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Research Article

New Trace Bounds for the Product of Two Matrices and Their Applications in the Algebraic Riccati Equation

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By using singular value decomposition and majorization inequalities, we propose new inequalities for the trace of the product of two arbitrary real square matrices. These bounds improve and extend the recent results. Further, we give their application in the algebraic Riccati equation. Finally, numerical examples have illustrated that our results are effective and superior.

1. Introduction

In the analysis and design of controllers and filters for linear dynamical systems, the Riccati equation is of great importance in both theory and practice (see [1–5]). Consider the following linear system (see [4]):

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1.1}$$

with the cost

$$J = \int_0^\infty \left(x^T Q x + u^T u\right) dt. \tag{1.2}$$

Moreover, the optimal control rate u^* and the optimal cost J^* of (1.1) and (1.2) are

$$\begin{gathered} u^* = Px, \quad P = B^T K, \\ J^* = x_0^T K x_0, \end{gathered} \tag{1.3}$$

where $x_0 \in R^n$ is the initial state of the systems (1.1) and (1.2), K is the positive definite solution of the following algebraic Riccati equation (ARE):

$$A^T K + KA - KRK = -Q, \tag{1.4}$$

with $R = BB^T$ and Q are symmetric positive definite matrices. To guarantee the existence of the positive definite solution to (1.4), we shall make the following assumptions: the pair (A, R) is stabilizable, and the pair (Q, A) is observable.

In practice, it is hard to solve the (ARE), and there is no general method unless the system matrices are special and there are some methods and algorithms to solve (1.4), however, the solution can be time-consuming and computationally difficult, particularly as the dimensions of the system matrices increase. Thus, a number of works have been presented by researchers to evaluate the bounds and trace bounds for the solution of the (ARE) [6–12]. In addition, from [2, 6], we know that an interpretation of $\mathrm{tr}(K)$ is that $\mathrm{tr}(K)/n$ is the average value of the optimal cost J^* as x_0 varies over the surface of a unit sphere. Therefore, consider its applications, it is important to discuss trace bounds for the product of two matrices. Most available results are based on the assumption that at least one matrix is symmetric [7, 8, 11, 12]. However, it is important and difficult to get an estimate of the trace bounds when any matrix in the product is nonsymmetric in theory and practice. There are some results in [13–15].

In this paper, we propose new trace bounds for the product of two general matrices. The new trace bounds improve the recent results. Then, for their application in the algebraic Riccati equation, we get some upper and lower bounds.

In the following, let $R^{n\times n}$ denote the set of $n \times n$ real matrices. Let $x = (x_1, x_2, \ldots, x_n)$ be a real n-element array which is reordered, and its elements are arranged in nonincreasing order. That is, $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. Let $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$. For $A = (a_{ij}) \in R^{n\times n}$, let $d(A) = (d_1(A), d_2(A), \ldots, d_n(A))$, $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$, $\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))$ denote the diagonal elements, the eigenvalues, the singular values of A, respectively, Let $\mathrm{tr}(A)$, A^T denote the trace, the transpose of A, respectively. We define $(A)_{ii} = a_{ii} = d_i(A)$, $\overline{A} = (A + A^T)/2$. The notation $A > 0$ ($A \geq 0$) is used to denote that A is a symmetric positive definite (semidefinite) matrix.

Let α, β be two real n-element arrays. If they satisfy

$$\sum_{i=1}^{k} \alpha_{[i]} \leq \sum_{i=1}^{k} \beta_{[i]}, \quad k = 1, 2, \ldots, n, \tag{1.5}$$

then it is said that α is controlled weakly by β, which is signed by $\alpha \prec_w \beta$.

If $\alpha \prec_w \beta$ and

$$\sum_{i=1}^{n} \alpha_{[i]} = \sum_{i=1}^{n} \beta_{[i]}, \tag{1.6}$$

then it is said that α is controlled by β, which is signed by $\alpha \prec \beta$.

Therefore, considering the application of the trace bounds, many scholars pay much attention to estimate the trace bounds for the product of two matrices.

Marshall and Olkin in [16] have showed that for any $A, B \in R^{n\times n}$, then

$$-\sum_{i=1}^{n} \sigma_{[i]}(A)\sigma_{[i]}(B) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \sigma_{[i]}(A)\sigma_{[i]}(B). \tag{1.7}$$

Xing et al. in [13] have observed another result. Let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decomposition:

$$B = U\operatorname{diag}\left(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)\right)V^T, \tag{1.8}$$

where $U, V \in R^{n\times n}$ are orthogonal. Then

$$\lambda_{[n]}(\overline{AS})\sum_{i=1}^{n} \sigma_{[i]}(B) \leq \operatorname{tr}(AB) \leq \lambda_{[1]}(\overline{AS})\sum_{i=1}^{n} \sigma_{[i]}(B), \tag{1.9}$$

where $S = UV^T$ is orthogonal.

Liu and He in [14] have obtained the following: let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decomposition:

$$B = U\operatorname{diag}\left(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)\right)V^T, \tag{1.10}$$

where $U, V \in R^{n\times n}$ are orthogonal. Then

$$\min_{1\leq i\leq n}\left(V^T AU\right)_{ii}\sum_{i=1}^{n} \sigma_{[i]}(B) \leq \operatorname{tr}(AB) \leq \max_{1\leq i\leq n}\left(V^T AU\right)_{ii}\sum_{i=1}^{n} \sigma_{[i]}(B). \tag{1.11}$$

F. Zhang and Q. Zhang in [15] have obtained the following: let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decomposition:

$$B = U\operatorname{diag}\left(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)\right)V^T, \tag{1.12}$$

where $U, V \in R^{n\times n}$ are orthogonal. Then

$$\sum_{i=1}^{n} \sigma_{[i]}(B)\lambda_{[n-i+1]}(\overline{AS}) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \sigma_{[i]}(B)\lambda_{[i]}(\overline{AS}), \tag{1.13}$$

where $S = UV^T$ is orthogonal. They show that (1.13) has improved (1.9).

2. Main Results

The following lemmas are used to prove the main results.

Lemma 2.1 (see [16, page 92, H.2.c]). *If $x_{[1]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq \cdots \geq y_{[n]}$ and $x \prec y$, then for any real array $u_{[1]} \geq \cdots \geq u_{[n]}$,*

$$\sum_{i=1}^{n} x_{[i]} u_{[i]} \leq \sum_{i=1}^{n} y_{[i]} u_{[i]}. \tag{2.1}$$

Lemma 2.2 (see [16, page 95, H.3.b]). *If $x_{[1]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq \cdots \geq y_{[n]}$ and $x \prec_w y$, then for any real array $u_{[1]} \geq \cdots \geq u_{[n]} \geq 0$,*

$$\sum_{i=1}^{n} x_{[i]} u_{[i]} \leq \sum_{i=1}^{n} y_{[i]} u_{[i]}. \tag{2.2}$$

Remark 2.3. Note that if $x \prec_w y$, then for $k = 1, 2, \ldots, n$, $(x_{[1]}, \ldots, x_{[k]}) \prec_w (y_{[1]}, \ldots, y_{[k]})$. Thus from Lemma 2.2, we have

$$\sum_{i=1}^{k} x_{[i]} u_{[i]} \leq \sum_{i=1}^{k} y_{[i]} u_{[i]}, \quad k = 1, 2, \ldots, n. \tag{2.3}$$

Lemma 2.4 (see [16, page 218, B.1]). *Let $A = A^T \in R^{n\times n}$, then*

$$d(A) \prec \lambda(A). \tag{2.4}$$

Lemma 2.5 (see [16, page 240, F.4.a]). *Let $A \in R^{n\times n}$, then*

$$\lambda\left(\frac{A + A^T}{2}\right) \prec_w \left|\lambda\left(\frac{A + A^T}{2}\right)\right| \prec_w \sigma(A). \tag{2.5}$$

Lemma 2.6 (see [17]). *Let $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$, $k = 1, 2, \ldots, n$, $1/p + 1/q = 1$. Then*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \leq c_{p,q} \sum_{k=1}^{n} a_k b_k, \tag{2.6}$$

where

$$c_{p,q} = \frac{M_1^p M_2^q - m_1^p m_2^q}{\left[p\left(M_1 m_2 M_2^q - m_1 M_2 m_2^q\right)\right]^{1/p} \left[q\left(m_1 M_2 M_1^p - M_1 m_2 m_1^p\right)\right]^{1/q}}. \tag{2.7}$$

Note that if $m_1 = 0, m_2 \neq 0$ *or* $m_2 = 0, m_1 \neq 0$*, obviously,* (2.6) *holds. If* $m_1 = m_2 = 0$*, choose* $c_{p,q} = +\infty$*, then* (2.6) *also holds.*

Remark 2.7. If $p = q = 2$, then we obtain Cauchy-Schwartz inequality

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \leq c_2 \sum_{k=1}^{n} a_k b_k, \tag{2.8}$$

where

$$c_2 = \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right). \tag{2.9}$$

Remark 2.8. Note that

$$\lim_{p\to\infty} \left(a_1^p + a_2^p + \cdots + a_n^p\right)^{1/p} = \max_{1\leq k\leq n}\{a_k\},$$

$$\begin{aligned}
\lim_{\substack{p\to\infty\\ q\to 1}} c_{p,q} &= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_1^p M_2^q - m_1^p m_2^q}{\left[p\left(M_1 m_2 M_2^q - m_1 M_2 m_2^q\right)\right]^{1/p} \left[q\left(m_1 M_2 M_1^p - M_1 m_2 m_1^p\right)\right]^{1/q}} \\
&= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_1^p \left[M_2^q - (m_1/M_1)^p m_2^q\right]}{M_1^{1/p}\left[p\left(m_2 M_2^q - (m_1/M_1) M_2 m_2^q\right)\right]^{1/p} M_1^{q/p} \left[q\left(m_1 M_2 - M_1 m_2 (m_1/M_1)^p\right)\right]^{1/q}} \\
&= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_2}{M_1^{1/p+p/q-p} m_1 M_2} = \lim_{\substack{p\to\infty\\ q\to 1}} \frac{1}{M_1^{1/p-1} m_1} = \frac{M_1}{m_1}.
\end{aligned} \tag{2.10}$$

Let $p \to \infty$, $q \to 1$ in (2.6), then we obtain

$$m_1 \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} a_k b_k \leq M_1 \sum_{k=1}^{n} b_k. \tag{2.11}$$

Lemma 2.9. *If* $q \geq 1$, $a_i \geq 0$ $(i = 1, 2, \ldots, n)$*, then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q \leq \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.12}$$

Proof. (1) Note that $q = 1$, or $a_i = 0$ $(i = 1, 2, \ldots, n)$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.13}$$

(2) If $q > 1$, $a_i > 0$, for $x > 0$, choose $f(x) = x^q$, then $f'(x) = qx^{q-1} > 0$ and $f''(x) = q(q-1)x^{q-2} > 0$. Thus, $f(x)$ is a convex function. As $a_i > 0$ and $(1/n)\sum_{i=1}^{n} a_i > 0$, from the property of the convex function, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(a_i) = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.14}$$

(3) If $q > 1$, without loss of generality, we may assume $a_i = 0$ $(i = 1, \ldots, r)$, $a_i > 0$ $(i = r+1, \ldots, n)$. Then from (2), we have

$$\left(\frac{1}{n-r}\right)^q \left(\sum_{i=1}^{n} a_i\right)^q = \left(\frac{1}{n-r}\sum_{i=1}^{n} a_i\right)^q \leq \frac{1}{n-r}\sum_{i=1}^{n} a_i^q. \tag{2.15}$$

Since $((n-r)/n)^q \leq (n-r)/n$, thus

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = \left(\frac{n-r}{n}\right)^q \left(\frac{1}{n-r}\right)^q \left(\sum_{i=1}^{n} a_i\right)^q \leq \frac{n-r}{n}\frac{1}{n-r}\sum_{i=1}^{n} a_i^q = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.16}$$

This completes the proof. □

Theorem 2.10. *Let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decomposition:*

$$B = U\operatorname{diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B))V^T, \tag{2.17}$$

where $U, V \in R^{n\times n}$ are orthogonal. Then

$$\sum_{i=1}^{n} \sigma_{[i]}(B) d_{[n-i+1]}\left(V^T AU\right) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) d_{[i]}\left(V^T AU\right). \tag{2.18}$$

Proof. By the matrix theory we have

$$\begin{aligned} \operatorname{tr}(AB) &= \operatorname{tr}\left[AU\operatorname{diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B))V^T\right] \\ &= \operatorname{tr}\left[V^T AU\operatorname{diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B))\right] \\ &= \sum_{i=1}^{n} \sigma_i(B)\left(V^T AU\right)_{ii}. \end{aligned} \tag{2.19}$$

Since $\sigma_{[1]}(B) \geq \sigma_{[2]}(B) \geq \cdots \geq \sigma_{[n]}(B) \geq 0$, without loss of generality, we may assume $\sigma(B) = (\sigma_{[1]}(B), \sigma_{[2]}(B), \ldots, \sigma_{[n]}(B))$. Next, we will prove the left-hand side of (2.18):

$$\sum_{i=1}^{n} \sigma_{[i]}(B) d_{[n-i+1]}(V^T AU) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) d_i(V^T AU). \tag{2.20}$$

If

$$d(V^T AU) = (d_{[n]}(V^T AU), d_{[n-1]}(V^T AU), \ldots, d_{[1]}(V^T AU)), \tag{2.21}$$

we obtain the conclusion. Now assume that there exists $j < k$ such that $d_j(V^T AU) > d_k(V^T AU)$, then

$$\begin{aligned} &\sigma_{[j]}(B) d_k(V^T AU) + \sigma_{[k]}(B) d_j(V^T AU) - \sigma_{[j]}(B) d_j(V^T AU) - \sigma_{[k]}(B) d_k(V^T AU) \\ &\quad = \left[\sigma_{[j]}(B) - \sigma_{[k]}(B)\right]\left[d_k(V^T AU) - d_j(V^T AU)\right] \leq 0. \end{aligned} \tag{2.22}$$

We use $\tilde{d}(V^T AU)$ to denote the vector of $d(V^T AU)$ after changing $d_j(V^T AU)$ and $d_k(V^T AU)$, then

$$\sum_{i=1}^{n} \sigma_{[i]}(B) \tilde{d}_i(V^T AU) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) d_i(V^T AU). \tag{2.23}$$

After limited steps, we obtain the the left-hand side of (2.18). For the right-hand side of (2.18),

$$\sum_{i=1}^{n} \sigma_{[i]}(B) d_i(V^T AU) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) d_{[i]}(V^T AU). \tag{2.24}$$

If

$$d(V^T AU) = (d_{[1]}(V^T AU), d_{[2]}(V^T AU), \ldots, d_{[n]}(V^T AU)), \tag{2.25}$$

we obtain the conclusion. Now assume that there exists $j > k$ such that $d_j(V^T AU) < d_k(V^T AU)$, then

$$\begin{aligned} &\sigma_{[j]}(B) d_k(V^T AU) + \sigma_{[k]}(B) d_j(V^T AU) - \sigma_{[j]}(B) d_j(V^T AU) - \sigma_{[k]}(B) d_k(V^T AU) \\ &\quad = \left[\sigma_{[j]}(B) - \sigma_{[k]}(B)\right]\left[d_k(V^T AU) - d_j(V^T AU)\right] \geq 0. \end{aligned} \tag{2.26}$$

We use $\widetilde{d}(V^T AU)$ to denote the vector of $d(V^T AU)$ after changing $d_j(V^T AU)$ and $d_k(V^T AU)$, then

$$\sum_{i=1}^{n} \sigma_{[i]}(B) d_i(V^T AU) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) \widetilde{d}_i(V^T AU). \tag{2.27}$$

After limited steps, we obtain the right-hand side of (2.18). Therefore,

$$\sum_{i=1}^{n} \sigma_{[i]}(B) d_{[n-i+1]}(V^T AU) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \sigma_{[i]}(B) d_{[i]}(V^T AU). \tag{2.28}$$

This completes the proof. □

Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, applying (2.18) with B in lieu of A, we immediately have the following corollary.

Corollary 2.11. *Let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decomposition:*

$$A = P\operatorname{diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))Q^T, \tag{2.29}$$

where $P, Q \in R^{n\times n}$ are orthogonal. Then

$$\sum_{i=1}^{n} \sigma_{[i]}(A) d_{[n-i+1]}(Q^T BP) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \sigma_{[i]}(A) d_{[i]}(Q^T BP). \tag{2.30}$$

Now using (2.18) and (2.30), one finally has the following theorem.

Theorem 2.12. *Let $A, B \in R^{n\times n}$ be arbitrary matrices with the following singular value decompositions, respectively:*

$$\begin{aligned} A &= P\operatorname{diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))Q^T, \\ B &= U\operatorname{diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B))V^T, \end{aligned} \tag{2.31}$$

where $P, Q, U, V \in R^{n\times n}$ are orthogonal. Then

$$\begin{aligned} &\max\left\{ \sum_{i=1}^{n} \sigma_{[i]}(A) d_{[n-i+1]}(Q^T BP), \sum_{i=1}^{n} \sigma_{[i]}(B) d_{[n-i+1]}(V^T AU) \right\} \\ &\quad\leq \operatorname{tr}(AB) \leq \min\left\{ \sum_{i=1}^{n} \sigma_{[i]}(B) d_{[i]}(V^T AU), \sum_{i=1}^{n} \sigma_{[i]}(A) d_{[i]}(Q^T BP) \right\}. \end{aligned} \tag{2.32}$$

Remark 2.13. We point out that (2.18) improves (1.11). In fact, it is obvious that

$$\begin{aligned}\min_{1\le i\le n}\left(V^T AU\right)_{ii}\sum_{i=1}^{n}\sigma_{[i]}(B) &\le \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[n-i+1]}\left(V^T AU\right)\\ &\le \operatorname{tr}(AB) \le \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[i]}\left(V^T AU\right) \le \max_{1\le i\le n}\left(V^T AU\right)_{ii}\sum_{i=1}^{n}\sigma_{[i]}(B).\end{aligned} \tag{2.33}$$

This implies that (2.18) improves (1.11).

Remark 2.14. We point out that (2.18) improves (1.13). Since for $i = 1,\dots,n$, $\sigma_i(B) \ge 0$ and $d_i(V^T AU) = d_i((V^T AU + (V^T AU)^T)/2)$, from Lemmas 2.1 and 2.4, then (2.18) implies

$$\begin{aligned}&\sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[n-i+1]}\left(\frac{V^T AU + \left(V^T AU\right)^T}{2}\right)\\ &\quad\le \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[n-i+1]}\left(\frac{V^T AU + \left(V^T AU\right)^T}{2}\right)\\ &\quad\le \operatorname{tr}(AB)\\ &\quad\le \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[i]}\left(\frac{V^T AU + \left(V^T AU\right)^T}{2}\right)\\ &\quad\le \sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[i]}\left(\frac{V^T AU + \left(V^T AU\right)^T}{2}\right).\end{aligned} \tag{2.34}$$

In fact, for $i = 1,2,\dots,n$, we have

$$\begin{aligned}\lambda_i\left(\frac{V^T AU + \left(V^T AU\right)^T}{2}\right) &= \lambda_i\left[V^T\frac{AUV^T + \left(AUV^T\right)^T}{2}V\right]\\ &= \lambda_i\left(\frac{AUV^T + \left(AUV^T\right)^T}{2}\right)\\ &= \lambda_i\left(\overline{AS}\right).\end{aligned} \tag{2.35}$$

Then (2.34) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[n-i+1]}(\overline{AS}) &\leq \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[n-i+1]}\left(V^{T}AU\right)\\ &\leq \operatorname{tr}(AB)\\ &\leq \sum_{i=1}^{n}\sigma_{[i]}(B)d_{[i]}\left(V^{T}AU\right)\\ &\leq \sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[i]}(\overline{AS}).\end{aligned} \tag{2.36}$$

This implies that (2.18) improves (1.13).

Remark 2.15. We point out that (1.13) improves (1.7). In fact, from Lemma 2.5, we have

$$\lambda(\overline{AS})\prec_{w}\sigma(AS). \tag{2.37}$$

Since S is orthogonal, $\sigma(AS) = \sigma(A)$. Then (2.37) is rewritten as follows: $\lambda(\overline{AS})\prec_{w}\sigma(A)$. By using $\sigma_{[1]}(B) \geq \sigma_{[2]}(B) \geq \cdots \geq \sigma_{[n]}(B) \geq 0$ and Lemma 2.2, we obtain

$$\sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[i]}(\overline{AS}) \leq \sum_{i=1}^{n}\sigma_{[i]}(B)\sigma_{[i]}(A). \tag{2.38}$$

Note that $\lambda_{i}(-\overline{AS}) = -\lambda_{n-i+1}(\overline{AS})$, from Lemma 2.2 and (2.38), we have

$$\begin{aligned}-\sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[n-i+1]}(\overline{AS}) &= \sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[i]}(-\overline{AS})\\ &\leq \sum_{i=1}^{n}\sigma_{[i]}(B)\left|\lambda_{[i]}(\overline{AS})\right| \leq \sum_{i=1}^{n}\sigma_{[i]}(B)\sigma_{[i]}(A).\end{aligned} \tag{2.39}$$

Thus, we obtain

$$-\sum_{i=1}^{n}\sigma_{[i]}(B)\sigma_{[i]}(A) \leq \sum_{i=1}^{n}\sigma_{[i]}(B)\lambda_{[n-i+1]}(\overline{AS}). \tag{2.40}$$

Both (2.38) and (2.40) show that (1.13) is tighter than (1.7).

3. Applications of the Results

Wang et al. in [6] have obtained the following: let K be the positive semidefinite solution of the ARE (1.4). Then the trace of matrix K has the lower and upper bounds given by

$$\frac{\lambda_{[n]}(\overline{A})+\sqrt{\left[\lambda_{[n]}(\overline{A})\right]^2+\lambda_{[1]}(R)\operatorname{tr}(Q)}}{\lambda_{[1]}(R)} \le \operatorname{tr}(K) \le \frac{\lambda_{[1]}(\overline{A})+\sqrt{\left[\lambda_{[1]}(\overline{A})\right]^2+(\lambda_{[n]}(R)/n)\operatorname{tr}(Q)}}{\lambda_{[n]}(R)/n}. \tag{3.1}$$

In this section, we obtain the application in the algebraic Riccati equation of our results including (3.1). Some of our results and (3.1) cannot contain each other.

Theorem 3.1. *If* $1/p+1/q=1$ *and* K *is the positive semidefinite solution of the ARE* (1.4), *then*

(1) *the trace of matrix* K *has the lower and upper bounds given by*

$$\begin{aligned}&\frac{\lambda_{[n]}(\overline{A})+\sqrt{\left[\lambda_{[n]}(\overline{A})\right]^2+\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}\operatorname{tr}(Q)}}{\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}}\\&\quad\le \operatorname{tr}(K)\le\frac{\lambda_{[1]}(\overline{A})+\sqrt{\left[\lambda_{[1]}(\overline{A})\right]^2+\left[\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}\right]\operatorname{tr}(Q)}}{\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}}.\end{aligned} \tag{3.2}$$

(2) *If* $(A+A^T)/2\ge 0$, *then the trace of matrix* K *has the lower and upper bounds given by*

$$\begin{aligned}&\frac{\left(1/c'_{p,q}n^{1-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(\overline{A})\right]^{1/p}+\sqrt{\left[\left(1/c'_{p,q}n^{1-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(\overline{A})\right]^{1/p}\right]^2+\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}\operatorname{tr}(Q)}}{\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}}\\&\le\operatorname{tr}(K)\\&\le\frac{\left[\sum_{i=1}^n\lambda_{[i]}^p(\overline{A})\right]^{1/p}+\sqrt{\left[\sum_{i=1}^n\lambda_{[i]}^p(\overline{A})\right]^{2/p}+\left[\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}\right]\operatorname{tr}(Q)}}{\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^n\lambda_{[i]}^p(R)\right]^{1/p}}.\end{aligned} \tag{3.3}$$

(3) *If* $(A+A^T)/2 \le 0$, *then the trace of matrix* K *has the lower and upper bounds given by*

$$\frac{-\left[\sum_{i=1}^{n}\left|\lambda_{[n-i+1]}(\overline{A})\right|^{p}\right]^{1/p}+\sqrt{\left[\sum_{i=1}^{n}\left|\lambda_{[n-i+1]}(\overline{A})\right|^{p}\right]^{2/p}+\left[\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\right]\operatorname{tr}(Q)}}{\left(1/c_{p,q}n^{2-1/q}\right)\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}}$$
$$\le \operatorname{tr}(K)$$
$$\le \frac{\left(-1/c'_{p,q}n^{1-1/q}\right)\left[\sum_{i=1}^{n}\left|\lambda_{[n-i+1]}(\overline{A})\right|^{p}\right]^{1/p}}{\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}}$$
$$+\frac{\sqrt{\left[\left(1/c'_{p,q}n^{1-1/q}\right)\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(\overline{A})\right|^{p}\right]^{1/p}\right]^{2}+\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\operatorname{tr}(Q)}}{\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}}, \tag{3.4}$$

where

$$\begin{gathered}
c_{p,q}=\frac{M_r^pM_k^q-m_r^pm_k^q}{\left[p\left(M_rm_kM_k^q-m_rM_km_k^q\right)\right]^{1/p}\left[q\left(m_rM_kM_r^p-M_rm_km_r^p\right)\right]^{1/q}},\\
M_r=\lambda_{[1]}(R),\quad m_r=\lambda_{[n]}(R),\quad M_k=\lambda_{[1]}(K),\quad m_k=\lambda_{[n]}(K),\\
c'_{p,q}=\frac{M_1^pM_k^q-m_1^pm_k^q}{\left[p\left(M_1m_kM_k^q-m_1M_km_k^q\right)\right]^{1/p}\left[q\left(m_1M_kM_1^p-M_1m_km_1^p\right)\right]^{1/q}},\\
M_1=\lambda_{[1]}(\overline{A}),\quad m_1=\lambda_{[n]}(\overline{A}).
\end{gathered} \tag{3.5}$$

Proof. (1) Take the trace in both sides of the matrix ARE (1.4) to get

$$\operatorname{tr}\left(A^TK\right)+\operatorname{tr}(KA)-\operatorname{tr}(KRK)=-\operatorname{tr}(Q). \tag{3.6}$$

Since K is symmetric positive definite matrix, $\lambda(K)=\sigma(K)$, $\operatorname{tr}(K)=\sum_{i=1}^{n}\sigma_{[i]}(K)$, and from Lemma 2.9, we have

$$\frac{\operatorname{tr}(K)}{n^{1-1/q}}\le\left[\operatorname{tr}\left(K^q\right)\right]^{1/q}\le\operatorname{tr}(K), \tag{3.7}$$

$$\sum_{i=1}^{n}\sigma_{[i]}(KK)=\sum_{i=1}^{n}\sigma_{[i]}^{2}(K)\le\left[\sum_{i=1}^{n}\sigma_{[i]}(K)\right]^{2}=\left[\operatorname{tr}(K)\right]^{2}. \tag{3.8}$$

By the Cauchy-Schwartz inequality (2.8), it can be shown that

$$\sum_{i=1}^{n}\sigma_{[i]}(KK)=\sum_{i=1}^{n}\sigma_{[i]}^{2}(K)\geq\frac{\left[\sum_{i=1}^{n}\sigma_{[i]}(K)\right]^{2}}{n}=\frac{\left[\operatorname{tr}(K)\right]^{2}}{n}. \tag{3.9}$$

Note that

$$K^{2}=U\operatorname{diag}(\lambda_{1}^{2}(K),\lambda_{2}^{2}(K),\ldots,\lambda_{n}^{2}(K))U^{T}, \tag{3.10}$$

$K,Q,R>0$, $\lambda_{[i]}(U^{T}RU)=\lambda_{[i]}(R)$ $(i=1,\ldots,n)$, then by (2.34), use (2.6), considering (3.7) and (3.9), we have

$$\begin{aligned}\operatorname{tr}(KRK)=\operatorname{tr}(K^{2}R)&\geq\sum_{i=1}^{n}\lambda_{[i]}(R)\sigma_{[i]}(K^{2})\\&\geq\frac{1}{c_{p,q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\left[\sum_{i=1}^{n}\sigma_{[i]}^{q}(K^{2})\right]^{1/q}\\&\geq\frac{1}{c_{p,q}n^{2-1/q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\left[\operatorname{tr}(K)\right]^{2}.\end{aligned} \tag{3.11}$$

From (2.34), note that $\lambda_{[i]}(U^{T}AU)=\lambda_{[i]}(A)$ and $\lambda_{[i]}(U^{T}A^{T}U)=\lambda_{[i]}(A^{T})$ $(i=1,\ldots,n)$, then we obtain

$$\begin{aligned}\operatorname{tr}(AK)&\leq\sum_{i=1}^{n}\lambda_{[i]}(A)\sigma_{[i]}(K)\leq\lambda_{[1]}(A)\sum_{i=1}^{n}\sigma_{[i]}(K),\\\operatorname{tr}(A^{T}K)&\leq\sum_{i=1}^{n}\lambda_{[i]}(A^{T})\sigma_{[i]}(K)\leq\lambda_{[1]}(A^{T})\sum_{i=1}^{n}\sigma_{[i]}(K).\end{aligned} \tag{3.12}$$

It is easy to see that

$$\begin{aligned}\operatorname{tr}(A^{T}K)+\operatorname{tr}(KA)&\leq\left[\lambda_{[1]}(A^{T})\operatorname{tr}(K)+\lambda_{[1]}(A)\operatorname{tr}(K)\right]\operatorname{tr}(K)\\&=2\lambda_{[1]}\left(\frac{A^{T}+A}{2}\right)\operatorname{tr}(K)=2\lambda_{[1]}(\overline{A})\operatorname{tr}(K).\end{aligned} \tag{3.13}$$

Combine (3.11) and (3.13), we obtain

$$\frac{1}{c_{p,q}n^{2-1/q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\left[\operatorname{tr}(K)\right]^{2}-2\operatorname{tr}(K)\lambda_{[n]}(\overline{A})-\operatorname{tr}(Q)\leq 0. \tag{3.14}$$

Solving (3.14) for $\operatorname{tr}(K)$ yields the right-hand side of the inequality (3.2). Similarly, we can obtain the left-hand side of the inequality (3.2).

(2) Note that when $(A + A^T)/2 \geq 0$, $\lambda_{[i]}(U^T A U) = \lambda_{[i]}(A)$ and $\lambda_{[i]}(U^T A^T U) = \lambda_{[i]}(A^T)$ $(i = 1, \ldots, n)$, by (2.34), (2.6) and (3.7), we have

$$\begin{aligned}
\operatorname{tr}(A^T K) &\leq \left[\sum_{i=1}^{n} \lambda_{[i]}^{p}(A^T)\right]^{1/p} \operatorname{tr}(K), \\
\operatorname{tr}(KA) &\leq \left[\sum_{i=1}^{n} \lambda_{[i]}^{p}(A)\right]^{1/p} \operatorname{tr}(K).
\end{aligned} \tag{3.15}$$

Thus,

$$\begin{aligned}
\operatorname{tr}(A^T K) + \operatorname{tr}(KA) &\leq \left[\left[\sum_{i=1}^{n} \lambda_{[i]}^{p}(A^T)\right]^{1/p} + \left[\sum_{i=1}^{n} \lambda_{[i]}^{p}(A)\right]\right] \operatorname{tr}(K) \\
&\leq 2\left[\sum_{i=1}^{n} \lambda_{[i]}^{p}\left(\frac{A^T + A}{2}\right)\right]^{1/p} \operatorname{tr}(K) \\
&= 2\left[\sum_{i=1}^{n} \lambda_{[i]}^{p}(\overline{A})\right]^{1/p} \operatorname{tr}(K).
\end{aligned} \tag{3.16}$$

From (3.11) and (3.16), with similar argument to (1), we can obtain (3.3) easily.

(3) Note that when $(A + A^T)/2 \leq 0$, by (3.3), we obtain (3.4) immediately. This completes the proof. □

Remark 3.2. From Remark 2.7 and Theorem 3.1, let $p = 2$, $q = 2$ in (3.2), then we obtain

$$\begin{aligned}
&\frac{\lambda_{[n]}(\overline{A}) + \sqrt{\left[\lambda_{[n]}(\overline{A})\right]^2 + \left[\sum_{i=1}^{n} \lambda_{[i]}^{2}(R)\right]^{1/2} \operatorname{tr}(Q)}}{\left[\sum_{i=1}^{n} \lambda_{[i]}^{2}(R)\right]^{1/2}} \\
&\quad \leq \operatorname{tr}(K) \leq \frac{\lambda_{[1]}(\overline{A}) + \sqrt{\left[\lambda_{[1]}(\overline{A})\right]^2 + \left[\left(1/c_1 n^{3/2}\right)\left[\sum_{i=1}^{n} \lambda_{[i]}^{2}(R)\right]^{1/2}\right] \operatorname{tr}(Q)}}{\left(1/c_1 n^{3/2}\right)\left[\sum_{i=1}^{n} \lambda_{[i]}^{2}(R)\right]^{1/2}},
\end{aligned} \tag{3.17}$$

where $c_1 = \left(\sqrt{M_r M_k / m_r m_k} + \sqrt{m_r m_k / M_r M_k}\right)$.

Remark 3.3. From Remark 2.7 and Theorem 3.1, let $p \to \infty$, $q \to 1$ in (3.2), then we obtain (3.1) immediately.

4. Numerical Examples

In this section, firstly, we will give two examples to illustrate that our new trace bounds are better than the recent results. Then, to illustrate the application in the algebraic Riccati equation of our results will have different superiority if we choose different p and q, we will give two examples when $p = 2$, $q = 2$, and $p \to \infty$, $q \to 1$.

Example 4.1 (see [13]). Now let

$$A = \begin{pmatrix} 0.9140 & 0.6989 & 0.6062 \\ 0.2309 & 0.0169 & 0.04501 \\ 0.3471 & 0.5585 & 0.0304 \end{pmatrix}, \quad B = \begin{pmatrix} 0.9892 & 0.1140 & 0.1233 \\ 0.0410 & 0.3096 & 0.5125 \\ 0.0476 & 0.7097 & 0.0962 \end{pmatrix}. \tag{4.1}$$

Neither A nor B is symmetric. In this case, the results of [6–12] are not valid.

Using (1.9) we obtain

$$0.78 \leq \operatorname{tr}(AB) \leq 1.97. \tag{4.2}$$

Using (1.11) yields

$$0.8611 \leq \operatorname{tr}(AB) \leq 1.9090. \tag{4.3}$$

By (2.18), we obtain

$$1.0268 \leq \operatorname{tr}(AB) \leq 1.7524, \tag{4.4}$$

where both lower and upper bounds are better than those of (4.2) and (4.3).

Example 4.2. Let

$$A = \begin{pmatrix} 0.0624 & 0.8844 & 0.2782 & 0.0389 \\ 0.7163 & 0.6565 & 0.2923 & 0.5980 \\ 0.5502 & 0.2660 & 0.5486 & 0.3376 \\ 0.1134 & 0.5739 & 0.3999 & 0.2792 \end{pmatrix}, \quad B = \begin{pmatrix} 1.7205 & 0.6542 & 1.3030 & 0.8813 \\ 0.6542 & 0.0631 & 0.6191 & 0.2696 \\ 1.3030 & 0.6191 & 0.4715 & 0.7551 \\ 0.8813 & 0.2696 & 0.7551 & 0.4584 \end{pmatrix}. \tag{4.5}$$

Neither A nor B is symmetric. In this case, the results of [6–12] are not valid.

Using (1.7) yields

$$-6.1424 \leq \operatorname{tr}(AB) \leq 6.1424. \tag{4.6}$$

From (1.9) we have

$$-1.5007 \leq \operatorname{tr}(AB) \leq 5.0110. \tag{4.7}$$

Using (1.11) yields

$$-3.1058 \leq \operatorname{tr}(AB) \leq 6.0736. \tag{4.8}$$

By (1.13), we obtain

$$-0.7267 \leq \operatorname{tr}(AB) \leq 4.3399. \tag{4.9}$$

The bound in (2.18) yields

$$-0.5375 \leq \operatorname{tr}(AB) \leq 4.2659. \tag{4.10}$$

Obviously, (4.10) is tighter than (4.6), (4.7), (4.8) and (4.9).

Example 4.3. Consider the systems (1.1), (1.2) with

$$A = \begin{pmatrix} -5 & -2 & 4 \\ 2 & 3 & -1 \\ 1 & 0 & -3 \end{pmatrix}, \qquad BB^T = \begin{pmatrix} 8 & 2 & 3 \\ 2 & 7 & 4 \\ 3 & 4 & 9 \end{pmatrix}, \qquad Q = \begin{pmatrix} 538 & 440 & 266 \\ 440 & 441 & 321 \\ 266 & 321 & 296 \end{pmatrix}. \tag{4.11}$$

Moreover, the corresponding ARE (1.4) with $R = BB^T$, (A, R) is stabilizable and (Q, A) is observable.

Using (3.17) yields

$$8.5498 \leq \operatorname{tr}(K) \leq 47.9041. \tag{4.12}$$

Using (3.1) we obtain

$$9.0132 \leq \operatorname{tr}(K) \leq 19.0099, \tag{4.13}$$

where both lower and upper bounds are better than those of (4.12).

Example 4.4. Consider the systems (1.1), (1.2) with

$$A = \begin{pmatrix} -6.0 & 1.5 & 2.0 \\ 0.0 & -2.0 & -3.0 \\ 2.5 & 4.0 & -1.5 \end{pmatrix}, \qquad BB^T = \begin{pmatrix} 4.0 & 1.0 & 2.0 \\ 1.0 & 2.0 & 0.5 \\ 2.0 & 0.5 & 2.5 \end{pmatrix}, \qquad Q = \begin{pmatrix} 17.5 & 7.45 & 3.465 \\ 7.45 & 9.7 & 7.845 \\ 3.465 & 7.845 & 9.905 \end{pmatrix}. \tag{4.14}$$

Moreover, the corresponding ARE (1.4) with $R = BB^T$, (A, R) is stabilizable and (Q, A) is observable.

Using (3.1) we obtain

$$1.6039 \leq \operatorname{tr}(K) \leq 5.6548. \tag{4.15}$$

Using (3.17) yields

$$1.6771 \leq \operatorname{tr}(K) \leq 5.5757, \tag{4.16}$$

where both lower and upper bounds are better than those of (4.15).

5. Conclusion

In this paper, we have proposed lower and upper bounds for the trace of the product of two arbitrary real matrices. We have showed that our bounds for the trace are the tightest among the parallel trace bounds in nonsymmetric case. Then, we have obtained the application in the algebraic Riccati equation of our results. Finally, numerical examples have illustrated that our bounds are better than the recent results.

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