Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 649427, 7 pages
doi:10.1155/2009/649427

Research Article

An Exponential Inequality for Negatively Associated Random Variables

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Received 15 October 2008; Revised 16 February 2009; Accepted 7 May 2009

Recommended by Jewgeni Dshalalow

An exponential inequality is established for identically distributed negatively associated random variables which have the finite Laplace transforms. The inequality improves the results of Kim and Kim (2007), Nooghabi and Azarnoosh (2009), and Xing et al. (2009). We also obtain the convergence rate $O(1)n^{1/2}(\log n)^{-1/2}$ for the strong law of large numbers, which improves the corresponding ones of Kim and Kim, Nooghabi and Azarnoosh, and Xing et al.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The concept of negatively associated random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2]. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\left(f_1(X_i, i \in A), f_2\left(X_j, j \in B\right)\right) \leq 0, \tag{1.1}$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated. As pointed out and proved by Joag-Dev and Proschan [2], a number of well-known multivariate distributions possess the negative association property, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks.

The exponential inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of convergence rate for the strong law of large numbers. The counterpart of the negative association is positive association. The concept of positively associated random variables was introduced by Esary et al. [3]. The exponential inequalities for positively associated random variables were obtained by Devroye [4], Ioannides and Roussas [5], Oliveira [6], Sung [7], Xing and Yang [8], and Xing et al. [9]. On the other hand, Kim and Kim [10], Nooghabi and Azarnoosh [11], and Xing et al. [12] obtained exponential inequalities for negatively associated random variables.

In this paper, we establish an exponential inequality for identically distributed negatively associated random variables by using truncation method (not using a block decomposition of the sums). Our result improves those of Kim and Kim [10], Nooghabi and Azarnoosh [11], and Xing et al. [12]. We also obtain the convergence rate $O(1)n^{1/2}(\log n)^{-1/2}$ for the strong law of large numbers.

2. Preliminary lemmas

To prove our main results, the following lemmas are needed. We start with a well known lemma. The constant C_p can be taken as that of Marcinkiewicz-Zygmund (see Shao [13]).

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with mean zero and finite pth moments, where $1 < p \leq 2$. Then there exists a positive constant C_p depending only on p such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_p \sum_{i=1}^{n} E|X_i|^p. \tag{2.1}$$

If $p = 2$, then it is possible to take $C_2 = 1$.

The following lemma is due to Joag-Dev and Proschan [2]. It is still valid for any $t \leq 0$.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables. Then for any $t > 0$,*

$$E \exp\left\{t\sum_{i=1}^{n} X_i\right\} \leq \prod_{i=1}^{n} Ee^{tX_i}. \tag{2.2}$$

The following lemma plays an essential role in our main results.

Lemma 2.3. *Let $X_1, \ldots, X_n$ be negatively associated mean zero random variables such that*

$$|X_i| \leq d_i, \quad 1 \leq i \leq n, \tag{2.3}$$

for a sequence of positive constants $d_1, \ldots, d_n$. Then for any $\lambda > 0$,

$$E \exp\left\{\lambda\sum_{i=1}^{n} X_i\right\} \leq \exp\left\{\frac{\lambda^2}{2}\sum_{i=1}^{n} e^{\lambda d_i} EX_i^2\right\}. \tag{2.4}$$

Proof. From the inequality $e^x \le 1 + x + (x^2/2)e^{|x|}$ for all $x \in R$, we have

$$\begin{aligned} Ee^{\lambda X_i} &\le 1 + \lambda EX_i + \frac{\lambda^2}{2}E\left(X_i^2 e^{\lambda|X_i|}\right) \\ &= 1 + \frac{\lambda^2}{2}E\left(X_i^2 e^{\lambda|X_i|}\right) \text{(since the } X_i \text{ have mean zero)} \\ &\le 1 + \frac{\lambda^2}{2}e^{\lambda d_i}EX_i^2 \\ &\le \exp\left\{\frac{\lambda^2}{2}e^{\lambda d_i}EX_i^2\right\}, \end{aligned} \tag{2.5}$$

since $1 + x \le e^x$ for all $x \in R$. It follows by Lemma 2.2 that

$$E\ \exp\left\{\lambda\sum_{i=1}^{n}X_i\right\} \le \prod_{i=1}^{n}Ee^{\lambda X_i} \le \prod_{i=1}^{n}\exp\left\{\frac{\lambda^2}{2}e^{\lambda d_i}EX_i^2\right\} = \exp\left\{\frac{\lambda^2}{2}\sum_{i=1}^{n}e^{\lambda d_i}EX_i^2\right\}. \tag{2.6}$$

□

3. Main results

Let $\{X_n, n \ge 1\}$ be a sequence of random variables and $\{c_n, n \ge 1\}$ be a sequence of positive real numbers. Define for $1 \le i \le n, n \ge 1$,

$$\begin{aligned} X_{1,i,n} &= -c_n I(X_i < -c_n) + X_i I(-c_n \le X_i \le c_n) + c_n I(X_i > c_n), \\ X_{2,i,n} &= (X_i - c_n)I(X_i > c_n), \\ X_{3,i,n} &= (X_i + c_n)I(X_i < -c_n). \end{aligned} \tag{3.1}$$

Note that $X_{1,i,n} + X_{2,i,n} + X_{3,i,n} = X_i$ for $1 \le i \le n, n \ge 1$. For each fixed $n \ge 1, X_{1,1,n}, \dots, X_{1,n,n}$ are bounded by c_n. If $\{X_n, n \ge 1\}$ are negatively associated random variables, then $\{X_{q,i,n}, 1 \le i \le n\}, q = 1, 2, 3$, are also negatively associated random variables, since $\{X_{q,i,n}, 1 \le i \le n\}$ are monotone transformations of $\{X_i, 1 \le i \le n\}$.

Lemma 3.1. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively associated random variables. Let $X_{1,i,n}, 1 \le i \le n, n \ge 1$ be as in* (3.1). *Then for any $\lambda > 0$,*

$$E\ \exp\left\{\lambda\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right\} \le \exp\left\{\frac{\lambda^2 n}{2}e^{2\lambda c_n}E|X_1|^2\right\}. \tag{3.2}$$

Proof. Noting that $|X_{1,i,n} - EX_{1,i,n}| \le 2c_n$, we have by Lemma 2.3 that

$$\begin{aligned} E\exp\left\{\lambda\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right\} &\le \exp\left\{\frac{\lambda^2}{2}\sum_{i=1}^{n}e^{2\lambda c_n}\operatorname{Var}(X_{1,i,n})\right\} \\ &\le \exp\left\{\frac{\lambda^2 n}{2}e^{2\lambda c_n}E|X_{1,1,n}|^2\right\} \\ &\le \exp\left\{\frac{\lambda^2 n}{2}e^{2\lambda c_n}E|X_1|^2\right\}. \end{aligned} \tag{3.3}$$

□

The following lemma gives an exponential inequality for the sum of bounded terms.

Lemma 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively associated random variables. Let $X_{1,i,n}, 1 \le i \le n, n \ge 1$ be as in* (3.1). *Then for any $\epsilon > 0$ such that $\epsilon \le eE|X_1|^2/(2c_n)$,*

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| > \epsilon\right) \le 2\exp\left\{-\frac{n\epsilon^2}{2eE|X_1|^2}\right\}. \tag{3.4}$$

Proof. By Markov's inequality and Lemma 3.1, we have that for any $\lambda > 0$

$$\begin{aligned} P\left(\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n}) > \epsilon\right) &= P\left(\exp\left\{\lambda\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right\} > e^{\lambda n\epsilon}\right) \\ &\le e^{-\lambda n\epsilon}E\exp\left\{\lambda\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right\} \\ &\le \exp\left\{-\lambda n\epsilon + \frac{\lambda^2 n}{2}e^{2\lambda c_n}E|X_1|^2\right\}. \end{aligned} \tag{3.5}$$

Putting $\lambda = \epsilon/(eE|X_1|^2)$, note that $2\lambda c_n \le 1$, we get

$$P\left(\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n}) > \epsilon\right) \le \exp\left\{-\frac{n\epsilon^2}{2eE|X_1|^2}\right\}. \tag{3.6}$$

Since $\{-X_n, n \ge 1\}$ are also negatively associated random variables, we can replace $X_{1,i,n}$ by $-X_{1,i,n}$ in the above statement. That is,

$$P\left(-\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n}) > \epsilon\right) \le \exp\left\{-\frac{n\epsilon^2}{2eE|X_1|^2}\right\}. \tag{3.7}$$

Observing that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{1,i,n}-EX_{1,i,n})\right|>\epsilon\right)=P\left(\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n}-EX_{1,i,n})>\epsilon\right)+P\left(-\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n}-EX_{1,i,n})>\epsilon\right), \tag{3.8}$$

the result follows by (3.6) and (3.7). □

Remark 3.3. From [14, Lemma 3.5] in Yang, it can be obtained an upper bound $2\exp(-n\epsilon^2/(4E|X_1|^2+2eE|X_1|^2)$, which is greater than our upper bound.

The following lemma gives an exponential inequality for the sum of unbounded terms.

Lemma 3.4. *Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed negatively associated random variables with $Ee^{\delta|X_1|}<\infty$ for some $\delta>0$. Let $X_{q,i,n}, 1\leq i\leq n, n\geq 1, q=2,3$, be as in* (3.1). *Then, for any $\epsilon>0$, the following statements hold:*

(i) $P(1/n|\sum_{i=1}^{n}(X_{2,i,n}-EX_{2,i,n})|>\epsilon)\leq 2\delta^{-2}\epsilon^{-2}n^{-1}Ee^{\delta|X_1|}e^{-\delta c_n}$.

(ii) $P(1/n|\sum_{i=1}^{n}(X_{3,i,n}-EX_{3,i,n})|>\epsilon)\leq 2\delta^{-2}\epsilon^{-2}n^{-1}Ee^{\delta|X_1|}e^{-\delta c_n}$.

Proof. (i) By Markov's inequality and Lemma 2.1, we get

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{2,i,n}-EX_{2,i,n})\right|\right)\leq\frac{1}{\epsilon^2n^2}E\left|\sum_{i=1}^{n}(X_{2,i,n}-EX_{2,i,n})\right|^2\leq\frac{\operatorname{Var}(X_{2,1,n})}{\epsilon^2n}\leq\frac{E|X_{2,1,n}|^2}{\epsilon^2n}. \tag{3.9}$$

The rest of the proof is similar to that of [12, Lemma 4.1] in Xing et al. and is omitted.

(ii) The proof is similar to that of (i) and is omitted. □

Now we state and prove one of our main results.

Theorem 3.5. *Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed negatively associated random variables with $Ee^{\delta|X_1|}<\infty$ for some $\delta>0$. Let $\epsilon_n=\sqrt{2\delta eE|X_1|^2c_n/n}$, where $\{c_n, n\geq 1\}$ is a sequence of positive numbers such that*

$$0<c_n\leq\left(\frac{eE|X_1|^2n}{8\delta}\right)^{1/3}. \tag{3.10}$$

Then

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i-EX_i)\right|>3\epsilon_n\right)\leq 2\left(1+\frac{Ee^{\delta|X_1|}}{\delta^3eE|X_1|^2c_n}\right)e^{-\delta c_n}. \tag{3.11}$$

Proof. Note that $2\epsilon_n c_n \le eE|X_1|^2$ and $n\epsilon_n^2/(2eE|X_1|^2) = \delta c_n$. It follows by Lemmas 3.2 and 3.4 that

$$\begin{aligned}
&P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > 3\epsilon_n\right) \\
&\le \left[P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| > \epsilon_n\right) + P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{2,i,n} - EX_{2,i,n})\right| > \epsilon_n\right)\right. \\
&\quad \left. + P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{3,i,n} - EX_{3,i,n})\right| > \epsilon_n\right)\right] \\
&\le 2\exp\left\{-\frac{n\epsilon_n^2}{2eE|X_1|^2}\right\} + \frac{4Ee^{\delta|X_1|}}{\delta^2\epsilon_n^2 n}e^{-\delta c_n} \\
&= 2\left(1 + \frac{Ee^{\delta|X_1|}}{\delta^3 eE|X_1|^2 c_n}\right)e^{-\delta c_n}
\end{aligned} \tag{3.12}$$

□

In Theorem 3.5, the condition on c_n is (3.10). But, Kim and Kim [10], Nooghabi and Azarnoosh [11], and Xing et al. [12] used c_n as only $\log n$. We give some examples satisfying the condition (3.10) of Theorem 3.5.

Example 3.6. Let $c_n = (\log n)^3 p_n$, where $1 \le p_n = o(n^{1/3}/(\log n)^3)$. Then $c_n \to \infty$ as $n \to \infty$ and so the upper bound of (3.11) is $O(1)e^{-\delta p_n(\log n)^3}$. The corresponding upper bound $O(1)(1+n^2/p_n(\log n)^3)n^{-\delta}$ was obtained by Kim and Kim [10] and Nooghabi and Azarnoosh [11]. Since our upper bound is much lower than it, our result improves the theorem in Kim and Kim [10] and Nooghabi and Azarnoosh [11, Theorem 5.1].

Example 3.7. Let $c_n = (\log n)^3$. By Example 3.6 with $p_n = 1$, the upper bound of (3.11) is $O(1)e^{-\delta(\log n)^3}$. The corresponding upper bound $O(1)n^{-\delta}$ was obtained by Xing et al. [12]. Hence our result improves Xing et al. [12, Theorem 5.1].

By choosing $c_n = \log n$ and $\delta > 1$ in Theorem 3.5, we obtain the following result.

Theorem 3.8. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively associated random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 1$. Let $\epsilon_n = \sqrt{2\delta eE|X_1|^2 \log n/n}$. Then*

$$\sum_{n=1}^{\infty} P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > 3\epsilon_n\right) < \infty. \tag{3.13}$$

Remark 3.9. By the Borel-Cantelli lemma, $\sum_{i=1}^{n}(X_i - EX_i)/n$ converges almost surely with rate $(3\epsilon_n)^{-1} = O(1)n^{1/2}(\log n)^{-1/2}$. The convergence rate is faster than the rate $O(1)n^{1/2}(\log n)^{-3/2}$ obtained by Xing et al. [12].

The following example shows that the convergence rate $n^{1/2}(\log n)^{-1/2}$ is unattainable in Theorem 3.8.

Example 3.10. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables. Then $\{X_n\}$ are negatively associated random variables with $Ee^{\delta|X_1|} < \infty$ for any δ. Set $Z =: \sum_{i=1}^{n} X_i/\sqrt{n}$. Then Z is also $N(0,1)$. It is well known that $P(Z > \epsilon) \geq 1/\sqrt{2\pi}(1/\epsilon - 1/\epsilon^3)e^{-\epsilon^2/2}$ (see Feller [15, page 175]). Thus we have that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| > \sqrt{\frac{\log n}{n}}\right) = 2P\left(Z > \sqrt{\log n}\right) \geq \sqrt{\frac{2}{\pi}}\frac{(\log n - 1)}{\log n\sqrt{n \log n}}, \tag{3.14}$$

which implies that the series $\sum_{n=1}^{\infty} P(1/n|\sum_{i=1}^{n} X_i| > \sqrt{\log n/n})$ diverges.

Acknowledgments

The author would like to thank the referees for the helpful comments and suggestions that considerably improved the presentation of this paper. This work was supported by the Korea Science and Engineering Foundation(KOSEF) Grant funded by the Korea government(MOST) (no. R01-2007-000-20053-0).

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