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Research Article

Cauchy Means of the Popoviciu Type

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We discuss log-convexity for the differences of the Popoviciu inequalities and introduce some mean value theorems and related results. Also we give the Cauchy means of the Popoviciu type and we show that these means are monotonic.

1. Introduction and Preliminaries

Let $f(x)$ and $p(x)$ be two positive real valued functions with $\int_a^b p(x)dx = 1$, then from theory of convex means (cf. [1–3]), the well-known Jensen inequality gives that for $t < 0$ or $t > 1$,

$$\int_a^b p(x)(f(x))^t dx \geq \left(\int_a^b p(x)f(x)dx\right)^t, \tag{1.1}$$

and vise versa for $0 < t < 1$. In [4], Simic has considered the difference

$$D_s = D_s(a,b,f,p) = \int_a^b p(x)(f(x))^s dx - \left(\int_a^b p(x)f(x)dx\right)^s. \tag{1.2}$$

The following result was given in [4] (see also [5]).

Theorem 1.1. *Let $f(x)$, $p(x)$ be nonnegative and integrable functions for $x \in (a,b)$, with $\int_a^b p(x)dx = 1$, then for $0 < r < s < t$; $r, s, t \neq 1$, one has*

$$\left(\frac{D_s}{s(s-1)}\right)^{t-r} \leq \left(\frac{D_r}{r(r-1)}\right)^{t-s} \left(\frac{D_t}{t(t-1)}\right)^{s-r}. \tag{1.3}$$

Remark 1.2. For extension of Theorem 1.1 see (cf. [4]).

Popoviciu ([6–8], [9, pages 214-215]) has proved the following results.

Theorem 1.3. *Let $\phi : [a,b] \to \mathbb{R}$ be convex and $f : [0,1] \to \mathbb{R}$ be continuous, increasing, and convex such that $a \leq f(x) \leq b$ for $x \epsilon [0,1]$. Then*

$$\int_0^1 \phi(f(x))dx \leq \frac{b+a-2\tilde{f}}{b-a}\phi(a) + \frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b \phi(x)dx, \tag{1.4}$$

where

$$\tilde{f} = \int_0^1 f(x)dx. \tag{1.5}$$

If ϕ is strictly convex, then the equality in (1.4) *holds if and only if*

$$f(x) = a + (b-a)\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)}, \quad \text{where } \lambda = \frac{b+a-2\tilde{f}}{b-a}. \tag{1.6}$$

Theorem 1.4. *Let $\phi : [a,b] \to \mathbb{R}$ be continuous and convex, and let $f : [0,1] \to \mathbb{R}$ be convex of order $1,\ldots,n+1$ such that $a \leq f(x) \leq b$ for $x \epsilon [0,1]$.*

Then

$$\int_0^1 \phi(f(x))dx \leq \int_0^1 \phi(U_j(\tilde{f},x))dx \quad \text{for } \frac{ja+b}{j+1} \leq \tilde{f} \leq \frac{(j-1)a+b}{j}, \; 2 \leq j \leq n, \tag{1.7}$$

$$\int_0^1 \phi(f(x))dx \leq V(\tilde{f}) \quad \text{for } a \leq \tilde{f} \leq \frac{na+b}{n+1}, \tag{1.8}$$

where

$$U_j(t,x) = a + j(j+1)\left(\left(t - \frac{ja+b}{j+1}\right) + \left(\frac{(j-1)a+b}{j} - t\right)x\right)x^{j-1}, \tag{1.9}$$

$$V(x) = \frac{b+na-(n+1)x}{b-a}\phi(a) + \frac{(n+1)(x-a)}{n(b-a)^{(n+1)/n}}\int_a^b \frac{\phi(x)dx}{(x-a)^{(n-1)/n}}. \tag{1.10}$$

If ϕ is strictly convex, then equality in (1.7) *holds if and only if*

$$f(x) = U_j(\tilde{f}, x); \tag{1.11}$$

and equality in (1.8) *holds if*

$$f(x) = a + (b-a)\left(\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)}\right)^n, \quad \text{where } \lambda = \frac{b+na-(n+1)\tilde{f}}{b-a}. \tag{1.12}$$

With the help of the following useful lemmas we prove our results.

Lemma 1.5. *Define the function*

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0, 1; \\ -\log x, & s = 0; \\ x \log x, & s = 1. \end{cases} \tag{1.13}$$

Then $\varphi_s''(x) = x^{s-2}$, that is, $\varphi_s(x)$ is convex for $x > 0$.

The following lemma is equivalent to definition of convex function (see [9, page 2]).

Lemma 1.6. *If ϕ is a convex function on I for all $s_1, s_2, s_3 \in I$ for which $s_1 < s_2 < s_3$, the following is valid*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0. \tag{1.14}$$

We quote here another useful lemma from log-convexity theory (cf. [4]).

Lemma 1.7. *A positive function f is log-convex in the Jensen-sense on an open interval I, that is, for each $s, t \in I$,*

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right), \tag{1.15}$$

if and only if the relation

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0, \tag{1.16}$$

holds for each real u, w and $s, t \in I$.

The following lemma given in [10] gives the relation between Beta function β and Hypergeometric function F.

Lemma 1.8. *Suppose $a, b, c, \alpha, \gamma \in \mathbb{R}$ are such that $a + c > b > 0$ and $0 < \alpha < 2\gamma$, β and F are Beta and Hypergeometric functions, respectively. Then*

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a (1+\gamma x)^c} dx = \gamma^{-b} \beta(b, a+c-b) F\left(\begin{matrix} a, b \\ a+c \end{matrix} \,\middle|\, \frac{\gamma - \alpha}{\gamma} \right). \tag{1.17}$$

The paper is organized in the following way. After this introduction, in the second section we discuss the log-convexity of differences of the Popoviciu inequalities (1.4), (1.7), and (1.8). In the third section we introduce some mean value theorems and the Cauchy means of the Popoviciu-type and discuss its monotonicity.

2. Main Results

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}$ be continuous, increasing, and convex such that $0 < a \le f(x) \le b$ for $x \in [0,1]$, and let $\widetilde{f}$ be defined in* (1.5) *and*

$$\Omega_s(f) = \begin{cases} \dfrac{1}{s(s-1)} \left[\dfrac{b+a-2\widetilde{f}}{b-a} a^s + \dfrac{2(\widetilde{f}-a)}{(b-a)^2 (s+1)} \left(b^{s+1} - a^{s+1} \right) - \displaystyle\int_0^1 (f(x))^s dx \right], & s \ne 0, 1; \\ \dfrac{2(\widetilde{f}-a)}{b-a} + \displaystyle\int_0^1 \log(f(x)) dx - \dfrac{b+a-2\widetilde{f}}{b-a} \log a - \dfrac{2(\widetilde{f}-a)}{(b-a)^2} (b \log b - a \log a), & s = 0; \\ \dfrac{b+a-2\widetilde{f}}{b-a} a \log a + \dfrac{\widetilde{f}-a}{(b-a)^2} \left(b^2 \log b - a^2 \log a \right) & \\ \quad - \dfrac{(\widetilde{f}-a)(b+a)}{2(b-a)} - \displaystyle\int_0^1 f(x) \log(f(x)) dx, & s = 1, \end{cases} \tag{2.1}$$

and let $\Omega_s(f)$ be positive.

One has that $\Omega_s(f)$ is log-convex and the following inequality holds for $-\infty < r < s < t < +\infty$,

$$\Omega_s^{t-r}(f) \le \Omega_r^{t-s}(f) \Omega_t^{s-r}(f). \tag{2.2}$$

Proof. Consider the function defined by

$$\omega(x) = u^2 \varphi_s(x) + 2uw\varphi_r(x) + w^2 \varphi_t(x), \tag{2.3}$$

where $r = (s+t)/2$, φ_s is defined by (1.13) and $u, w \in \mathbb{R}$. We have

$$\begin{aligned} \omega''(x) &= u^2 x^{s-2} + 2uwx^{r-2} + w^2 x^{t-2} \\ &= \left(ux^{s/2-1} + wx^{t/2-1} \right)^2 > 0, \quad x > 0. \end{aligned} \tag{2.4}$$

Therefore, $\omega(x)$ is convex for $x > 0$. Using Theorem 1.3,

$$\begin{aligned}
&\frac{b+a-2\tilde{f}}{b-a}\left(u^2\varphi_s(a)+2uw\varphi_r(a)+w^2\varphi_t(a)\right)\\
&\qquad+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b\left(u^2\varphi_s(x)+2uw\varphi_r(x)+w^2\varphi_t(x)\right)dx\\
&\qquad\geq\int_0^1\left(u^2\varphi_s(f(x))+2uw\varphi_r(f(x))+w^2\varphi_t(f(x))\right)dx,\\
&u^2\left[\frac{b+a-2\tilde{f}}{b-a}\varphi_s(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b\varphi_s(x)dx-\int_0^1\varphi_s(f(x))dx\right]\\
&\qquad+2uw\left[\frac{b+a-2\tilde{f}}{b-a}\varphi_r(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b\varphi_r(x)dx-\int_0^1\varphi_r(f(x))dx\right]\\
&\qquad+w^2\left[\frac{b+a-2\tilde{f}}{b-a}\varphi_t(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b\varphi_t(x)dx-\int_0^1\varphi_t(f(x))dx\right]\geq 0,
\end{aligned}\tag{2.5}$$

since

$$\Omega_s(f)=\frac{b+a-2\tilde{f}}{b-a}\varphi_s(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b\varphi_s(x)dx-\int_0^1\varphi_s(f(x))dx,\quad \tilde{f}=\int_0^1 f(x)dx,\tag{2.6}$$

we have

$$u^2\Omega_s(f)+2uw\Omega_r(f)+w^2\Omega_t(f)\geq 0.\tag{2.7}$$

By Lemma 1.7, we have

$$\Omega_s(f)\Omega_t(f)\geq\Omega_r^2(f)=\Omega_{(s+t)/2}^2(f),\tag{2.8}$$

that is $\Omega_s(f)$ is log-convex in the Jensen-sense for $s\in\mathbb{R}$. Since

$$\lim_{s\to 0}\Omega_s(f)=\Omega_0(f),\qquad \lim_{s\to 1}\Omega_s(f)=\Omega_1(f).\tag{2.9}$$

This implies $\Omega_s(f)$ is continuous, therefore it is log-convex.

Since $\Omega_s(f)$ is log-convex, that is, $\log\Omega_s(f)$ is convex, therefore by Lemma 1.6 for $-\infty<r<s<t<+\infty$ and taking $\phi_s=\log\Omega_s$, we get

$$\log\Omega_s^{t-r}(f)\leq\log\Omega_r^{t-s}(f)+\log\Omega_t^{s-r}(f),\tag{2.10}$$

which is equivalent to (2.2). □

Theorem 2.2. *Let f, $\Omega_s(f)$ be defined in Theorem 2.1 and let $t,\ s,\ u,\ v$ be real numbers such that $s \le u$, $t \le v$, $s \ne t$, $u \ne v$, one has*

$$\left(\frac{\Omega_t(f)}{\Omega_s(f)}\right)^{1/(t-s)} \le \left(\frac{\Omega_v(f)}{\Omega_u(f)}\right)^{1/(v-u)}. \tag{2.11}$$

Proof. In (cf. [9, page 2]), we have the following result for convex function f with $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{2.12}$$

Since by Theorem 2.1, $\Omega_s(f)$ is log-convex, we can set in (2.12):
$f(x) = \log \Omega_x$ and $x_1 = s$, $x_2 = t$, $y_1 = u$, $y_2 = v$. We get

$$\begin{gathered}\frac{\log \Omega_t(f) - \log \Omega_s(f)}{t-s} \le \frac{\log \Omega_v(f) - \log \Omega_u(f)}{v-u}, \\ \log\left(\frac{\Omega_t(f)}{\Omega_s(f)}\right)^{1/(t-s)} \le \log\left(\frac{\Omega_v(f)}{\Omega_u(f)}\right)^{1/(v-u)},\end{gathered} \tag{2.13}$$

and after applying exponential function, we get (2.11). □

Theorem 2.3. *Let $f : [0,1] \to \mathbb{R}$ be convex of order $1, \ldots, n+1$ such that $0 < a \le f(x) \le b$ for $x \in [0,1]$, and let $\tilde{f}$ be defined in (1.5) and*

$$\Lambda_s(f) = \int_0^1 \varphi_s(U_j(\tilde{f}, x))dx - \int_0^1 \varphi_s(f(x))dx, \tag{2.14}$$

where

$$U_j(\tilde{f}, x) = a + j(j+1)\left(\left(\tilde{f} - \frac{ja+b}{j+1}\right) + \left(\frac{(j-1)a+b}{j} - \tilde{f}\right)x\right)x^{j-1}, \tag{2.15}$$

for

$$\frac{ja+b}{j+1} \le \tilde{f} \le \frac{(j-1)a+b}{j}, \quad 2 \le j \le n, \tag{2.16}$$

and let $\Lambda_s(f)$ be positive.

One has that $\Lambda_s(f)$ is log-convex and the following inequality holds for $-\infty < r < s < t < +\infty$,

$$\Lambda_s^{t-r}(f) \le \Lambda_r^{t-s}(f)\Lambda_t^{s-r}(f). \tag{2.17}$$

Proof. As in the proof of Theorem 2.1, we use Theorem 1.4 instead of Theorem 1.3. □

Theorem 2.4. *Let $f, \Lambda_s(f)$ be defined in Theorem 2.3 and t, s, u, v be real numbers such that $s \le u$, $t \le v$, $s \ne t$, $u \ne v$, one has*

$$\left(\frac{\Lambda_t(f)}{\Lambda_s(f)}\right)^{1/(t-s)} \le \left(\frac{\Lambda_v(f)}{\Lambda_u(f)}\right)^{1/(v-u)}. \tag{2.18}$$

Proof. Similar to the proof of Theorem 2.2. □

Lemma 2.5. *Let $f : [0,1] \to \mathbb{R}$ be convex of order $1,\ldots,n+1$ such that $0 < a \le f(x) \le b$ for $x \in [0,1]$, $\tilde{f}$ be defined in* (1.5) *and V be defined in* (1.10), *and let β and F are Beta and Hypergeometric functions respectively, and*

$$\Gamma_s(f) = V(\tilde{f}) - \int_0^1 \varphi_s(f(x))dx. \tag{2.19}$$

Then

$$\Gamma_s(f) = \begin{cases} \dfrac{1}{s(s-1)}\left[\dfrac{b+na-(n+1)\tilde{f}}{b-a}a^s + \dfrac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}a^{s+1/n}\left(\dfrac{b-a}{b}\right)^{1/n}\right. \\ \qquad \left. \times\beta\left(\dfrac{1}{n},1\right)F\left(\begin{matrix} s+\dfrac{1}{n}+1, \dfrac{1}{n} \\ \dfrac{1}{n}+1\end{matrix}\middle|\,\dfrac{b-a}{b}\right) - \displaystyle\int_0^1 (f(t))^s dt\right], & s \ne 0,1; \\ \dfrac{b+na-(n+1)\tilde{f}}{b-a}(-\log a) - \dfrac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}} \\ \times\left[n\log b(b-a)^{1/n} - na^{1/n}\left(\dfrac{b-a}{b}\right)^{1/n+1}\beta\left(\dfrac{1}{n}+1,1\right)F\left(\begin{matrix}\dfrac{1}{n}+1, \dfrac{1}{n}+1 \\ \dfrac{1}{n}+2\end{matrix}\middle|\,\dfrac{b-a}{b}\right)\right] \\ +\displaystyle\int_0^1 \log(f(t))dt, & s = 0; \\ \dfrac{b+na-(n+1)\tilde{f}}{b-a}a\log a + \dfrac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}} \\ \times\left[na(b-a)^{1/n}\log b + \dfrac{n}{n+1}\log b(b-a)^{1/n+1} - a^{1/n}\left(\dfrac{b-a}{b}\right)^{1/n+1}\beta\left(\dfrac{1}{n}+1,1\right)\right. \\ \qquad \left.\times F\left(\begin{matrix}\dfrac{1}{n}+1, \dfrac{1}{n}+1 \\ \dfrac{1}{n}+2\end{matrix}\middle|\,\dfrac{b-a}{b}\right)\left[\dfrac{n}{n+1}+na\right]\right] - \displaystyle\int_0^1 f(t)\log(f(t))dt, & s = 1, \end{cases} \tag{2.20}$$

for

$$a \leq \tilde{f} \leq \frac{na+b}{n+1}. \tag{2.21}$$

Proof. First, we solve these three integrals

$$I_1 = \int_a^b \frac{t^s}{(t-a)^{(n-1)/n}} dt, \qquad I_2 = \int_a^b \frac{\log t}{(t-a)^{(n-1)/n}} dt, \qquad I_3 = \int_a^b \frac{t \log t}{(t-a)^{(n-1)/n}} dt. \tag{2.22}$$

Take

$$I_1 = \int_a^b \frac{t^s}{(t-a)^{(n-1)/n}} dt. \tag{2.23}$$

Substitute

$$t = \frac{a+bx}{1+x}, \qquad dt = \frac{b-a}{(1+x)^2} dx, \tag{A}$$

and limits, when $t \to a$ then $x \to 0$, when $t \to b$ then $x \to \infty$. So,

$$\begin{aligned} I_1 &= \int_a^b \frac{t^s}{(t-a)^{(n-1)/n}} dt = \int_0^\infty \frac{((a+bx)/(1+x))^s}{((a+bx)/(1+x)-a)^{1-1/n}} \cdot \frac{b-a}{(1+x)^2} dx \\ &= (b-a)^{1/n} \int_0^\infty \frac{(a+bx)^s (1+x)^{1-1/n}}{(1+x)^{s+2} x^{1-1/n}} dx, \\ I_1 &= (b-a)^{1/n} a^s \int_0^\infty \frac{x^{1/n-1}}{(1+x)^{s+1/n+1} (1+(b/a)x)^{-s}} dx. \end{aligned} \tag{2.24}$$

By using Lemma 1.8 with $a = s + 1/n + 1$, $b = 1/n$, $c = -s$, $\alpha = 1$, $\gamma = b/a$ such that $1/n + 1 > 1/n > 0$ and $0 < 1 < 2b/a$, we get

$$I_1 = a^{s+1/n} (b-a)^{1/n} \beta\left(\frac{1}{n}, 1\right) F\left(\begin{array}{c} s + \dfrac{1}{n} + 1, \dfrac{1}{n} \\ \dfrac{1}{n} + 1 \end{array} \middle| \, \frac{b-a}{b} \right). \tag{2.25}$$

Take second integral

$$I_2 = \int_a^b \frac{\log t}{(t-a)^{(n-1)/n}} dt, \tag{2.26}$$

using integration by parts, we have

$$I_2 = n\log b(b-a)^{1/n} - n\int_a^b t^{-1}(t-a)^{1/n}dt. \tag{2.27}$$

Let

$$I_4 = \int_a^b t^{-1}(t-a)^{1/n}dt. \tag{2.28}$$

By using same substitution (A) as above, we get

$$I_4 = \int_0^\infty \frac{1+x}{a+bx}\cdot\left(\frac{a+bx}{1+x}-a\right)^{1/n}\cdot\frac{b-a}{(1+x)^2}dx = \frac{(b-a)^{1/n+1}}{a}\int_0^\infty \frac{x^{(1/n+1)-1}}{(1+x)^{1/n+1}(1+(b/a)x)}dx. \tag{2.29}$$

By using Lemma 1.8 with $a = 1/n+1$, $b = 1/n+1$, $c = 1$, $\alpha = 1$, $\gamma = b/a$ such that $1/n+2 > 1/n+1 > 0$ and $0 < 1 < 2b/a$, we get

$$I_2 = n\log b(b-a)^{1/n} - na^{1/n}\left(\frac{b-a}{b}\right)^{1/n+1}\beta\left(\frac{1}{n}+1,1\right)F\left(\begin{array}{c|c} \frac{1}{n}+1,\frac{1}{n}+1 & \frac{b-a}{b} \\ \frac{1}{n}+2 & \end{array}\right). \tag{2.30}$$

Now, take third integral

$$I_3 = \int_a^b \frac{t\log t}{(t-a)^{(n-1)/n}}dt. \tag{2.31}$$

Using integration by parts, we get

$$I_3 = na(b-a)^{1/n}\log b + \frac{n}{n+1}\log b(b-a)^{1/n+1} - \frac{n}{n+1}\int_a^b \frac{(t-a)^{1/n+1}}{t}dt - na\int_a^b \frac{(t-a)^{1/n}}{t}dt. \tag{2.32}$$

Let

$$I_5 = \int_a^b \frac{(t-a)^{1/n+1}}{t}dt. \tag{2.33}$$

By using same substitution (A) as above, we get

$$I_5 = \int_0^\infty \frac{((a+bx)/(1+x)-a)^{1/n+1}}{((a+bx)/(1+x))}\cdot\frac{b-a}{(1+x)^2}dx = \frac{(b-a)^{1/n+1}}{a}\int_0^\infty \frac{x^{(1/n+1)-1}}{(1+x)^{1/n+1}(1+(b/a)x)}dx. \tag{2.34}$$

By using Lemma 1.8 with $a = 1/n + 1$, $b = 1/n + 1$, $c = 1$, $\alpha = 1$, $\gamma = b/a$ such that $1/n + 2 > 1/n + 1 > 0$ and $0 < 1 < 2b/a$, we get

$$I_5 = a^{1/n}\left(\frac{b-a}{b}\right)^{1/n+1}\beta\left(\frac{1}{n}+1,1\right)F\left(\begin{array}{c} \frac{1}{n}+1, \frac{1}{n}+1 \\ \frac{1}{n}+2 \end{array}\middle| \frac{b-a}{b}\right). \tag{2.35}$$

Let

$$I_6 = \int_a^b \frac{(t-a)^{1/n}}{t}dt. \tag{2.36}$$

By using same substitution (A) as above, we have

$$I_6 = \int_0^\infty \frac{((a+bx)/(1+x)-a)^{1/n}}{((a+bx)/(1+x))}\cdot\frac{b-a}{(1+x)^2}dx = \frac{(b-a)^{1/n+1}}{a}\int_0^\infty \frac{x^{(1/n+1)-1}}{(1+x)^{1/n+1}(1+(b/a)x)}dx. \tag{2.37}$$

By using Lemma 1.8 with $a = 1/n + 1$, $b = 1/n + 1$, $c = 1$, $\alpha = 1$, $\gamma = b/a$ such that $1/n + 2 > 1/n + 1 > 0$ and $0 < 1 < 2b/a$, we get

$$I_6 = a^{1/n}\left(\frac{b-a}{a}\right)^{1/n+1}\beta\left(\frac{1}{n}+1,1\right)F\left(\begin{array}{c} \frac{1}{n}+1, \frac{1}{n}+1 \\ \frac{1}{n}+2 \end{array}\middle| \frac{b-a}{b}\right). \tag{2.38}$$

Then

$$\begin{aligned} I_3 = na(b-a)^{1/n}\log b + \frac{n}{n+1}\log b(b-a)^{1/n+1} - a^{1/n}\left(\frac{b-a}{a}\right)^{1/n+1} \\ \times\beta\left(\frac{1}{n}+1,1\right)F\left(\frac{1}{n}+1,\frac{1}{n}+1,\frac{1}{n}+2 \,\middle|\, \frac{b-a}{b}\right)\left[\frac{n}{n+1}+na\right]. \end{aligned} \tag{2.39}$$

For $s \neq 0, 1$,

$$\Gamma_s(f) = \frac{1}{s(s-1)}\left[\frac{b+na-(n+1)\tilde{f}}{b-a}a^s + \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}\int_a^b \frac{t^s}{(t-a)^{(n-1)/n}}dt - \int_0^1 (f(t))^s dt\right]. \tag{2.40}$$

Using I_1, we have

$$\Gamma_s(f) = \frac{1}{s(s-1)}\left[\frac{b+na-(n+1)\tilde{f}}{b-a}a^s + \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}a^{s+1/n}\left(\frac{b-a}{b}\right)^{1/n}\right.$$
$$\left.\times\beta\left(\frac{1}{n},1\right)F\left(\begin{array}{c|c} s+\frac{1}{n}+1, \frac{1}{n} & \frac{b-a}{b} \\ \frac{1}{n}+1 & \end{array}\right) - \int_0^1 (f(t))^s dt\right]. \tag{2.41}$$

For $s=0$,

$$\Gamma_0(f) = \frac{b+na-(n+1)\tilde{f}}{b-a}(-\log a) - \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}\int_a^b \frac{\log t}{(t-a)^{(n-1)/n}}dt + \int_0^1 \log(f(t))dt. \tag{2.42}$$

Using I_2, we have

$$\Gamma_0(f) = \frac{b+na-(n+1)\tilde{f}}{b-a}(-\log a) - \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}$$
$$\times\left[n\log b(b-a)^{1/n} - na^{1/n}\left(\frac{b-a}{b}\right)^{1/n+1}\beta\left(\frac{1}{n}+1,1\right)F\left(\begin{array}{c|c} \frac{1}{n}+1, \frac{1}{n}+1 & \frac{b-a}{b} \\ \frac{1}{n}+2 & \end{array}\right)\right]$$
$$+\int_0^1 \log(f(t))dt. \tag{2.43}$$

For $s=1$,

$$\Gamma_1(f) = \frac{b+na-(n+1)\tilde{f}}{b-a}a\log a + \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}\int_a^b \frac{t\log t}{(t-a)^{(n-1)/n}}dt - \int_0^1 f(t)\log(f(t))dt. \tag{2.44}$$

Using I_3, we have

$$\Gamma_1(f) = \frac{b+na-(n+1)\tilde{f}}{b-a}a\log a + \frac{(n+1)(\tilde{f}-a)}{n(b-a)^{(n+1)/n}}$$
$$\times\left[na(b-a)^{1/n}\log b + \frac{n}{n+1}\log b(b-a)^{1/n+1} - a^{1/n}\left(\frac{b-a}{b}\right)^{1/n+1}\beta\left(\frac{1}{n}+1,1\right)\right.$$
$$\left.\times F\left(\begin{array}{c|c} \frac{1}{n}+1, \frac{1}{n}+1 & \frac{b-a}{b} \\ \frac{1}{n}+2 & \end{array}\right)\left[\frac{n}{n+1}+na\right]\right] - \int_0^1 f(t)\log(f(t))dt. \tag{2.45}$$

□

Theorem 2.6. *Let $f : [0,1] \to \mathbb{R}$ be convex of order $1,\ldots,n+1$ such that $0 < a \le f(x) \le b$ for $x \in [0,1]$, $\tilde{f}$ be defined in* (1.5) *and let $\Gamma_s(f)$ defined in* (2.20) *be positive.*

One has that $\Gamma_s(f)$ is log-convex and the following inequality holds for $-\infty < r < s < t < +\infty$,

$$\Gamma_s^{t-r}(f) \le \Gamma_r^{t-s}(f)\Gamma_t^{s-r}(f). \tag{2.46}$$

Proof. As in the proof of Theorem 2.1, we use Theorem 1.4 instead of Theorem 1.3. □

Theorem 2.7. *Let $f, \Gamma_s(f)$ be defined in Theorem 2.6 and $t,\ s,\ u,\ v$ be real numbers such that $s \le u$, $t \le v$, $s \ne t$, $u \ne v$, one has*

$$\left(\frac{\Gamma_t(f)}{\Gamma_s(f)}\right)^{1/(t-s)} \le \left(\frac{\Gamma_v(f)}{\Gamma_u(f)}\right)^{1/(v-u)}. \tag{2.47}$$

Proof. Similar to the proof of Theorem 2.2. □

3. Cauchy Means

Let us note that (2.11) has the form of some known inequalities between means (e.g., Stolarsky's means, etc.). Here we prove that expressions on both sides of (2.11) are also means.

Lemma 3.1. *Let $h \in C^2(I)$ be such that h'' is bounded, that is, $m \le h'' \le M$. Then the functions ϕ_1, ϕ_2 defined by*

$$\phi_1(t) = \frac{M}{2}t^2 - h(t), \qquad \phi_2(t) = h(t) - \frac{m}{2}t^2, \tag{3.1}$$

are convex functions.

Theorem 3.2. *Let $h \in C^2(I_1)$, I_1 is a compact interval in $\mathbb{R}$ and f be a continuous, increasing and convex such that $a \le f(x) \le b$ for $x \in [0,1]$, $\tilde{f}$ be defined in* (1.5) *then $\exists \xi \in [a,b]$, $\xi \ne 0$ such that*

$$\begin{aligned} &\frac{b+a-2\tilde{f}}{b-a}h(a) + \frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b h(x)dx - \int_0^1 h(f(x))dx \\ &\quad = \frac{h''(\xi)}{2}\left[\frac{b+a-2\tilde{f}}{b-a}a^2 + \frac{2(\tilde{f}-a)(b^2+ba+a^2)}{3(b-a)} - \int_0^1 (f(x))^2 dx\right]. \end{aligned} \tag{3.2}$$

Proof. Suppose $m = \min h''(x) \le h''(x) \le M = \max h''(x)$ for $x \in I_1$. Then by applying ϕ_1 and ϕ_2 defined in Lemma 3.1 for ϕ in (1.4), we have

$$\begin{aligned} \frac{b+a-2\tilde{f}}{b-a}\phi_1(a) + \frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b \phi_1(x)dx \ge \int_0^1 \phi_1(f(x))dx, \\ \frac{b+a-2\tilde{f}}{b-a}\phi_2(a) + \frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b \phi_2(x)dx \ge \int_0^1 \phi_2(f(x))dx, \end{aligned} \tag{3.3}$$

that is,

$$\begin{aligned}
&\frac{M}{2}\left[\frac{b+a-2\tilde{f}}{b-a}a^2+\frac{2(\tilde{f}-a)(b^2+ba+a^2)}{3(b-a)}-\int_0^1 (f(x))^2dx\right] \\
&\quad\geq \frac{b+a-2\tilde{f}}{b-a}h(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b h(x)dx-\int_0^1 h(f(x))dx,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\frac{b+a-2\tilde{f}}{b-a}h(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b h(x)dx-\int_0^1 h(f(x))dx \\
&\quad\geq \frac{m}{2}\left[\frac{b+a-2\tilde{f}}{b-a}a^2+\frac{2(\tilde{f}-a)(b^2+ba+a^2)}{3(b-a)}-\int_0^1 (f(x))^2dx\right].
\end{aligned} \tag{3.5}$$

By combining (3.4) and (3.5) and using the fact that for $m \leq \rho \leq M$ there exists $\xi \in I_1$ such that $h''(\xi) = \rho$ we get (3.2). □

Theorem 3.3. *Let $k, l \in C^2(I_1)$ and satisfy* (3.2)*, f be a continuous, increasing and convex such that $a \leq f(x) \leq b$ for $x \in [0,1]$, $\tilde{f}$ be defined in* (1.5)*, and*

$$f(x) \neq a+(b-a)\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)}, \quad \textit{where } \lambda = \frac{b+a-2\tilde{f}}{b-a}, \tag{3.6}$$

then there exists $\xi \in I_1$ such that

$$\frac{k''(\xi)}{l''(\xi)} = \frac{((b+a-2\tilde{f})/(b-a))k(a)+\left(2(\tilde{f}-a)/(b-a)^2\right)\int_a^b k(x)dx-\int_0^1 k(f(x))dx}{((b+a-2\tilde{f})/(b-a))l(a)+\left(2(\tilde{f}-a)/(b-a)^2\right)\int_a^b l(x)dx-\int_0^1 l(f(x))dx}. \tag{3.7}$$

Provided that denominators are non-zero.

Proof. Consider the linear functionals Ψ and η such that $\Psi(m) = \eta(m)(\xi)$ for some function $m \in C^2(I_1)$ and $\xi \in I_1$. Consider the following linear combination

$$m = c_1k - c_2l, \tag{3.8}$$

where c_1 and c_2 are defined as follows:

$$\begin{aligned}
c_1 = \Psi(l) &= \frac{b+a-2\tilde{f}}{b-a}l(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b l(x)dx-\int_0^1 l(f(x))dx, \\
c_2 = \Psi(k) &= \frac{b+a-2\tilde{f}}{b-a}k(a)+\frac{2(\tilde{f}-a)}{(b-a)^2}\int_a^b k(x)dx-\int_0^1 k(f(x))dx.
\end{aligned} \tag{3.9}$$

Since $k, l \in C^2(I_1)$ and satisfy (3.2), therefore m as linear combination of k and l should also satisfy (3.2).

Let η be defined as follows:

$$\eta(k)(\xi) = \frac{k''(\xi)}{2}\left[\frac{b+a-2\widetilde{f}}{b-a}a^2 + \frac{2(\widetilde{f}-a)(b^2+ba+a^2)}{3(b-a)} - \int_0^1 (f(x))^2 dx\right]. \tag{3.10}$$

Obviously, we have

$$\Psi(m) = 0. \tag{3.11}$$

On the other hand, there is an $\xi \in I_1$ such that

$$\eta(m)(\xi) = \Psi(m) = 0. \tag{3.12}$$

By using the linearity property of the operator η

$$0 = \Psi(l)\eta(k)(\xi) - \Psi(k)\eta(l)(\xi). \tag{3.13}$$

Now $\Psi(l) \neq 0$ and $\eta(l)(\xi) \neq 0$, we have from the last equation

$$\frac{\Psi(k)}{\Psi(l)} = \frac{\eta(k)(\xi)}{\eta(l)(\xi)}. \tag{3.14}$$

After putting values, we get (3.7). □

Corollary 3.4. *Let f be a continuous, increasing and convex such that $a \leq f(x) \leq b$ for $x \in [0,1]$, then for $-\infty < s \neq t \neq 0,\ 1 \neq s < +\infty$ there exists $\xi \in I_1$ such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)}\left(\frac{\mathfrak{D}a^t + \mathcal{L}\left(b^{t+1}-a^{t+1}\right)/(t+1) - \int_0^1 (f(x))^t\,dx}{\mathfrak{D}a^s + \mathcal{L}\left(b^{s+1}-a^{s+1}\right)/(s+1) - \int_0^1 (f(x))^s\,dx}\right), \tag{3.15}$$

where $\mathfrak{D}$ denotes $(b+a-2\widetilde{f})/(b-a)$ and $\mathcal{L}$ denotes $2(\widetilde{f}-a)/(b-a)^2$.

Proof. Set $k(x) = \varphi_t(x)$ and $l(x) = \varphi_s(x),\ t \neq s \neq 0, 1$ in (3.7) we get (3.15). □

Remark 3.5. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, therefore from (3.15) we have

$$a \leq \left(\frac{s(s-1)}{t(t-1)} \cdot \frac{\mathfrak{D}a^t + \mathcal{L}\left(b^{t+1}-a^{t+1}\right)/(t+1) - \int_0^1 (f(x))^t\,dx}{\mathfrak{D}a^s + \mathcal{L}\left(b^{s+1}-a^{s+1}\right)/(s+1) - \int_0^1 (f(x))^s\,dx}\right)^{1/(t-s)} \leq b. \tag{3.16}$$

In fact, similar result can also be given for (3.7). Namely, suppose that k''/l'' has inverse function. Then from (3.7) we have

$$\xi = \left(\frac{k''}{l''}\right)^{-1}\left(\frac{\mathfrak{D}k(a) + \mathcal{L}\int_a^b k(x)dx - \int_0^1 k(f(x))dx}{\mathfrak{D}l(a) + \mathcal{L}\int_a^b l(x)dx - \int_0^1 l(f(x))dx}\right). \tag{3.17}$$

The expression on the right-hand side of (3.17) is also a mean.

From the inequality (3.16), we can define means $M_{t,s}(f)$ as follows:

$$M_{t,s}(f) = \left(\frac{s(s-1)}{t(t-1)} \cdot \frac{\mathfrak{D}a^t + \mathcal{L}\left(b^{t+1} - a^{t+1}\right)/(t+1) - \int_0^1 (f(x))^t\, dx}{\mathfrak{D}a^s + \mathcal{L}\left(b^{s+1} - a^{s+1}\right)/(s+1) - \int_0^1 (f(x))^s\, dx}\right)^{1/(t-s)}, \tag{3.18}$$

for $-\infty < s \neq t \neq 0,\ \ 1 \neq s < +\infty$. Moreover we can extend these means in other cases. By limit we have

$$\begin{aligned}
M_{s,s}(f) &= \exp\Bigg(\frac{\mu a^s \log a + (\nu/(s+1))\left(b^{s+1}\log b - a^{s+1}\log a\right) - \left(\nu/(s+1)^2\right)\left(b^{s+1} - a^{s+1}\right) - E}{\mu a^s + (\nu/(s+1))\left(b^{s+1} - a^{s+1}\right) - \int_0^1 (f(x))^s\, dx} \\
&\qquad - \frac{2s-1}{s(s-1)}\Bigg), \qquad s \neq 0, 1; \\
M_{0,0}(f) &= \exp\left(\frac{\int_0^1 (\log(f(x)))^2\, dx + 2\int_0^1 \log(f(x))dx - \mathcal{H} - \nu\left(b(\log b)^2 - a(\log a)^2\right) - G}{2\nu(b-a) + 2\int_0^1 \log(f(x))dx - 2\mu\log a - 2\nu(b\log b - a\log a)}\right); \\
M_{1,1}(f) &= \exp\Bigg(\frac{\mathcal{K} + (\nu/2)\left(b^2(\log b)^2 - a^2(\log a)^2\right) - (3/2)\nu\left(b^2(\log b) - a^2(\log a)\right) - H}{2\mu a\log a + \nu\left(b^2\log b - a^2\log a\right) - (\nu/2)\left(b^2 - a^2\right) - 2\int_0^1 f(x)\log(f(x))dx} \\
&\qquad + \frac{(3/4)\nu\left(b^2 - a^2\right) - 2\mu a\log a + 2\int_0^1 f(x)\log(f(x))dx}{2\mu a\log a + \nu\left(b^2\log b - a^2\log a\right) - (\nu/2)\left(b^2 - a^2\right) - 2\int_0^1 f(x)\log(f(x))dx}\Bigg); \\
M_{1,0}(f) &= \left(\frac{\mu a\log a + (\nu/2)\left(b^2\log b - a^2\log a\right) - (\nu/4)\left(b^2 - a^2\right) - \int_0^1 f(x)\log(f(x))dx}{\nu(b-a) + \int_0^1 \log(f(x))dx - \mu\log a - \nu(b\log b - a\log a)}\right); \\
M_{s,0}(f) &= \left(\frac{\mu a^s + (\nu/(s+1))\left(b^{s+1} - a^{s+1}\right) - \int_0^1 (f(x))^s dx}{s(s-1)\left[\nu(b-a) + \int_0^1 \log(f(x))dx - \mu\log a - \nu(b\log b - a\log a)\right]}\right)^{1/s}, \quad s \neq 0, 1; \\
M_{s,1}(f) &= \left(\frac{\mu a^s + (\nu/(s+1))\left(b^{s+1} - a^{s+1}\right) - \int_0^1 (f(x))^s dx}{s(s-1)\left[\mu a\log a + (\nu/2)\left(b^2\log b - a^2\log a\right) - (\nu/4)\left(b^2 - a^2\right) - O\right]}\right)^{1/(s-1)}, \quad s \neq 0, 1,
\end{aligned} \tag{3.19}$$

where E denotes $\int_0^1 (f(x))^s \log(f(x))dx$, $\mathscr{H}$ denotes $\mu(\log a)^2$, G denotes $2\mu \log a$, $\mathscr{K}$ denotes $\mu a(\log a)^2$, H denotes $\int_0^1 (f(x))(\log(f(x)))^2 dx$, and O denotes $\int_0^1 f(x)\log(f(x))dx$, where

$$\mu = \frac{b + a - 2\tilde{f}}{b - a}, \qquad \nu = \frac{2(\tilde{f} - a)}{(b - a)^2}. \tag{3.20}$$

In our next result we prove that this new mean is monotonic.

Theorem 3.6. *Let $t \leq u$, $r \leq s$, then the following inequality is valid*

$$M_{t,r}(f;w) \leq M_{u,s}(f;w). \tag{3.21}$$

Proof. Since $\Omega_s(f)$ is log-convex, therefore by (2.11) we get (3.21). □

Remark 3.7. Similar results of the Cauchy means and related results can also proved for Theorems 2.3 and 2.6.

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