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Research Article

Some Modulus and Normal Structure in Banach Space

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Recommended by Jong Kim

We present some sufficient conditions for which a Banach space X has normal structure in terms of the modulus of U-convexity, modulus of W^*-convexity, and the coefficient $R(1, X)$, which generalized some well-known results. Furthermore the relationship between modulus of convexity, modulus of smoothness, and Gao's constant is considered, meanwhile the exact value of Milman modulus has been obtained for some Banach space.

1. Introduction

We assume that X and X^* stand for a Banach space and its dual space, respectively. By S_X and B_X, we denote the unit sphere and the unit ball of a Banach space X, respectively. Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive provided the inequality

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point, where C is a nonempty bounded closed convex subset of a Banach space X.

Recall that a Banach space X is called to be uniformly nonsquare if there exists $\delta > 0$ such that $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$ whenever $x, y \in S_X$. A bounded convex subset K of a Banach space X is said to have normal structure if, for every convex subset H

of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}. \tag{1.2}$$

A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x - y\| : x, y \in K\}. \tag{1.3}$$

It was proved by Kirk that every reflexive Banach space with normal structure has the fixed point property (see [1]).

The modulus of convexity of X is the function $\delta_X(\epsilon) : [0,2] \to [0,1]$ defined by

$$\begin{aligned} \delta_X(\epsilon) &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\} \\ &= \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}. \end{aligned} \tag{1.4}$$

The function $\delta_X(\epsilon)$ strictly increasing on $[\epsilon_0(X), 2]$. Here $\epsilon_0(X) = \sup\{\epsilon : \delta_X(\epsilon) = 0\}$ is the characteristic of convexity of X. Also, X is uniformly nonsquare provided $\epsilon_0(X) < 2$. Various geometrical properties and the geometric conditions sufficient for normal structure in terms of the modulus of convexity have been widely studied in [2–7].

The following modulus

$$\begin{aligned} \rho_1(\epsilon) &= \sup\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\} \\ &= \sup\left\{1 - \frac{\|x + y\|}{2} : \|x\| \ge 1, \|y\| \ge 1, \|x - y\| \le \epsilon\right\} \end{aligned} \tag{1.5}$$

has been considered first in [8]. It turns out to be a modulus of smoothness in sense that the following holds (see [8]). A space X is uniformly smooth if and only if

$$\lim_{\epsilon \to 0} \frac{\rho_1(\epsilon)}{\epsilon} = 0. \tag{1.6}$$

Clearly $\delta(\epsilon) \le \rho_1(\epsilon)$. Various geometrical properties concerning the modulus $\rho_1(\epsilon)$ also have been studied by many authors, for more details see [8–11].

For $t \ge 0$, Milman's modulus $d_X(t)$ and $\beta_X(t)$ are defined as follows:

$$\begin{aligned} d_X(t) &= \inf\{\max\{\|x + ty\|, \|x - ty\|\} - 1 : x, y \in S_X\}, \\ \beta_X(t) &= \sup\{\min\{\|x + ty\|, \|x - ty\|\} - 1 : x, y \in S_X\}. \end{aligned} \tag{1.7}$$

$J(t,X) = \beta_X(t) + 1$ and $S(t,X) = d_X(t) + 1$are called the parameterized James constant and parameterized Schäffer constant, respectively. Some properties on which were studied in [7, 12, 13]. Obviously the James constant $J(X)$ and Schäffer constant $S(X)$ are the case of $t = 1$. The following coefficient is defined by Domínguez Benavides [14]:

$$R(1,X) = \sup\left\{\liminf_{n\to\infty}\|x + x_n\|\right\}, \tag{1.8}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and all weakly null sequence (x_n) in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty}\left(\limsup_{n\to\infty}\|x_n - x_m\|\right) \leq 1. \tag{1.9}$$

Obvious, $1 \leq R(1,X) \leq 2$. Some geometric conditions sufficient for normal structure in terms of the coefficient have been studied in [5, 15]. In [16, 17], Gao introduced the modulus of U-convexity and modulus of W^* -convexity of a Banach space X, respectively, as follows:

$$\begin{aligned} U_X(\epsilon) &:= \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X,\ f(x - y) \geq \epsilon \text{ for some } f \in \nabla_x\right\},\\ W_X^*(\epsilon) &:= \inf\left\{\frac{1}{2}f(x - y) : x, y \in S_X,\ \|x - y\| \geq \epsilon \text{ for some } f \in \nabla_x\right\}, \end{aligned} \tag{1.10}$$

where $\nabla_x := \{f \in S_{X^*} : f(x) = \|x\|\}$. It is easily to prove that $U_X(\epsilon) \geq \delta_X(\epsilon)$ and $W_X^*(\epsilon) \geq \delta_X(\epsilon)$. Saejung (see [13, 18, 19]) studied the above modulus extensively, and got some useful results as follows.

(i) If $U_X(\epsilon) > 0$ or $W^*(\epsilon) > 0$ for some $\epsilon \in (0,2)$, then X is uniformly nonsquare.

(ii) If $U_X(\epsilon) > (1/2)\max\{0, \epsilon - 1\}$ for some $\epsilon \in (0,2)$, then X and X^* have normal structure.

(iii) If $W_X^*(\epsilon) > (1/2)\max\{0, \epsilon - 1\}$ for some $\epsilon \in (0,2)$, then X and X^* have normal structure.

In a recent paper [20], Gao introduced two quadratic parameters, which are defined as

$$\begin{aligned} E(X) &= \sup\left\{\|x + y\|^2 + \|x - y\|^2 : x, y \in S_X\right\},\\ f(X) &= \inf\left\{\|x + y\|^2 + \|x - y\|^2 : x, y \in S_X\right\}. \end{aligned} \tag{1.11}$$

The two constants are also significant tools in the geometric theory of Banach spaces. Furthermore, Gao obtained the values of $E(X)$ and $f(X)$ for some classical Banach spaces. In terms of these constants, he got some sufficient conditions for a Banach space X to have normal structure, which plays an important role in fixed point theory.

This paper is organized as follows. In Section 2, some geometrical conditions sufficient for normal structure in terms of $U_X(\epsilon), W_X^*(\epsilon)$, and $R(1,X)$ are given, which improve Saejung's results and the results in [5]; furthermore we consider the relationship between

$\delta_X(\epsilon)$ and $E(X)$ and give some new results which improve the results in [20]. In Section 3, we consider the relationship between $\rho_1(\epsilon)$ and $f(X)$; meanwhile some exact value of $J(t,X)$ and $S(t,X)$ are computed in some concrete Banach space.

2. Normal Structure

First we recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$ if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. A filter U on $\mathbb{N}$ is called to be an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $A : A \subset \mathbb{N}$, $i_0 \in A$ for some fixed $i_0 \in \mathbb{N}$; otherwise, it is called nontrivial. Let $l_\infty(X)$ denote the subspace of the product space $\coprod_{n\in\mathbb{N}} X$ equipped with the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \left\{ (x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0 \right\}. \tag{2.1}$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically.

Lemma 2.1 (see [2]). *Let X be a Banach space without weak normal structure, then there exists a weakly null sequence $\{x_n\}_{n=1}^\infty \subseteq S_X$ such that*

$$\lim_n \|x_n - x\| = 1 \quad \forall x \in co\{x_n\}_{n=1}^\infty. \tag{2.2}$$

Theorem 2.2. *If $U_X(1+\epsilon) > f(\epsilon)$ for some $\epsilon \in [0,1]$. Then X has normal structure. Where the function $f(\epsilon)$ is defined as*

$$f(\epsilon) := \begin{cases} \left(R(1,X) - 1\right)\dfrac{\epsilon}{2}, & 0 \le \epsilon \le \dfrac{1}{R(1,X)}, \\ \dfrac{1}{2}\left(1 - \dfrac{1-\epsilon}{R(1,X)-1}\right), & \dfrac{1}{R(1,X)} < \epsilon \le 1. \end{cases} \tag{2.3}$$

Proof. Observe that X is uniformly nonsquare (see (i)) and then X is superreflexive, $U_X(\epsilon) = U_{\widetilde{X}}(\epsilon)$ (see [18]). Therefore it suffices to prove that X has weak normal structure. Now suppose that X fails to have weak normal structure. Then, by the Lemma 2.1, there exists a weakly null sequence $\{x_n\}_{n=1}^\infty$ in S_X such that

$$\lim_n \|x_n - x\| = 1 \quad \forall x \in \operatorname{co}\{x_n\}_{n=1}^\infty. \tag{2.4}$$

Take $\{f_n\} \subset S_{X^*}$ such that $f_n \in \nabla_{x_n}$ for all $n \in \mathbb{N}$. By the reflexivity of X^*, without loss of generality we may assume that $f_n \rightharpoonup f$ for some $f \in B_{X^*}$ (where $\rightharpoonup$ denotes weak star convergence). We now choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$\lim_n \|x_{n+1} - x_n\| = 1, \qquad |(f_{n+1} - f)(x_n)| < \frac{1}{n}, \qquad f_n(x_{n+1}) < \frac{1}{n}, \tag{2.5}$$

for all $n \in \mathbb{N}$. It follows that $\lim_n f_{n+1}(x_n) = \lim_n (f_{n+1} - f)(x_n) + f(x_n) = 0$. Note that the sequence $\{x_n\}$ is weakly null and verifies $D[\{x_n\}] = 1$. It follows from the definition of $R(1, X)$ that

$$\liminf_n \|x_{n+1} + x_n\| \le R(1, X). \tag{2.6}$$

Therefore we can choose a subsequence $\{x_n\}$ still denoted by $\{x_n\}$ such that

$$\|x_{n+1} + x_n\| \le R(1, X). \tag{2.7}$$

Next denote that $R := R(1, X)$ and consider two cases for $\epsilon \in [0, 1]$. First if $\epsilon \in [0, 1/R]$, put

$$\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}, \qquad \tilde{y} = \{[1 - (R-1)\epsilon] x_{n+1} + \epsilon x_n\}_{\mathcal{U}}, \qquad \tilde{f} = (-f_n)_{\mathcal{U}}. \tag{2.8}$$

By (2.5) and (2.7), then

$$\begin{aligned} \|\tilde{f}\| &= \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1, \\ \|\tilde{y}\| &= \|[1 - (R-1)\epsilon] x_{n+1} + \epsilon x_n\| \\ &= \|\epsilon(x_n + x_{n+1}) + (1 - R\epsilon) x_{n+1}\| \\ &\le R\epsilon + (1 - R\epsilon) = 1. \end{aligned} \tag{2.9}$$

Meanwhile we have

$$\begin{aligned} \tilde{f}(\tilde{x} - \tilde{y}) &= \lim_{\mathcal{U}} (-f_n)((R-1)\epsilon x_{n+1} - (1+\epsilon) x_n) \\ &= 1 + \epsilon, \\ \|\tilde{x} + \tilde{y}\| &= \lim_{\mathcal{U}} \|[2 - (R-1)\epsilon] x_{n+1} - (1-\epsilon) x_n\| \\ &\ge \lim_{\mathcal{U}} (f_{n+1})([2 - (R-1)\epsilon] x_{n+1} - (1-\epsilon) x_n) \\ &= 2 - (R-1)\epsilon. \end{aligned} \tag{2.10}$$

From the definition of $U_X(\epsilon)$, then we get that $U_X(1+\epsilon) = U_{\tilde{X}}(1+\epsilon) \le (R-1)\epsilon/2$, a contradiction.

If $\epsilon \in (1/R, 1]$, in this case $R > 1$, other $\epsilon > 1$. Let

$$\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}, \qquad \tilde{y} = \{[1 - (R-1)\epsilon'] x_n + \epsilon' x_{n+1}\}_{\mathcal{U}}, \qquad \tilde{f} = (-f_n)_{\mathcal{U}}, \tag{2.11}$$

where $\epsilon' = 1-(R-1)\epsilon \in [0, 1/R)$. It follows from the first case we have that

$$\|\widetilde{f}\| = \|\widetilde{x}\| = 1, \qquad \|\widetilde{y}\| \le 1. \tag{2.12}$$

Furthermore, we have

$$\begin{aligned}
\widetilde{f}(\widetilde{x}-\widetilde{y}) &= \lim_{\mathcal{U}}(-f_n)\big((1-\epsilon')x_{n+1} - [2-(R-1)\epsilon']x_n\big) \\
&= 2-(R-1)\epsilon', \\
\|\widetilde{x}+\widetilde{y}\| &= \lim_{\mathcal{U}}\big\|(1+\epsilon')x_{n+1} - (R-1)\epsilon' x_n\big\| \\
&\ge \lim_{\mathcal{U}}(f_{n+1})\big((1+\epsilon')x_{n+1} - (R-1)\epsilon' x_n\big) \\
&= 1+\epsilon'.
\end{aligned} \tag{2.13}$$

From the definition of $U_X(\epsilon)$. Then

$$U_X(2-(R-1)\epsilon') = U_{\widetilde{X}}(2-(R-1)\epsilon') \le \frac{1-\epsilon'}{2}, \tag{2.14}$$

which is equivalent to

$$U_X(1+\epsilon) = U_{\widetilde{X}}(1+\epsilon) \le \frac{1}{2}\left(1-\frac{1-\epsilon}{R-1}\right), \tag{2.15}$$

a contradiction. □

Remark 2.3. Let $\epsilon_1 = 1+\epsilon$, by Theorem 2.2, we get that X has normal structure whenever $U_X(\epsilon_1) > f(\epsilon_1)$, where $f(\epsilon_1)$ is defined as

$$f(\epsilon_1) := \begin{cases} 0, & 0 \le \epsilon_1 \le 1, \\ (R(1,X)-1)\dfrac{\epsilon_1 - 1}{2}, & 1 \le \epsilon_1 \le \dfrac{1}{R(1,X)} + 1, \\ \dfrac{1}{2}\left(1 - \dfrac{2-\epsilon_1}{R(1,X)-1}\right), & \dfrac{1}{R(1,X)} + 1 \le \epsilon_1 \le 2, \end{cases} \tag{2.16}$$

obviously $f(\epsilon_1) \le (\epsilon_1 - 1)/2$ for any $\epsilon_1 \in [0,2]$, therefore Theorem 2.2 is a generalization of the result of Saejung (ii).

Since $U_X(\epsilon) \ge \delta_X(\epsilon)$ (see [18]), we have the following corollary in [5].

Corollary 2.4. *If $\delta_X(1+\epsilon) > f(\epsilon)$, where $f(\epsilon)$ is the same as Theorem 2.2, then X has normal structure.*

Remark 2.5. In fact from the discussion in Remark 2.3, Corollary 2.4 is equivalent to if $\delta_X(\epsilon) > f_1(\epsilon)$ for some $\epsilon \in [0,2]$, then X has normal structure, where $f_1(\epsilon)$ is defined as

$$f_1(\epsilon) := \begin{cases} 0, & 0 \le \epsilon \le 1, \\ (R(1,X)-1)\dfrac{\epsilon-1}{2}, & 1 \le \epsilon \le \dfrac{1}{R(1,X)}+1, \\ \dfrac{1}{2}\left(1-\dfrac{2-\epsilon}{R(1,X)-1}\right), & \dfrac{1}{R(1,X)}+1 \le \epsilon \le 2. \end{cases} \tag{2.17}$$

Theorem 2.6. *If $W_X^*(1+\epsilon) > f(\epsilon)$, where $f(\epsilon)$ is the same as Theorem 2.2, then X has normal structure.*

Proof. Observe that X is uniformly nonsquare (see (i)) and then X is superreflexive, $W_X^*(\epsilon) = W_{\tilde{X}}^*(\epsilon)$ (see [19]). Therefore it suffices to prove that X has weak normal structure. Repeat the arguments in the proof of Theorem 2.2. Firstly if $\epsilon \in [0,1/R]$, let

$$\tilde{x} = (x_{n+1}-x_n)_{\mathcal{U}}, \qquad \tilde{y} = \{[1-(R-1)\epsilon]x_{n+1}+\epsilon x_n\}_{\mathcal{U}}, \qquad \tilde{f} = (f_{n+1})_{\mathcal{U}}. \tag{2.18}$$

By (2.5) and (2.7), then

$$\|\tilde{f}\| = \|\tilde{x}\| = 1, \qquad \|\tilde{y}\| \le 1. \tag{2.19}$$

Meanwhile, we have

$$\begin{aligned} \frac{1}{2}\tilde{f}(\tilde{x}-\tilde{y}) &= \frac{1}{2}\lim_{\mathcal{U}}(f_{n+1})\big((R-1)\epsilon x_{n+1}-(1+\epsilon)x_n\big) \\ &= \frac{(R-1)\epsilon}{2}, \\ \|\tilde{x}-\tilde{y}\| &= \lim_{\mathcal{U}}\big\|[(R-1)\epsilon]x_{n+1}-(1+\epsilon)x_n\big\| \\ &\ge \lim_{\mathcal{U}}(-f_n)\big([(R-1)\epsilon]x_{n+1}-(1+\epsilon)x_n\big) \\ &= 1+\epsilon. \end{aligned} \tag{2.20}$$

From the definition of $W_X^*(\epsilon)$, then $W_X^*(1+\epsilon) = W_{\tilde{X}}^*(1+\epsilon) \le (R-1)\epsilon/2$. A contradiction.

If $\epsilon \in (1/R,1]$, in this case $R>1$, other $\epsilon > 1$. Let

$$\tilde{x} = (x_{n+1}-x_n)_{\mathcal{U}}, \qquad \tilde{y} = \{[1-(R-1)\epsilon']x_n+\epsilon' x_{n+1}\}_{\mathcal{U}}, \qquad \tilde{f} = (f_{n+1})_{\mathcal{U}}, \tag{2.21}$$

where $\epsilon' = 1-(R-1)\epsilon \in [0,1/R)$. It follows from the first case we have that

$$\|\tilde{f}\| = \|\tilde{x}\| = 1, \qquad \|\tilde{y}\| \le 1. \tag{2.22}$$

Furthermore, we have

$$\begin{aligned}\frac{1}{2}\tilde{f}(\tilde{x}-\tilde{y}) &= \frac{1}{2}\lim_{\mathcal{U}}(f_{n+1})\left((1-\epsilon')x_{n+1}-\left[2-(R-1)\epsilon'\right]x_n\right)\\ &= \frac{1-\epsilon'}{2},\\ \|\tilde{x}-\tilde{y}\| &= \lim_{\mathcal{U}}\left\|(1-\epsilon')x_{n+1}-\left[2-(R-1)\epsilon'\right]x_n\right\|\\ &\geq \lim_{\mathcal{U}}(-f_n)\left((1-\epsilon')x_{n+1}-\left[2-(R-1)\epsilon'\right]x_n\right)\\ &= 2-(R-1)\epsilon'.\end{aligned} \tag{2.23}$$

It follows from the definition of $W_X^*(\epsilon)$. Then

$$W_X^*(2-(R-1)\epsilon') = W_{\tilde{X}}^*(2-(R-1)\epsilon') \leq \frac{1-\epsilon'}{2}, \tag{2.24}$$

which is equivalent to

$$W_X^*(1+\epsilon) = W_{\tilde{X}}^*(1+\epsilon) \leq \frac{1}{2}\left(1-\frac{1-\epsilon}{R-1}\right), \tag{2.25}$$

a contradiction. □

Remark 2.7. Similarly, we also get Corollary 2.4 because of $W_X^*(\epsilon) \geq \delta_X(\epsilon)$. Meanwhile Theorem 2.6 also strengthens the result of Saejung (iii).

The following proposition can be found in [21].

Proposition 2.8. *Let X be a Banach space, one has the following equality:*

$$E(X) = \sup\left\{\epsilon^2 + 4\left(1-\delta_X(\epsilon)\right)^2 : \epsilon \in (0,2]\right\}. \tag{2.26}$$

Corollary 2.9. *Let X be a Banach space, if $E(X)$ satisfies one of the following conditions:*

(i) $E(X) < 4+\epsilon^2$ *for arbitrary* $\epsilon \in [0,1]$,

(ii) $E(X) < [2-(R(1,X)-1)(\epsilon-1)]^2+\epsilon^2$ *for arbitrary* $\epsilon \in [1, 1+1/R(1,X)]$,

(iii) $E(X) < [(R(1,X)+1-\epsilon)/(R(1,X)-1)]^2+\epsilon^2$ *for arbitrary* $\epsilon \in [1+1/R(1,X), 2]$.

Then X has normal structure. Specially $E(X) < 2(1+1/R(1,X))^2$ implies that X has normal structure.

Proof. In fact from Remark 2.3, if $\delta_X(\epsilon) > f_1(\epsilon)$ for some $\epsilon \in [0,2]$, then X has normal structure, where $f_1(\epsilon)$ is defined as

$$f_1(\epsilon) := \begin{cases} 0, & 0 \le \epsilon \le 1, \\ (R(1,X)-1)\dfrac{\epsilon-1}{2}, & 1 \le \epsilon \le \dfrac{1}{R(1,X)}+1, \\ \dfrac{1}{2}\left(1-\dfrac{2-\epsilon}{R(1,X)-1}\right), & \dfrac{1}{R(1,X)}+1 \le \epsilon \le 2. \end{cases} \tag{2.27}$$

By Proposition 2.8, if $E(X) < 4+\epsilon^2$ for arbitrary $\epsilon \in [0,1]$, there exist a $\epsilon \in [0,1]$, such that

$$\epsilon^2 + 4(1-\delta_X(\epsilon))^2 < 4+\epsilon^2, \tag{2.28}$$

which implies that there exist a $\epsilon \in [0,1]$, such that $\delta_X(\epsilon) > 0$, therefore X has normal structure from Remark 2.5.

Repeating the arguments as above, we can prove that the conditions $E(X) < [2-(R(1,X)-1)(\epsilon-1)]^2+\epsilon^2$ for arbitrary $\epsilon \in [1, 1+1/R(1,X)]$ and $E(X) < [(R(1,X)+1-\epsilon)/(R(1,X)-1)]^2+\epsilon^2$ for arbitrary $\epsilon \in [1+1/R(1,X),2]$ imply that $\delta_X(\epsilon) > (R(1,X)-1)((\epsilon-1)/2)$ for some $1 \le \epsilon \le 1/R(1,X)+1$ and $\delta_X(\epsilon) > (1/2)[1-(2-\epsilon)/(R(1,X)-1)]$ for some $\epsilon \in [1+1/R(1,X),2]$, respectively. The desired conclusion follows from Remark 2.5 . In particular, let $\epsilon = 1+1/R(1,X)$, we get that $E(X) < 2(1+1/R(1,X))^2$ implies that X has normal structure. □

Remark 2.10. In [13], Saejung in fact has obtained that $E(X) < 3+\sqrt{5}$ implies that X has the normal structure which improves the results of Gao (see [20]). However we have the following example. Given $\beta \ge 1$, consider in l_2 the equivalent norm $|\cdot|$ given by

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\}, \tag{2.29}$$

and let $X_\beta := (l_2, |\cdot|_\beta)$. The space X_β has normal structure if and only if $\beta < \sqrt{2}$ and verifies that $E(X_\beta) = \min\{8, 4\beta^2\}$ and $R(1,X_\beta) = \max\{\beta/\sqrt{2}, \sqrt{3}/\sqrt{2}\}$. Then, for any $\beta \in [(1+\sqrt{5})/2\sqrt{2}, (1+\sqrt{2/3})/\sqrt{2})$ we have

$$3+\sqrt{5} \le 4\beta^2 = E(X_\beta) < 2\left(1+\sqrt{\frac{2}{3}}\right)^2 = 2\left(1+\frac{1}{R(1,X_\beta)}\right)^2. \tag{2.30}$$

Then X has normal structure by Corollary 2.9 but lies out of the scope of $E(X) < 3+\sqrt{5}$.

3. The Modulus of $\rho_1(\epsilon), f(X)$ and $J(t,X), S(t,X)$

Proposition 3.1. *Let X be a Banach space, for $\epsilon \in [0,2]$, then*

$$f(X) = \inf\{\epsilon^2 + 4(1-\rho_1(\epsilon))^2, \epsilon \in [0,2]\}. \tag{3.1}$$

Proof. Take $x \in S_X$, $y \in S_X$, and $\|x-y\| = \epsilon$. It follows from the definition of $f(X)$, then we get

$$f(X) \le \epsilon^2 + \|x+y\|^2. \tag{3.2}$$

Thus we have

$$1 - \frac{\|x+y\|}{2} \le 1 - \frac{\sqrt{f(X) - \epsilon^2}}{2}. \tag{3.3}$$

From the definition of $\rho_1(\epsilon)$, we have that

$$\rho_1(\epsilon) \le 1 - \frac{\sqrt{f(X)-\epsilon^2}}{2}, \quad \text{or equivalently} \quad f(X) \le \inf\left\{\epsilon^2 + 4\left(1-\rho_1(\epsilon)\right)^2\right\}. \tag{3.4}$$

On the other hand for every $x, y \in S_X$, $\rho_1(\|x-y\|) \ge 1 - \|x+y\|/2$, which implies that

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &\ge \epsilon^2 + 4\left(1-\rho_1(\|x-y\|)\right)^2 \\ &\ge \left\{\epsilon^2 + 4\left(1-\rho_1(\epsilon)\right)^2 : \epsilon \in (0,2]\right\}. \end{aligned} \tag{3.5}$$

Hence $f(X) \ge \inf\{\epsilon^2 + 4(1-\rho_1(\epsilon))^2 : \epsilon \in (0,2]\}$. Finally we obtain the desired equality.

From Proposition 3.1, we can compute the exact value of $f(X)$ for some Banach space. □

Example 3.2. Let $X = R^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & x_1x_2 \ge 0, \\ \|x\|_1 & x_1x_2 \le 0. \end{cases} \tag{3.6}$$

Then we have $f(X) = 16/5$.

Proof. It is well known that $\rho_1(\epsilon) = \max\{\epsilon/4, \epsilon - 1\}$ (see [11]). From Proposition 3.1, we have the following:

(1) if $0 \le \epsilon \le 4/3$, $\epsilon^2 + 4(1-\rho_1(\epsilon))^2 = (5/4)(\epsilon - 4/5)^2 + 16/5$, therefore we get $f(X) = 16/5$;

(2) if $4/3 \le \epsilon \le 2$, similarly we have $\epsilon^2 + 4(1-\rho_1(\epsilon))^2 = 5(\epsilon - 8/5)^2 + 16/5$, then $f(X) = 16/5$. □

Proposition 3.3 (see [8]). *Let X be a Banach space, for $\epsilon \in [0,2]$, then*

$$\rho_1(\epsilon) = \left(1-\rho_1(\epsilon)\right)\beta_X\left(\frac{\epsilon}{2\left(1-\rho_1(\epsilon)\right)}\right), \tag{3.7}$$

where $\beta_X(\epsilon) = \sup\{\min\{\|x+\epsilon y\|, \|x-\epsilon y\|\} : x, y \in S_X\} - 1$ is Milman's moduli.

From the above relationship we can compute the exact value of $\rho_1(\epsilon)$ and $f(X)$ for some Banach space.

Example 3.4. Let $X = b_{2,\infty}$ is Bynum space, it is well known that $\beta_X(\epsilon) = J(\epsilon, X) - 1 = \epsilon/\sqrt{2}$ (see [22]). By Proposition 3.3 we have

$$\rho_1(\epsilon) = \frac{\epsilon}{2\sqrt{2}}. \tag{3.8}$$

By Proposition 3.1, we get

$$\begin{aligned} f(X) &= \inf\left\{\epsilon^2 + 4(1-\rho_1(\epsilon))^2,\ \epsilon \in [0,2]\right\} \\ &= \inf\left\{\epsilon^2 + 4 + \frac{\epsilon^2}{2} - 2\sqrt{2}\epsilon,\ \epsilon \in [0,2]\right\} \\ &= \inf\left\{\frac{3}{2}\left(\epsilon - \frac{2\sqrt{2}}{3}\right)^2 + \frac{8}{3},\ \epsilon \in [0,2]\right\} \\ &= \frac{8}{3}. \end{aligned} \tag{3.9}$$

Remark 3.5. The constants $E(X)$ and $f(X)$ have been studied in [20]; some exact values of these constants have been obtained for some Banach space (see [20, 23]). We observe that for Banach Space L_p, l_p, Day-James space $l_{1,\infty}$ and the example in [23], the following equality holds as similarly as $J(X)S(X) = 2$:

$$E(X)f(X) = 16. \tag{3.10}$$

Equality (3.10) obviously holds for Banach space with nouniformly nonsquare (see [20]). Initially we conjecture that equality (3.10) holds for any Banach space. Unfortunately, Example 3.4 gives us a counterexample. In fact it is well known that $E(b_{2,\infty}) = 3 + 2\sqrt{2}$ (see [22]). Therefore by Example 3.4 obviously $E(b_{2,\infty})f(b_{2,\infty}) \neq 16$.

Recently, some geometric properties on the parameterized James constant and parameterized Schäffer constant have been studied (see [7, 12, 13]). In the sequel, we compute the exact values of $J(t,X)$ and $S(t,X)$ for some contrete Banach space. Note that $J(t,X) = \beta_X(t) + 1$ and $S(t,X) = d_X(t) + 1$.

Example 3.6. Let $X = R^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & x_1x_2 \geq 0, \\ \|x\|_1 & x_1x_2 \leq 0. \end{cases} \tag{3.11}$$

Then we have

$$J(t,X) = \begin{cases} \dfrac{t}{2} + 1 & 0 \leq t \leq 1, \\ t + \dfrac{1}{2} & 1 < t < \infty. \end{cases} \tag{3.12}$$

Proof. It is well known that $\rho_1(\epsilon) = \max\{\epsilon/4, \epsilon - 1\}$ (see [11]). From Proposition 3.3, we have the following:

(1) if $0 \leq \epsilon \leq 4/3$, then $\epsilon/(4-\epsilon) = \beta_X(2\epsilon/(4-\epsilon))$; set $t = 2\epsilon/(4-\epsilon)$, then $0 \leq t \leq 1$. Therefore

$$\beta_X(t) = \frac{t}{2}, \qquad J(t, X) = \frac{t}{2} + 1; \tag{3.13}$$

(2) if $4/3 < \epsilon \leq 2$, then $(\epsilon - 1)/(2-\epsilon) = \beta_X(\epsilon/2(2-\epsilon))$; set $t = \epsilon/2(2-\epsilon)$, then $1 \leq t < \infty$. Therefore

$$\beta_X(t) = t - \frac{1}{2}, \qquad J(t, X) = t + \frac{1}{2}. \tag{3.14}$$

In particular, we get $J(X) = 3/2$ (see [24]). □

Proposition 3.7 (see [25]). *Let X be a Banach space, for $\epsilon \in [0, 2]$, then*

$$\delta_X(\epsilon) = (1 - \delta_X(\epsilon))d_X\left(\frac{\epsilon}{2(1 - \delta_X(\epsilon))}\right), \tag{3.15}$$

where $d_X(\epsilon) = \inf\{\max\{\|x + \epsilon y\|, \|x - \epsilon y\|\} : x, y \in S_X\} - 1$ is Milman's moduli.

Example 3.8. Let $X = R^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty, & x_1x_2 \geq 0, \\ \|x\|_1, & x_1x_2 \leq 0. \end{cases} \tag{3.16}$$

Then we have

$$S(t, X) = \begin{cases} 1, & 0 \leq t \leq \dfrac{1}{2}, \\ \dfrac{2(1+t)}{3}, & \dfrac{1}{2} < t \leq 2, \\ t, & 2 < t < \infty. \end{cases} \tag{3.17}$$

Proof. It is well known that $\delta_X(\epsilon) = \max\{0, (\epsilon - 1)/2\}$. From Proposition 3.7, we have the following:

(1) if $0 \leq \epsilon \leq 1$, then $d_X(\epsilon/2) = 0$; set $t = \epsilon/2$, then $0 \leq t \leq 1/2$. Therefore

$$d_X(t) = 0, \qquad S(t, X) = 1, \tag{3.18}$$

(2) if $1 < \epsilon \leq 2$, then $(\epsilon - 1)/(3-\epsilon) = d_X(\epsilon/(3-\epsilon))$; set $t = \epsilon/(3-\epsilon)$, then $1/2 \leq t \leq 2$. Therefore

$$d_X(t) = \frac{2t-1}{3}, \qquad S(t,X) = \frac{2(t+1)}{3}. \tag{3.19}$$

(3) for the case of $2 < t < \infty$, let $x = (1,1), y = (-1/2, 1/2)$, for any $2 < t < \infty$ we have

$$\|x + ty\| = \|x - ty\| = t. \tag{3.20}$$

From the definition of $S(t,X)$, we get that $S(t,X) \leq t$ for $2 < t < \infty$. On the other hand $t \leq S(t,X)$ for any $0 < t < \infty$. Finally we get that $S(t,X) = t$.

In particular, we get $S(X) = 4/3 = 2/J(X)$. □

Example 3.9. Fixed a number λ, $\lambda > 1$, and consider the plane $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\| = \max\left\{\lambda|x_1|, \sqrt{x_1^2 + x_2^2}\right\}. \tag{3.21}$$

Then from [10] we have known that

$$\delta_X(\epsilon) = \begin{cases} 0, & 0 \leq \epsilon \leq 2\sqrt{1 - \dfrac{1}{\lambda^2}}, \\ 1 - \lambda\sqrt{1 - \dfrac{\epsilon^2}{4}}, & 2\sqrt{1-\dfrac{1}{\lambda^2}} \leq \epsilon \leq \dfrac{2\lambda}{\sqrt{1+\lambda^2}}, \\ 1 - \sqrt{1 - \dfrac{\epsilon^2}{4\lambda^2}}, & \dfrac{2\lambda}{\sqrt{1+\lambda^2}} \leq \epsilon \leq 2. \end{cases} \tag{3.22}$$

Repeating the same arguments, we get

$$S(t,X) = \begin{cases} 1, & 0 \leq t \leq \sqrt{1 - \dfrac{1}{\lambda^2}}, \\ \dfrac{\sqrt{1+\lambda^2 t^2}}{\lambda}, & \sqrt{1 - \dfrac{1}{\lambda^2}} \leq t \leq 1, \\ \dfrac{\sqrt{\lambda^2 + t^2}}{\lambda}, & 1 \leq t \leq \dfrac{\lambda}{\sqrt{\lambda^2 - 1}}. \end{cases} \tag{3.23}$$

In particular, we get $S(X) = \sqrt{1+\lambda^2}/\lambda$

Example 3.10. Let X be $\mathbb{R}^2$ with the $l_{2,1}$ norm defined by

$$\|x\| = \begin{cases} \|x\|_2, & x_1 x_2 \geq 0\ , \\ \|x\|_1, & x_1 x_2 \leq 0. \end{cases} \tag{3.24}$$

From [2] we have known that

$$\delta_X(\epsilon) = \begin{cases} 0, & 0 \le \epsilon \le \sqrt{2}, \\ 1 - \sqrt{2 - \dfrac{\epsilon^2}{2}}, & \sqrt{2} \le \epsilon \le \sqrt{\dfrac{8}{3}}, \\ 1 - \sqrt{1 - \dfrac{\epsilon^2}{8}}, & \sqrt{\dfrac{8}{3}} \le \epsilon \le 2. \end{cases} \tag{3.25}$$

Similarly, we get

$$S(t, X) = \begin{cases} 1, & 0 \le t \le \dfrac{\sqrt{2}}{2}, \\ \dfrac{\sqrt{2 + 4t^2}}{2}, & \dfrac{\sqrt{2}}{2} < t \le 1, \\ \dfrac{\sqrt{4 + 2t^2}}{2}, & 1 < t \le \sqrt{2}. \end{cases} \tag{3.26}$$

In particular, we get $S(X) = \sqrt{6}/2 = 2/J(X)$.

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