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Research Article

Existence of Solutions to the System of Generalized Implicit Vector Quasivariational Inequality Problems

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We study the system of generalized implicit vector quasivariational inequality problems and prove a new existence result of its solutions by Kakutani-Fan-Glicksberg's fixed points theorem. As a special case, we also derive a new existence result of solutions to the generalized implicit vector quasivariational inequality problems.

1. Introduction

The system of generalized implicit vector quasivariational inequality problems generalizes the generalized implicit vector quasivariational inequality problems, and the latter had been studied in [1–3]. In this paper, we study the system of generalized implicit vector quasivariational inequality problems and prove a new existence result of its solutions by Kakutani-Fan-Glicksberg's fixed points theorem. For other existence results with respect to the system of generalized implicit vector quasivariational inequality problems, we refer the reader to [4–6] and references therein.

Let I be an index set (finite or infinite). For each $i \in I$, let X_i and Y_i be two Hausdorff topological vector spaces, K_i a nonempty subset of X_i, and C_i a closed, convex and pointed cone of Y_i with $\operatorname{int} C_i \neq \emptyset$, where $\operatorname{int} C_i$ denotes the interior of C_i. Denote that $K_{\hat{i}} = \prod_{j \in I, j \neq i} K_j$, $K = \prod_{i \in I} K_i = K_i \times K_{\hat{i}}$, $X = \prod_{i \in I} X_i$. For each $x \in K$, we can write $x = (x_i, x_{\hat{i}})$. For each $i \in I$, let D_i be a nonempty subset of the continuous linear operators space $L(X_i, Y_i)$ from X_i into Y_i and let $F : D_i \times K_i \times K_i \longrightarrow Y_i$, $G_i : K \longrightarrow 2^{K_i}$, $T_i : K \longrightarrow 2^{D_i}$ be three set-valued maps, where 2^{D_i} and 2^{K_i} denote the family of all nonempty subsets of D_i and K_i, respectively. The system of generalized implicit vector quasivariational inequality problems (briefly, SGIVQIP) is as follows: find $\overline{x} = (\overline{x}_i, \overline{x}_{\hat{i}}) \in K$ such that for each $i \in I, \overline{x}_i \in G_i(\overline{x})$, and

$$\forall y_i \in G_i(\overline{x}),\ \exists \overline{u}_i \in T_i(\overline{x}), \quad F_i(\overline{u}_i, \overline{x}_i, y_i) \not\subset -\operatorname{int} C_i. \tag{1.1}$$

$\overline{x} = (\overline{x}_i, \overline{x}_{\hat{i}})$ is said to be a solution of the SGIVQIP. An SGIVQIP is usually denoted by $\{K_i, D_i, G_i, T_i, F_i\}_{i\in I}$.

If I is a singleton, then the SGIVQIP coincides with the generalized implicit vector quasivariational inequality problems (briefly, GIVQIP). A GIVQIP is usually denoted by $\{K, D, G, T, F\}$.

Throughout this paper, unless otherwise specified, assume that for each $i \in I$, K_i is a nonempty convex compact subset of a Banach space X_i, Y_i is a Hausdorff topological vector space, and C_i is a closed, convex, and pointed cone of Y_i with $\operatorname{int} C_i \neq \emptyset$, where $\operatorname{int} C_i$ denotes the interior of C_i.

2. Preliminaries

In this section, we introduce some useful notations and results.

Definition 2.1. Let X and Y be two topological spaces and K a nonempty convex subset of X. $F : K \longrightarrow 2^Y$ is a set-valued map.

(1) F is called upper semicontinuous at $x_0 \in K$ if, for any open set $G \supset F(x_0)$, there exists an open neighborhood U of x_0 in K such that for all $x \in U$,

$$G \supset F(x); \tag{2.1}$$

and upper semicontinuous on K if it is upper semicontinuous at every point of K.

(2) F is called lower semicontinuous at $x_0 \in K$ if, for any open set $G \cap F(x_0) \neq \emptyset$, there exists an open neighborhood U of x_0 in K such that for all $x \in U$,

$$G \cap F(x) \neq \emptyset; \tag{2.2}$$

and lower semicontinuous on K if it is lower semicontinuous at every point of K.

(3) F is called continuous at $x_0 \in K$ if, it is both upper semicontinuous and lower semicontinuous at x_0; and continuous on K if it is continuous at every point of K.

Definition 2.2. Let X and Y be two topological vector spaces and K a nonempty convex subset of X.Also $F : K \longrightarrow 2^Y$ is a set-valued map.

(1) F is called upper C-semicontinuous at $x_0 \in K$ if, for any open neighborhood V of the zero element θ in Y, there exists an open neighborhood U of x_0 in K such that, for all $x \in U$,

$$F(x) \subset F(x_0) + V + C; \tag{2.3}$$

and upper C-semicontinuous on K if it is upper C-semicontinuous at every point of K.

(2) F is called lower C-semicontinuous at $x_0 \in K$ if, for any open neighborhood V of the zero element θ in Y, there exists an open neighborhood U of x_0 in K such that, for all $x \in U$,

$$F(x) \cap (F(x_0) + V + C) \neq \emptyset; \tag{2.4}$$

and lower C-semicontinuous on K if it is lower C-semicontinuous at every point of K.

(3) F is called C-continuous at $x_0 \in K$ if it is upper C-semicontinuous and lower C-semicontinuous at $x_0 \in K$; and C-continuous on K if it is C-continuous at every point of K.

Definition 2.3. Let X and Y be two topological vector spaces and K a nonempty convex subset of X. Let $F : K \to 2^Y$ be a set-valued map.

(1) F is called C-convex if, for each $x_1, x_2 \in K, t \in [0,1]$,

$$F(tx_1 + (1-t)x_2) \subset [tF(x_1) + (1-t)F(x_2)] - C; \tag{2.5}$$

and C-concave if $-F$ is C-convex.

(2) F is called C-quasiconvex-like if, for each $x_1, x_2 \in K, t \in [0,1]$,

$$\text{either } F(tx_1 + (1-t)x_2) \subset F(x_1) - C \quad \text{or} \quad F(tx_1 + (1-t)x_2) \subset F(x_2) - C; \tag{2.6}$$

and C-quasiconcave-like if $-F$ is C-quasiconvex-like.

Lemma 2.4 ([7, Theorem 1]). *Let K be a nonempty paracompact subset of a Hausdorff topological space X and, Z be a nonempty subset of a Hausdorff topological vector space Y. Suppose that $S, T : K \mapsto 2^Z$ be two set-valued maps with following conditions:*

(1) *for each $x \in K$, $coS(x) \subset T(x)$;*

(2) *for each $y \in Z$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is open.*

Then T has a continuous selection, that is, there is a continuous map $f : K \mapsto Z$ such that $f(x) \in T(x)$ for each $x \in K$.

3. Existence of Solutions to the SGIVQIP

Lemma 3.1. *Let D, W, X be three Hausdorff topological spaces, Z a topological vector space, and C a closed, convex, and pointed cone of Z. Let $T : W \times X \mapsto 2^D$ and $F : D \times W \times W \mapsto 2^Z$ be two set-valued maps. Assume that $(w, x, y) \in W \times X \times W$ and*

(1) *$T(\cdot,\cdot)$ is upper semicontinuous on $W \times X$ with nonempty and compact values;*

(2) *$F(\cdot,\cdot,\cdot)$ is upper C-semicontinuous on $D \times W \times W$ with nonempty and compact values;*

(3) *for each $u \in T(w, x), F(u, w, y) \subset -\operatorname{int} C$.*

Then there exist open neighborhood $U(w)$ of w and open neighborhood $U(x)$ of x, and open neighborhood $U(y)$ of y such that $\{F(u,w',y') : u \in T(w',x')\} \subset -\operatorname{int} C$ whenever $w' \in U(w)$, $x' \in U(x)$, $y' \in U(y)$.

Proof. By (3) and compactness of $F(u,w,y)$, there exists an open neighborhood $V(u)$ of the zero element θ of Z such that $F(u,w,y)+V(u) \subset -\operatorname{int} C$. By (2), there exist open neighborhood $O(u)$ of u and open neighborhood $O_u(w)$ of w, open neighborhood $O_u(y)$ of y such that $F(u',w',y') \subset F(u,w,y)+V(u)-C \subset -\operatorname{int} C-C \subset -\operatorname{int} C$ whenever $u' \in O(u), w' \in O_u(w), y' \in O_u(y)$. Since $T(w,x)$ is compact and $\cup_{u\in T(w,x)} O(u) \supset T(w,x)$, there exist finite $u^1, u^2, \dots, u^M \in T(w,x)$ such that $\bigcup_{j=1}^M O(u^j) \supset T(w,x)$. Taking

$$O(w) = \cap_{j=1}^M O_{u^j}(w), \quad U(y) = \cap_{j=1}^M O_{u^j}(y). \tag{3.1}$$

Clearly, $O(w)$ and $U(y)$ are open neighborhood of w and y, respectively. Thus for each $u \in \cup_{j=1}^M O(u^j)$, we have $F(u,w',y') \subset -\operatorname{int} C$ whenever $w' \in O(w)$, $y' \in U(y)$. By (1), there exist open neighborhood $U(w)$ of w with $U(w) \subset O(w)$ and open neighborhood $U(x)$ of x such that $T(w',x') \subset \cup_{j=1}^M O(u^j)$ whenever $w' \in U(w), x' \in U(x)$, which implies that

$$\{F(u,w',y') : u \in T(w',x')\} \subset \left\{F(u,w',y') : u \in \cup_{j=1}^M O\left(u^j\right)\right\} \subset -\operatorname{int} C. \tag{3.2}$$

whenever $w' \in U(w)$, $x' \in U(x)$, $y' \in U(y)$.

The proof is finished. □

By Lemma 3.1, we obtain the following result.

Theorem 3.2. *Consider an SGIVQIP $\{K_i, D_i, G_i, T_i, F_i\}_{i\in I}$. For each $i \in I$, assume that*

(1) *$G_i(\cdot)$ is continuous on K with convex compact values and for each $x \in K, \operatorname{int} G_i(x) \neq \emptyset$;*

(2) *$T_i(\cdot)$ is upper semicontinuous on K with nonempty and compact values;*

(3) *$F_i(\cdot,\cdot,\cdot)$ is upper C_i- semicontinuous on $D_i \times K_i \times K_i$ with nonempty and compact values;*

(4) *for each $x \in K$ and each $u_i \in T_i(x)$, $F_i(u_i, x_i, \cdot)$ is C_i- convex or C_i- quasiconvex-like;*

(5) *for each $x \in K$ and each $u_i \in T_i(x)$, if $x_i \in \operatorname{int} \mathrm{G_i(x)}$, then $F_i(u_i, x_i, x_i) \not\subset -\operatorname{int} \mathrm{C_i}$, where x_i is the ith component of x.*

Then the SGIVQIP has a solution, that is, there exists $\overline{x} = (\overline{x}_i, \overline{x}_{\hat{i}}) \in K$ such that for each $i \in I, \overline{x}_i \in G_i(\overline{x})$, and

$$\forall y_i \in G_i(\overline{x}), \ \exists \overline{u}_i \in T_i(\overline{x}), \quad F_i(\overline{u}_i, \overline{x}_i, y_i) \not\subset -\operatorname{int} C_i. \tag{3.3}$$

Proof. For each $i \in I$, define a set-valued map $S_i : K \to 2^{K_i} \cup \{\emptyset\}$ by

$$S_i(x) = \{y_i \in K_i : F_i(u_i, x_i, y_i) \subset -\operatorname{int} C_i,\ \forall u_i \in T_i(x)\}. \tag{3.4}$$

□

Step 1. We prove that the set $J_i = \{x \in K : G_i(x) \cap S_i(x) = \emptyset\}$ is closed. For any sequence $x^n \in J_i = \{x \in K : G_i(x) \cap S_i(x) = \emptyset\}$ with $x^n \to x^0$, we have

$$\forall y_i^n \in G_i(x^n),\ \exists u_i^n \in T_i(x^n), \quad F_i(u_i^n, x_i^n, y_i^n) \not\subset -\operatorname{int} C_i. \tag{3.5}$$

If $x^0 \notin J_i$, then there exists $z_i^0 \in G_i(x^0)$ such that for each $u_i \in T_i(x^0)$, $F_i(u_i, x_i^0, z_i^0) \subset -\operatorname{int} C_i$. By Lemma 3.1, there exist open neighborhood $U(x^0)$ of x^0 and open neighborhood $U(z_i^0)$ of z_i^0, such that $\{F(u_i, x_i', z_i') : u_i \in T(x')\} \subset -\operatorname{int} C_i$ whenever $x' \in U(x^0), z_i' \in U(z_i^0)$. By (1), there exist $z_i^n \in G_i(x^n)$ such that $z_i^n \to z_i^0$ $(n \to +\infty)$, which implies that there exists a positive integer N such that $x^n \in U(x^0), z_i^n \in U(z_i^0)$ whenever $n > N$. Thus we have $F_i(u_i, x^n, z_i^n) \subset -\operatorname{int} C_i$, for all $u_i \in T_i(x^n)$ whenever $n > N$, a contradiction. This shows that J_i is closed,that is, $W_i = \{x \in K : G_i(x) \cap S_i(x) \neq \emptyset\}$ is open.

Without loss of generality, assume that $W_i \neq \emptyset$.

Define a set-valued map $P_i : K \mapsto 2^{K_i} \cup \{\emptyset\}$ by

$$P_i(x) = \operatorname{int} G_i(x) \cap S_i(x) \quad \text{for each } x \in K. \tag{3.6}$$

Step 2. We prove that for each $x \in W_i, P_i(x)$ is nonempty and convex.

For each $y_i \in S_i(x)$, we have $F_i(u_i, x_i, y_i) \subset -\operatorname{int} C_i$, for all $u_i \in T_i(x)$. By Lemma 3.1, there exists an open neighborhood $U(y_i)$ of y_i such that $\{F_i(u_i, x_i, y_i') : u_i \in T_i(x)\} \subset -\operatorname{int} C_i$ whenever $y_i' \in U(y_i)$, which implies that $U(y_i) \subset S_i(x)$, that is, $S_i(x)$ is open. By (4), it is easy to verify that $S_i(x)$ is convex.

Since $G_i(x)$ is convex and $\operatorname{int} G_i(x) \neq \emptyset$, then for each $x \in W_i, P_i(x)$ is nonempty and convex.

Step 3. We prove that $P_i|_{W_i}$ has a continuous selection $f_i : W_i \mapsto 2^{K_i}$.

For each $y_i^0 \in P_i(x)$, we have $y_i^0 \in \operatorname{int} G_i(x)$ and $y_i^0 \in S_i(x)$. By $y_i^0 \in \operatorname{int} G_i(x)$, there exists $\varepsilon_0 > 0$ such that $y_i^0 + \varepsilon_0 \subset \operatorname{int} G_i(x)$, where $y_i^0 + \varepsilon_0 = \{z_i \in K_i : d_i(z_i, y_i^0) < \varepsilon_0\}$. Since $G_i(x)$ is continuous with convex compact values, then there exists an open neighborhood $O(x)$ of x such that

$$G_i(x) \subset G_i(x') + \frac{1}{2}\varepsilon_0, \tag{3.7}$$

whenever $x' \in O(x)$, where $G_i(x') + (1/2)\varepsilon_0 = \{z_i \in K_i : d_i(z_i, G_i(x')) < (1/2)\varepsilon_0\}$. Thus $y_i^0 + \varepsilon_0 \subset \operatorname{int} G_i(x) \subset G_i(x) \subset G_i(x') + (1/2)\varepsilon_0$ whenever $x' \in O(x)$, which implies that $y_i^0 + (1/2)\varepsilon_0 \subset G_i(x')$ whenever $x' \in O(x)$, that is, $y_i^0 \in \operatorname{int} G_i(x')$ whenever $x' \in O(x)$. This shows that the set $\{x \in K : y_i^0 \in \operatorname{int} G_i(x)\}$ is open. By $y_i^0 \in S_i(x)$, we have $F_i(u_i, x_i, y_i^0) \subset -\operatorname{int} C_i$, $\forall u_i \in T_i(x)$. By Lemma 3.1, there exists an open neighborhood $O(x)$ of x such that

$$\left\{F_i\left(u_i, x_i', y_i^0\right) : u_i \in T_i(x')\right\} \subset -\operatorname{int} C_i, \tag{3.8}$$

whenever $x' \in O(x)$, which implies that $O(x) \subset \{x \in K : y_i^0 \in S_i(x)\}$, that is, $\{x \in K : y_i^0 \in S_i(x)\}$ is open. Hence, for each $y_i \in P_i(x)$, the set $P_i^{-1}(y_i) = \{x \in K : y_i \in \operatorname{int} G_i(x) \cap S_i(x)\}$ is open.

By Lemma 2.4, $P_i|_{W_i}$ has a continuous selection $f_i : W_i \mapsto 2^{K_i}$.

Step 4. We prove that the SGIVQIP has a solution.

For each $i \in I$, define the set-valued map $H_i : K \mapsto 2^{K_i}$ by

$$H_i(x) = \begin{cases} f_i(x), & \text{if } x \in W_i, \\ G_i(x), & \text{if } x \in J_i. \end{cases} \tag{3.9}$$

Note that $H_i(x)$ is upper semicontinuous when $x \in \operatorname{int} J_i$ and $H_i(x)$ is upper semicontinuous when $x \in W_i$, and it is easy to verify that $H_i(x)$ is also upper semicontinuous when $x \in \partial J_i$, where ∂J_i denotes the boundary of J_i. Thus, $H_i(x)$ is upper semicontinuous with nonempty convex compact values. By [8, Theorem 7.1.15], the set-valued map $H : K \mapsto 2^K$ defined by $H(x) = \prod_{i \in I} H_i(x)$ is closed with nonempty convex values. By Kakutani-Fan-Glicksberg's fixed points theorem (see [9, pages 550]), H has a fixed point, that is, there exists $\overline{x} \in H(\overline{x})$. The condition (5) implies that for each $i \in I$, $\overline{x}_i \notin \operatorname{int} G_i(\overline{x}) \cap S_i(\overline{x})$, that is, $\overline{x}_i \neq f_i(\overline{x})$ for each $i \in I$. Thus we have that for each $i \in I$, $\overline{x}_i \in G_i(\overline{x})$, and

$$\forall y_i \in G_i(\overline{x}),\ \exists \overline{u}_i \in T_i(\overline{x}), \quad F_i(\overline{u}_i, \overline{x}_i, y_i) \not\subset -\operatorname{int} C_i. \tag{3.10}$$

The proof is finished.

If I is a singleton, we obtain the following existence result of solutions to the GIVQIP by Theorem 3.2.

Corollary 3.3. *Consider a GIVQIP $\{K, D, G, T, F\}$. Assume that*

(1) $G(\cdot)$ *is continuous on K with convex compact values and for each $x \in K$, $\operatorname{int} G(x) \neq \emptyset$;*

(2) $T(\cdot)$ *is upper semicontinuous on K with nonempty and compact values;*

(3) $F(\cdot,\cdot,\cdot)$ *is upper C-semicontinuous on $D \times K \times K$ with nonempty and compact values;*

(4) *for each $x \in K$ and each $u \in T(x)$, $F(u, x, \cdot)$ is C-convex or C-quasiconvex-like;*

(5) *for each $x \in K$ and each $u \in T(x)$, if $x \in \operatorname{int} G(x)$, then $F(u, x, x) \not\subset -\operatorname{int} C$.*

Then the GIVQIP has a solution, that is, there exists $\overline{x} \in K$ such that $\overline{x} \in G(\overline{x})$,

$$\forall y \in G(\overline{x}),\ \exists \overline{u} \in T(\overline{x}), \quad F(\overline{u}, \overline{x}, y) \not\subset -\operatorname{int} C. \tag{3.11}$$

Remark 3.4. Theorem 3.2, Corollary 3.3, and each corresponding result in literatures [1–6] do not include each other as a special case.

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