Research Article

Fractional Calculus and p-Valently Starlike Functions

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In this investigation, the authors prove coefficient bounds, distortion inequalities for fractional calculus of a family of multivalent functions with negative coefficients, which is defined by means of a certain nonhomogenous Cauchy-Euler differential equation.

1. Introduction and Definitions

Let $\mathcal{T}_n(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0;\ n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* and *multivalent* in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

The fractional calculus are defined as follows (e.g., [1, 2]).

Definition 1.1. The fractional integral of order δ is defined by

$$\mathcal{D}_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z - \xi)^{1-\delta}} \, d\xi \quad (\delta > 0), \tag{1.2}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z - \xi)^{\delta-1}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 1.2. The fractional derivative of order δ is defined by

$$\mathfrak{D}_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\delta} \, d\xi \quad (0 \le \delta < 1), \tag{1.3}$$

where $f(z)$ is constrained and multiplicity of $(z-\xi)^{-\delta}$ is removed as in Definition 1.1.

Definition 1.3. Under the hypotheses of Definition 1.1, the fractional derivative of order $(n+\delta)$ is defined by

$$\mathfrak{D}_z^{n+\delta} f(z) = \frac{d^n}{dz^n} \mathfrak{D}_z^\delta f(z) \quad (0 \le \delta < 1, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.4}$$

$(a)_v$ denotes the *Pochhammer* symbol *(or the shifted factorial)*, since

$$(1)_n = n! \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.5}$$

defined (for $a, v \in \mathbb{C}$ and in terms of the Gamma function) by

$$(a)_v := \frac{\Gamma(a+v)}{\Gamma(a)} = \begin{cases} 1, & (v = 0, \ a \in \mathbb{C} \setminus \{0\}), \\ a(a+1)\cdots(a+n-1), & (v = n \in \mathbb{N}; \ a \in \mathbb{C}). \end{cases} \tag{1.6}$$

The earlier investigations by Goodman [3, 4] and Ruscheweyh [5], we define the (n, p, ε)-neighborhood of a function $f \in \mathcal{T}_n(p)$ by

$$\mathcal{N}_{n,p}^\varepsilon(\mathfrak{D}_z^\delta f, \mathfrak{D}_z^\delta g) := \left\{ g \in \mathcal{T}_n(p) : g(z) = z^p - \sum_{k=n+p}^\infty b_k z^k, \ \sum_{k=n+p}^\infty (k+1-\delta)_\delta k |a_k - b_k| \le \varepsilon \right\}, \tag{1.7}$$

so that, obviously,

$$\mathcal{N}_{n,p}^\varepsilon(\mathfrak{D}_z^\delta h, \mathfrak{D}_z^\delta g) := \left\{ g \in \mathcal{T}_n(p) : g(z) = z^p - \sum_{k=n+p}^\infty b_k z^k, \ \sum_{k=n+p}^\infty (k+1-\delta)_\delta k |b_k| \le \varepsilon \right\}, \tag{1.8}$$

where

$$h(z) := z^p. \tag{1.9}$$

The class $\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$ denote the subclass of $\mathcal{T}_n(p)$ consisting of functions $f(z)$ which satisfy

$$\mathfrak{Re}\, \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} > \alpha \quad (0 \le \alpha < p, \ p \in \mathbb{N}), \tag{1.10}$$

where

$$\mathcal{F}(z) = \lambda z \mathfrak{D}_z^{1+\delta} f(z) + (1-\lambda) \mathfrak{D}_z^{\delta} f(z) \quad (0 \le \lambda \le 1, \ 0 \le \delta < 1). \tag{1.11}$$

We note that the class $\mathcal{S}_{1,1}^{0}(\lambda, \alpha)$ was investigated by Altıntaş [6] and the class $\mathcal{S}_{n,p}^{0}(\lambda, \alpha)$ was studied by Altıntaş et al. [7, 8] .

We donote by

$$\mathcal{S}_{n,p}^{0}(0, \alpha) = \mathcal{S}_n^*(p, \alpha), \qquad \mathcal{S}_{n,p}^{0}(1, \alpha) = \mathcal{C}_n(p, \alpha) \tag{1.12}$$

the classes of p-valently starlike functions of order α in $\mathbb{U}$ $(0 \le \alpha < p)$ and p-valently convex functions of order α in $\mathbb{U}$ $(0 \le \alpha < p)$, respectively [see, [2, 9]].

Finally $\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$ denote the subclass of the general class $\mathcal{T}_n(p)$ consisting of functions $f(z) \in \mathcal{T}_n(p)$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$z^2 \mathfrak{D}_z^{2+\delta} \omega + 2(1+\mu) z \mathfrak{D}_z^{1+\delta} \omega + \mu(1+\mu) \mathfrak{D}_z^{\delta} \omega = (p-\delta+\mu)(p-\delta+\mu+1) \mathfrak{D}_z^{\delta} g, \tag{1.13}$$

where $\omega = f(z), f(z) \in \mathcal{T}_n(p), g = g(z) \in \mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$ and $\mu > \delta - p$.

The main object of the present paper is to give coefficients bounds and distortion inequalities for functions in the classes $\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$ and $\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$.

2. Coefficient Bounds and Distortion Inequalities

We begin by proving the following result.

Lemma 2.1. *Let the function $f(z) \in \mathcal{T}_n(p)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} (k+1-\delta)_{\delta}(k-\alpha-\delta)[1+\lambda(k-1-\delta)] a_k \le (p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)]$$

$$(0 \le \lambda \le 1; \ 0 \le \alpha < p-\delta; \ 0 \le \delta < 1; \ p \in \mathbb{N}). \tag{2.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p+1-\delta)_{\delta}(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]} z^{n+p}. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{T}_n(p)$ and $\mathcal{F}(z)$ be defined by (1.11). Suppose that $f(z) \in \mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$. Then, in conjunction with (1.10) and (1.11) yields

$$\mathfrak{Re} \, \frac{(p+1-\delta)_{\delta}(p-\delta)[1+\lambda(p-1-\delta)]z^{p-\delta} - \sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}(k-\delta)[1+\lambda(k-1-\delta)]a_k z^{k-\delta}}{(p+1-\delta)_{\delta}[1+\lambda(p-1-\delta)]z^{p-\delta} - \sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}[1+\lambda(k-1-\delta)]a_k z^{k-\delta}} > \alpha.$$

$$(2.3)$$

By letting $z \to 1^-$ along the real axis, we arrive easily at the inequality in (2.1). $\qquad \square$

Lemma 2.2. *Let the function $f(z)$ given by* (1.1) *be in the class $\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$. Then*

$$\sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}\, a_k \le \frac{(p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]},$$

$$(2.4)$$

$$\sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}\, k a_k \le \frac{(p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)](n+p)}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]}.$$

$$(2.5)$$

Proof. By using **Lemma 2.1**, we find from (2.1) that

$$(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]\sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}a_k$$

$$\le \sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}(k-\alpha-\delta)[1+\lambda(k-1-\delta)]a_k$$

$$(2.6)$$

$$\le (p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)],$$

which immediately yields the first assertion (2.4) of **Lemma 2.2**. $\qquad \square$

For the proof of second assertion, by appealing to (2.1), we also have

$$[1+\lambda(n+p-\delta-1)]\left\{\sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}k a_k - (\alpha+\delta)\sum_{k=n+p}^{\infty}(k+1-\delta)_{\delta}a_k\right\}$$

$$\le (p+1-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)],$$

$$(2.7)$$

by using (2.4) in (2.7), we can easily get the assertion (2.5) of **Lemma 2.2**.

The distortion inequalities for functions in the class $\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$ are given by **Theorem 2.3** below.

Theorem 2.3. *Let a function $f(z) \in \mathcal{T}_n(p)$ be in the class $\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$. Then*

$$|f(z)| \leq |z|^p + \frac{(p + 1 - \delta)_{\delta}(p - \alpha - \delta)(p - \delta + \mu)(p - \delta + \mu + 1)}{(n + p + 1 - \delta)_{\delta}(n + p - \alpha - \delta)(n + p - \delta + \mu)} \frac{1 + \lambda(p - \delta - 1)}{1 + \lambda(n + p - \delta - 1)}|z|^{n+p},$$
(2.8)

$$|f(z)| \geq |z|^p - \frac{(p + 1 - \delta)_{\delta}(p - \alpha - \delta)(p - \delta + \mu)(p - \delta + \mu + 1)}{(n + p + 1 - \delta)_{\delta}(n + p - \alpha - \delta)(n + p - \delta + \mu)} \frac{1 + \lambda(p - \delta - 1)}{1 + \lambda(n + p - \delta - 1)}|z|^{n+p}.$$
(2.9)

Proof. Suppose that a function $f(z) \in \mathcal{T}_n(p)$ is given by (1.1) and also let the function $g(z) \in \mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$ occurring in the nonhomogenous differential equation (1.13) be given as in the Definitions (1.2) or (1.3) with of course

$$b_k \geq 0 \quad (k = n + p, n + p + 1, \ldots).$$
(2.10)

Then we easily see from (1.13) that

$$a_k = \frac{(p - \delta + \mu)(p - \delta + \mu + 1)}{(k - \delta + \mu)(k - \delta + \mu + 1)}b_k \quad (k = n + p, n + p + 1, \ldots).$$
(2.11)

So that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p - \delta + \mu)(p - \delta + \mu + 1)}{(k - \delta + \mu)(k - \delta + \mu + 1)}b_k z^k,$$
(2.12)

$$|f(z)| \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p - \delta + \mu)(p - \delta + \mu + 1)}{(k - \delta + \mu)(k - \delta + \mu + 1)}b_k.$$
(2.13)

Since $g(z) \in \mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$, the first assertion (2.4) of Lemma 2.2 yields the following inequality:

$$|b_k| \leq \frac{(p + 1 - \delta)_{\delta}(p - \alpha - \delta)[1 + \lambda(p - \delta - 1)]}{(n + p + 1 - \delta)_{\delta}(n + p - \alpha - \delta)[1 + \lambda(n + p - \delta - 1)]}.$$
(2.14)

From (2.13) and (2.14) we have

$$|f(z)| \leq |z|^p + |z|^{n+p} \frac{(p + 1 - \delta)_{\delta}(p - \alpha - \delta)[1 + \lambda(p - \delta - 1)]}{(n + p + 1 - \delta)_{\delta}(n + p - \alpha - \delta)[1 + \lambda(n + p - \delta - 1)]}$$
$$\cdot \sum_{k=n+p}^{\infty} \frac{1}{(k - \delta + \mu)(k - \delta + \mu + 1)},$$
(2.15)

and also note that

$$
\sum_{k=n+p}^{\infty} \frac{1}{(k-\delta+\mu)(k-\delta+\mu+1)} = \sum_{k=n+p}^{\infty} \left(\frac{1}{k-\delta+\mu} - \frac{1}{k-\delta+\mu+1} \right)
$$
$$
= \frac{1}{n+p-\delta+\mu},
$$

(2.16)

where $\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, \ldots\}$. The assertion (2.8) of Theorem 2.3 follows at once from (2.15). The assertion (2.9) of Theorem 2.3 can be proven by similarly applying (2.12), (2.14), and (2.15), and also (2.16). □

By setting $\delta := 0$ in Theorem 2.3, we obtain the following Corollary 2.4.

Corollary 2.4 (See Altıntaş et al. [8, Theorem 1]). *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation* (1.13), *then*

$$
|f(z)| \leq |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)[1+\lambda(p-1)]}{(n+p-\alpha)(n+p+\mu)[1+\lambda(n+p-1)]} |z|^{n+p},
$$
$$
|f(z)| \geq |z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)[1+\lambda(p-1)]}{(n+p-\alpha)(n+p+\mu)[1+\lambda(n+p-1)]} |z|^{n+p}.
$$

(2.17)

By letting $\delta := 0$, $\lambda := 0$ and $\delta := 0$, $\lambda := 1$ in Theorem 2.3. We arrive at Corollaries 2.5 and 2.6 (see, [8]).

Corollary 2.5. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation* (1.13) *with $g \in \mathcal{S}_n^*(p,\alpha)$, then*

$$
|f(z)| \leq |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)} |z|^{n+p},
$$
$$
|f(z)| \geq |z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)} |z|^{n+p}.
$$

(2.18)

Corollary 2.6. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation* (1.13) *with $g \in \mathcal{C}_n(p,\alpha)$, then*

$$
|f(z)| \leq |z|^p + \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)(n+p)} |z|^{n+p},
$$
$$
|f(z)| \geq |z|^p - \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)(n+p)} |z|^{n+p}.
$$

(2.19)

3. Neighborhoods for the Classes $\mathcal{S}_{n,p}^\delta(\lambda,\alpha)$ and $\mathcal{K}_{n,p}^\delta(\lambda,\alpha,\mu)$

In this section, we determine inclusion relations for the classes $\mathcal{S}_{n,p}^\delta(\lambda,\alpha)$ and $\mathcal{K}_{n,p}^\delta(\lambda,\alpha,\mu)$ concerning the (n,p,ϵ)-neighborhoods is defined by (1.7) and (1.8).

Theorem 3.1. *Let a function $f(z) \in \mathcal{T}_n(p)$ be in the class $\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$. Then*

$$\mathcal{S}_{n,p}^{\delta}(\lambda, \alpha) \subset \mathcal{N}_{n,p}^{\varepsilon}(\mathfrak{D}_z^{\delta} h, \mathfrak{D}_z^{\delta} f), \tag{3.1}$$

where $h(z)$ is given by (1.9) and the parameter ε is the given by

$$\varepsilon := \frac{(n+p)(p-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]}. \tag{3.2}$$

Proof. Assertion (3.1) would follow easily from the definition of $\mathcal{N}_{n,p}^{\varepsilon}(\mathfrak{D}_z^{\delta} h, \mathfrak{D}_z^{\delta} f)$, which is given by (1.8) with $g(z)$ replaced by $f(z)$ and the second assertion (2.5) of Lemma 2.2. □

Theorem 3.2. *Let a function $f(z) \in \mathcal{T}_n(p)$ be in the class $\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$. Then*

$$\mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu) \subset \mathcal{N}_{n,p}^{\varepsilon}(\mathfrak{D}_z^{\delta} g, \mathfrak{D}_z^{\delta} f), \tag{3.3}$$

where $g(z)$ is given by (1.13) and the parameter ε is the given by

$$\varepsilon := \frac{(n+p)(p-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)][n+(p-\delta+\mu)(p-\delta+\mu+2)]}{(n+p-\alpha-\delta)(n+p-\delta+\mu)[1+\lambda(n+p-1-\delta)]}. \tag{3.4}$$

Proof. Suppose that $f(z) \in \mathcal{K}_{n,p}^{\delta}(\lambda, \alpha, \mu)$. Then, upon substituting from (2.11) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} (k-\delta)_{\delta} k |b_k - a_k| \le \sum_{k=n+p}^{\infty} (k-\delta)_{\delta} k b_k + \sum_{k=n+p}^{\infty} (k-\delta)_{\delta} k a_k, \tag{3.5}$$

where $a_k \ge 0$ and $b_k \ge 0$, we obtain that

$$\begin{aligned}
\sum_{k=n+p}^{\infty} (k-\delta)_{\delta} k |b_k - a_k| &\le \sum_{k=n+p}^{\infty} (k-\delta)_{\delta} k b_k \\
&+ \sum_{k=n+p}^{\infty} \frac{(p-\delta+\mu)(p-\delta+\mu+1)}{(k-\delta+\mu)(k-\delta+\mu+1)} (k-\delta)_{\delta} k b_k.
\end{aligned} \tag{3.6}$$

Since $g(z) \in \mathcal{S}_{n,p}^{\delta}(\lambda, \alpha)$, the second assertion (2.5) of Lemma 2.2 yields that

$$(k-\delta)_{\delta} k b_k \le \frac{(n+p)(p-\delta)_{\delta}(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]} \quad (k = n+p, n+p+1, \ldots). \tag{3.7}$$

Finally, by making use of (2.5) as well as (3.7) on the right-hand side of (3.6), we find that

$$
\begin{aligned}
\sum_{k=n+p}^{\infty} (k - \delta)_\delta k |b_k - a_k| \\
\leq \frac{(n+p)(p-\delta)_\delta(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]} \left(1 + \frac{(p-\delta+\mu)(p-\delta+\mu+1)}{(k-\delta+\mu)(k-\delta+\mu+1)} \right),
\end{aligned}
\tag{3.8}
$$

which, by virtue of the identity (2.16), immediately yields that

$$
\begin{aligned}
\sum_{k=n+p}^{\infty} (k - \delta)_\delta k |b_k - a_k| \\
\leq \frac{(n+p)(p-\delta)_\delta(p-\alpha-\delta)[1+\lambda(p-1-\delta)]}{(n+p-\alpha-\delta)[1+\lambda(n+p-1-\delta)]} \cdot \frac{[n+(p-\delta+\mu)(p-\delta+\mu+2)]}{(n+p-\delta+\mu)} =: \varepsilon.
\end{aligned}
\tag{3.9}
$$

Thus, by definition (1.7) with $g(z)$ interchanged by $f(z)$, $f(z) \in \mathcal{N}_{n,p}^\varepsilon(\mathfrak{D}_z^\delta g, \mathfrak{D}_z^\delta f)$. This evidently completes the proof of Theorem 3.2.   □

By setting $\delta = 0$ in Theorem 3.2, we receive the following result.

Corollary 3.3. *If the function $f(z) \in \mathcal{T}_n(p)$ is in the class $\mathcal{K}_{n,p}^0(\lambda, \alpha, \mu)$. Then*

$$
\mathcal{K}_{n,p}^0(\lambda, \alpha, \mu) \subset \mathcal{N}_{n,p}^\varepsilon(g, f),
\tag{3.10}
$$

where $g(z)$ is given by (1.13) and the parameter ε is the given by

$$
\varepsilon := \frac{(n+p)(p-\alpha)[1+\lambda(p-1)][n+(p+\mu)(p+\mu+2)]}{(n+p-\alpha)(n+p+\mu)[1+\lambda(n+p-1)]}.
\tag{3.11}
$$

Acknowledgment

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