairs of integers (x_1, y_1) and (x_2, y_2) properly.

2. Lemmas

It is well known that the Bernoulli numbers B_n is defined [8] in general by

$$\frac{1}{e^t-1} + \frac{1}{2} - \frac{1}{t} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n}}{(2n)!} B_n. \tag{2.1}$$

In particular, we have

$$B_1 = \frac{1}{6}, \qquad B_2 = \frac{1}{30}, \qquad B_3 = \frac{1}{42}, \qquad B_4 = \frac{1}{30}. \tag{2.2}$$

In [9], the following summation formula is given:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{\pi^{2k+1} E_k}{2^{2k+2}(2k)!} \tag{2.3}$$

for nonnegative integer k, where E_k denotes the Euler number, which implies

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}}, \quad n \in \mathbb{N}. \tag{2.4}$$

Recently, the Bernoulli and Euler numbers and polynomials are generalized in [10–13]. The following two Lemmas were established by Qi and Guo in [3, 14].

Lemma 2.1 (see [3]). *For real number $x > 0$ and natural number m, one has*

$$\ln\Gamma(x) = \frac{1}{2}\ln(2\pi) + \left(x - \frac{1}{2}\right)\ln x - x + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{2(2n-1)n}\cdot\frac{1}{x^{2n-1}} + (-1)^m\theta_1\frac{B_{m+1}}{(2m+1)(2m+2)}\cdot\frac{1}{x^{2m+1}}, \quad 0 < \theta_1 < 1; \tag{2.5}$$

$$\psi(x) = \ln x - \frac{1}{2x} + \sum_{n=1}^{m}(-1)^n\frac{B_n}{2n}\cdot\frac{1}{x^{2n}} + (-1)^{m+1}\theta_2\frac{B_{m+1}}{(2m+2)}\cdot\frac{1}{x^{2m+2}}, \quad 0 < \theta_2 < 1; \tag{2.6}$$

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\theta_3\cdot\frac{B_{m+1}}{x^{2m+3}}, \quad 0 < \theta_3 < 1; \tag{2.7}$$

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} + \sum_{n=1}^{m}(-1)^n(2n+1)\frac{B_n}{x^{2n+2}} + (-1)^{m+1}(2m+3)\theta_4\cdot\frac{B_{m+1}}{x^{2m+4}}, \quad 0 < \theta_4 < 1. \tag{2.8}$$

Lemma 2.2 (see [14]). *Inequalities*

$$\ln x - \frac{1}{x} \le \psi(x) \le \ln x - \frac{1}{2x}, \tag{2.9}$$

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \le (-1)^{k+1}\psi^{(k)}(x) \le \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{2.10}$$

hold in $(0,\infty)$ for $k \in \mathbb{N}$.

Lemma 2.3. *Let $r(x,y) = \psi(x+y+1) - \psi(y+1) - \alpha x/(x+y+1)$, then the following statements are true:*

(1) *if $0 \le \alpha \le 1$, then $r(x,y) \ge 0$ for $(x,y) \in (0,\infty)\times[0,\infty)$;*

(2) *if $\alpha > 1$, then $r(\alpha,y) < 0$ for $y \in (2/(\alpha-1),\infty)$.*

Proof. (1) Making use of (2.6) we get

$$\lim_{y\to\infty} r(x,y) = \lim_{y\to\infty}\left[\ln(x+y+1) - \ln(y+1)\right] = 0 \tag{2.11}$$

for any fixed $x > 0$.

Since $\psi(x+1) = 1/x + \psi(x)$ and $0 \le \alpha \le 1$, we have

$$r(x,y) - r(x,y+1) = \frac{x\left[(1-\alpha)y + x + 2 - \alpha\right]}{(y+1)(x+y+1)(x+y+2)} > 0 \tag{2.12}$$

for all $(x,y) \in (0,\infty)\times[0,\infty)$.

Therefore, Lemma 2.3(1) follows from (2.11) and (2.12).

(2) If $\alpha > 1$, then (2.12) leads to

$$r(\alpha, y) - r(\alpha, y+1) < 0 \tag{2.13}$$

for $y \in (2/(\alpha - 1), \infty)$.

Therefore, Lemma 2.3(2) follows from (2.11) and (2.13). □

Lemma 2.4. *If* $g(x, y) = 2x\psi(y+1) - 2[\ln\Gamma(x+y+1) - \ln\Gamma(y+1)] + x^2\psi'(y+1)$*, then* $g(x, y) > 0$ *for* $(x, y) \in (0, \infty) \times (0, \infty)$.

Proof. It is easy to see that

$$g(0, y) = 0 \tag{2.14}$$

for all $y \in (0, \infty)$.

Let $g_1(x, y) = \partial g(x, y)/\partial x$, then

$$g_1(x, y) = 2\left[x\psi'(y+1) - \psi(x+y+1) + \psi(y+1)\right], \tag{2.15}$$

$$g_1(0, y) = 0, \tag{2.16}$$

$$\frac{\partial g_1(x, y)}{\partial x} = 2\left[\psi'(y+1) - \psi'(x+y+1)\right] > 0 \tag{2.17}$$

for $x > 0$. On the other hand, from (2.10) we know that $\psi'(x)$ is strictly decreasing on $(0, \infty)$.

Therefore, Lemma 2.4 follows from (2.14)–(2.17). □

Remark 2.5. Let

$$\begin{aligned} a(x, y) &= \frac{2}{x^3}\left[\ln\Gamma(x+y+1) - \ln\Gamma(y+1)\right] - \frac{2}{x^2}\psi(x+y+1), \\ b(x, y) &= -\frac{1}{x^2}\left[\psi(x+y+1) - \psi(y+1)\right], \\ c(x, y) &= -\frac{1}{x}\psi'(y+1). \end{aligned} \tag{2.18}$$

Then simple computation shows that

$$g(x, y) = x^3\left[2b(x, y) - a(x, y) - c(x, y)\right]. \tag{2.19}$$

Lemma 2.6. *Let* $d(x, y) = (1/x)\psi'(x+y+1) + \alpha/(x+y+1)^2$*, then the following statements are true:*

(1) *if* $0 \le \alpha \le 1/4$*, then*

$$\left[a(x, y) + d(x, y)\right]\left[c(x, y) + d(x, y)\right] > \left[b(x, y) + d(x, y)\right]^2 \tag{2.20}$$

for $(x, y) \in (0, \infty) \times (0, \infty)$*;*

(2) *if* $\alpha \geq 1$, *then*

$$[a(x,y)+d(x,y)][c(x,y)+d(x,y)] < [b(x,y)+d(x,y)]^2 \tag{2.21}$$

for $(x,y) \in (0,\infty)\times(0,\infty)$.

Proof. Let

$$f(x,y) = 2\psi'(y+1)[x\psi(x+y+1) - \ln\Gamma(x+y+1) + \ln\Gamma(y+1)] - [\psi(x+y+1)-\psi(y+1)]^2,$$
$$p(x,y) = f(x,y) - g(x,y)\left[\psi'(x+y+1) + \frac{\alpha x}{(x+y+1)^2}\right]. \tag{2.22}$$

Then it is not difficult to verify

$$p(0,y) = 0, \tag{2.23}$$

$$p(x,y) = x^4\left\{[a(x,y)+d(x,y)][c(x,y)+d(x,y)] - [b(x,y)+d(x,y)]^2\right\}, \tag{2.24}$$

$$\frac{\partial p(x,y)}{\partial x} = -\frac{\alpha x}{(x+y+1)^2}\frac{\partial g(x,y)}{\partial x} - g(x,y)\left[\psi''(x+y+1) + \frac{\alpha}{(x+y+1)^2} - \frac{2\alpha x}{(x+y+1)^3}\right]. \tag{2.25}$$

(1) If $0 \leq \alpha \leq 1/4$, then making use of Lemmas 2.2, 2.4 and (2.25) we get

$$\begin{aligned}\frac{\partial p(x,y)}{\partial x} &> -\frac{\alpha x}{(x+y+1)^2}\frac{\partial g(x,y)}{\partial x}\\ &\quad + g(x,y)\left[\frac{1}{(x+y+1)^2} + \frac{1}{(x+y+1)^3} - \frac{\alpha}{(x+y+1)^2} + \frac{2\alpha x}{(x+y+1)^3}\right],\\ &> \frac{1}{(x+y+1)^2}\left[(1-\alpha)g(x,y) - \alpha x\frac{\partial g(x,y)}{\partial x}\right]\end{aligned} \tag{2.26}$$

for $(x,y) \in (0,\infty)\times(0,\infty)$.

Let $g_i(x,y) = \partial^i g(x,y)/\partial x^i$, $i = 1, 2, 3, 4$, $q(x,y) = (1-\alpha)g(x,y) - \alpha x(\partial g(x,y)/\partial x)$, and $q_j(x,y) = \partial^j q(x,y)/\partial x^j$, $j = 1,2$. Then simple computation leads to

$$g_3(x,y) = -2\psi''(x+y+1), \tag{2.27}$$

$$g_4(x,y) = -2\psi'''(x+y+1), \tag{2.28}$$

$$\frac{\partial q_2(x,y)}{\partial x} = (1-4\alpha)g_3(x,y) - \alpha x g_4(x,y), \tag{2.29}$$

$$q_2(0,y) = q_1(0,y) = q(0,y) = 0 \tag{2.30}$$

for all $y \in (0,\infty)$.

It is well known that $\ln \Gamma(x) = -cx + \sum_{k=1}^{\infty}[x/k - \ln(1 + x/k)] - \ln x$, where $c = 0.577215\cdots$ is the Euler's constant. From this we get

$$\psi^{(n)} = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+x)^{n+1}}. \tag{2.31}$$

From Lemma 2.2, (2.27)–(2.29), (2.31) and the assumption $0 \leq \alpha \leq 1/4$, we conclude that

$$\frac{\partial q_2(x,y)}{\partial x} > 0. \tag{2.32}$$

Therefore, Lemma 2.6(1) follows from (2.23)–(2.26), (2.30), and (2.32).

(2) If $\alpha \geq 1$, then making use of (2.8), Lemma 2.4 and (2.25) we obtain

$$\begin{aligned}\frac{\partial p(x,y)}{\partial x} &< -\frac{\alpha x}{(x+y+1)^2}\frac{\partial g(x,y)}{\partial x} + g(x,y)\left[\frac{1}{(x+y+1)^3} + \frac{1}{2(x+y+1)^4} + \frac{2\alpha x}{(x+y+1)^3}\right]\\ &< -\frac{\alpha x}{(x+y+1)^2}\frac{\partial g(x,y)}{\partial x} + g(x,y)\frac{2\alpha(x+1)}{(x+y+1)^3}\\ &< \frac{\alpha(x+1)}{(x+y+1)^3}\left[2g(x,y) - x\frac{\partial g(x,y)}{\partial x}\right].\end{aligned} \tag{2.33}$$

Let

$$v(x,y) = 2g(x,y) - x\frac{\partial g(x,y)}{\partial x}, \qquad v_i(x,y) = \frac{\partial^i v(x,y)}{\partial x^i}, \quad i = 1,2. \tag{2.34}$$

Then

$$v_2(x,y) = 2x\psi''(x+y+1) < 0 \tag{2.35}$$

for $(x,y) \in (0,\infty) \times (0,\infty)$ by Lemma 2.2, and

$$v(0,y) = v_1(0,y) = 0 \tag{2.36}$$

for $y \in (0,\infty)$.

Therefore, Lemma 2.6(2) follows from (2.23)–(2.25) and (2.33)–(2.36). □

3. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. (1) Let $G(x,y) = \ln F_\alpha(x,y)$ and $G_1(x,y) = x^2(\partial G(x,y)/\partial x)$, then

$$G_1(x,y) = -\left[\ln\Gamma(x+y+1) - \ln\Gamma(y+1)\right] + x\psi(x+y+1) - \frac{\alpha x^2}{x+y+1}. \tag{3.1}$$

The following three cases will complete the proof of Theorem 1.4(1).

Case 1. If $0 \le \alpha \le 1/2$, then (3.1) and Lemma 2.2 imply

$$\begin{aligned}\frac{\partial G_1(x,y)}{\partial x} &= x\left[\psi'(x+y+1) - \frac{\alpha(x+2y+2)}{(x+y+1)^2}\right]\\ &> x\left[\frac{1}{x+y+1} + \frac{1}{2(x+y+1)^2} - \frac{\alpha(x+2y+2)}{(x+y+1)^2}\right]\\ &= \frac{x}{2(x+y+1)^2}\left[(2-2\alpha)x + (2-4\alpha)y + 3 - 4\alpha\right]\\ &> 0\end{aligned} \tag{3.2}$$

for $(x,y) \in (0,\infty)\times[0,\infty)$.

From (3.2) and the fact that $G_1(0,y) = 0$ for all $y \in [0,\infty)$ we know that $F_\alpha(x,y)$ is strictly increasing with respect to x on $(0,\infty)$ for any fixed $y \in [0,\infty)$.

Case 2. If $\alpha \ge 1$, then (3.1) and (2.7) imply

$$\begin{aligned}\frac{\partial G_1(x,y)}{\partial x} &< x\left[\frac{1}{x+y+1} + \frac{1}{2(x+y+1)^2} + \frac{1}{6(x+y+1)^3} - \frac{\alpha(x+2y+2)}{(x+y+1)^2}\right]\\ &= \frac{x}{6(x+y+1)^3}\left[(6-6\alpha)x^2 + \lambda_1(y)x + \lambda_2(y)\right]\\ &< 0\end{aligned} \tag{3.3}$$

for $(x,y) \in (0,\infty)\times[0,\infty)$, where $\lambda_1(y) = (12-18\alpha)y + 15 - 18\alpha < 0$ and $\lambda_2(y) = 6(1-2\alpha)y^2 + (15-24\alpha)y + 10 - 12\alpha < 0$.

From (3.3) and the fact that $G_1(0,y) = 0$ for all $y \in [0,\infty)$ we know that $F_\alpha(x,y)$ is strictly decreasing with respect to x on $(0,\infty)$ for any fixed $y \in [0,\infty)$.

Case 3. If $1/2 < \alpha < 1$, let

$$G_2(x,y) = \psi'(x+y+1) - \frac{\alpha(x+2y+2)}{(x+y+1)^2}. \tag{3.4}$$

Then

$$\frac{\partial G_1(x,y)}{\partial x} = xG_2(x,y), \tag{3.5}$$

$$\begin{aligned} G_2(0,y) &< \frac{1}{y+1} + \frac{1}{2(y+1)^2} + \frac{1}{6(y+1)^3} - \frac{2\alpha}{y+1} \\ &= \frac{1}{6(y+1)^3}\left[6(1-2\alpha)y^2 + (15-24\alpha)y + 10 - 12\alpha\right] < 0 \end{aligned} \tag{3.6}$$

for $y \geq (15 - 24\alpha + \sqrt{48\alpha - 15})/(24\alpha - 12)$.

It is obvious that (3.6) implies

$$G_2\left(0, \frac{15 + \sqrt{48\alpha - 15}}{24\alpha - 12}\right) < 0. \tag{3.7}$$

The continuity of $G_2(x,y)$ with respect to $x \in (0,\infty)$ for any fixed $y \in [0,\infty)$ and (3.7) imply that there exists $\delta = \delta(\alpha) > 0$ such that

$$G_2\left(x, \frac{15 + \sqrt{48\alpha - 15}}{24\alpha - 12}\right) < 0 \tag{3.8}$$

for $x \in (0,\delta)$.

From (3.5), (3.8) and $G_1(0,(15 + \sqrt{48\alpha - 15})/(24\alpha - 12)) = 0$ we know that $F_\alpha(x,y)$ is strictly decreasing with respect to x on $(0,\delta)$ for $y = (15 + \sqrt{48\alpha - 15})/(24\alpha - 12)$.

On the other hand, making use of (2.5) and (2.6) we have

$$\begin{aligned} \lim_{x \to +\infty} G_1(x,y) &= \lim_{x \to +\infty} x\left[1 - \left(y + \frac{1}{2}\right)\frac{\ln(x+y+1)}{x} - \frac{\alpha x}{x+y+1}\right] + C(y,\theta_1) \\ &= \lim_{x \to +\infty} (1-\alpha)x + C(y,\theta_1) \\ &= +\infty, \end{aligned} \tag{3.9}$$

where

$$C(y,\theta_1) = \left(y + \frac{1}{2}\right)\ln(y+1) + \frac{1}{12(y+1)} - \frac{1}{2} - \frac{\theta_1}{360(y+1)^3} \tag{3.10}$$

for $y \in [0,\infty)$ and $0 < \theta_1 < 1$.

Equation (3.9) implies that there exists $M = M(\alpha) > \delta(\alpha)$ such that

$$G_1\left(x, \frac{15 + \sqrt{48\alpha - 15}}{24\alpha - 12}\right) > 0 \tag{3.11}$$

for $x \in (M,\infty)$.

Hence, from (3.11) we know that $F_\alpha(x,y)$ is strictly increasing with respect to x on (M,∞) for $y=(15+\sqrt{48\alpha-15})/(24\alpha-12)$.

(2) Since

$$x\frac{\partial G(x,y)}{\partial y}=\psi(x+y+1)-\psi(y+1)-\frac{\alpha x}{x+y+1}=r(x,y), \tag{3.12}$$

then, Theorem 1.4(2) follows from (3.12) and Lemma 2.3. □

Proof of Theorem 1.5. Let $G(x,y)=\ln F_\alpha(x,y)$, $G''_{11}(x,y)=\partial^2 G(x,y)/\partial x^2$, $G''_{12}=\partial^2 G(x,y)/\partial x\partial y$ and $G''_{22}(x,y)=\partial^2 G(x,y)/\partial y^2$, then simple calculation yields

$$\begin{aligned} G''_{11}(x,y) &= \frac{2}{x^3}\left[\ln\Gamma(x+y+1)-\ln\Gamma(y+1)\right]-\frac{2}{x^2}\psi(x+y+1)\\ &\quad+\frac{1}{x}\psi'(x+y+1)+\frac{\alpha}{(x+y+1)^2}\\ &= a(x,y)+d(x,y), \end{aligned} \tag{3.13}$$

$$\begin{aligned} G''_{12}(x,y) &= -\frac{1}{x^2}\left[\psi(x+y+1)-\psi(y+1)\right]+\frac{1}{x}\psi'(x+y+1)+\frac{\alpha}{(x+y+1)^2}\\ &= b(x,y)+d(x,y), \end{aligned} \tag{3.14}$$

$$\begin{aligned} G''_{22}(x,y) &= \frac{1}{x}\left[\psi'(x+y+1)-\psi'(y+1)\right]+\frac{\alpha}{(x+y+1)^2}\\ &= c(x,y)+d(x,y), \end{aligned} \tag{3.15}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$, and $d(x,y)$ are defined in Remark 2.5 and Lemma 2.6.

According to the Definition 1.2 and Proposition 1.3, to prove Theorem 1.5 we need only to show that

$$G''_{11}(x,y)\le 0, \tag{3.16}$$

$$G''_{11}(x,y)G''_{22}(x,y)-\left[G''_{12}(x,y)\right]^2\ge 0 \tag{3.17}$$

for $0\le\alpha\le 1/4$ and $(x,y)\in(0,\infty)\times(0,\infty)$, and

$$G''_{11}(x,y)G''_{22}(x,y)-\left[G''_{12}(x,y)\right]^2< 0 \tag{3.18}$$

for $\alpha\ge 1$ and $(x,y)\in(0,\infty)\times(0,\infty)$.

Next, let $w(x,y) = x^3 G_{11}''(x,y)$, then

$$w(x,y) = 2\left[\ln\Gamma(x+y+1) - \ln\Gamma(y+1)\right] - 2x\psi(x+y+1) + x^2\psi'(x+y+1) + \frac{\alpha x^3}{(x+y+1)^2},$$
$$w(0,y) = 0, \tag{3.19}$$

$$\begin{aligned}\frac{\partial w(x,y)}{\partial x} &= x^2\left[\psi''(x+y+1) + \frac{\alpha(x+3y+3)}{(x+y+1)^3}\right] \\ &< x^2\left[\frac{\alpha(x+3y+3)}{(x+y+1)^3} - \frac{1}{(x+y+1)^2} - \frac{1}{(x+y+1)^3}\right] \\ &= \frac{x^2}{(x+y+1)^3}\left[(\alpha-1)x + (3\alpha-1)y + 3\alpha - 2\right] \\ &< 0\end{aligned} \tag{3.20}$$

for $(x,y) \in (0,\infty) \times [0,\infty)$ by Lemma 2.2 and $0 \le \alpha \le 1/4$.

Therefore, (3.16) follows from (3.19) and (3.20), and (3.17) and (3.18) follow from Lemma 2.6. The proof of Theorem 1.5 is completed. □

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