Research Article
Conditions for Carathéodory Functions

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The purpose of the present paper is to derive some sufficient conditions for Carathéodory functions in the open unit disk. Our results include several interesting corollaries as special cases.

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1. Introduction

Let $\mathcal{P}$ be the class of functions $p$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. If $p$ in $\mathcal{P}$ satisfies

$$\text{Re}\{p(z)\} > 0 \quad (z \in U),$$

then we say that $p$ is the Carathéodory function.

Let $\mathcal{A}$ denote the class of all functions $f$ analytic in the open unit disk $U = \{ z : |z| < 1 \}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f < g$ or $f(z) < g(z)$, if $g$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $0 < \alpha \leq 1$, let $\mathcal{S}_{\mathcal{C}}(\alpha)$ and $\mathcal{S}_{\mathcal{S}}(\alpha)$ denote the classes of functions $f \in \mathcal{A}$ which are strongly convex and starlike of order $\alpha$; that is, which satisfy

$$1 + \frac{zf''(z)}{f'(z)} < \left( \frac{1+z}{1-z} \right)^{\alpha} \quad (z \in U),$$

$$\frac{zf'(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^{\alpha} \quad (z \in U),$$

(1.3) (1.4)
respectively. We note that (1.3) and (1.4) can be expressed, equivalently, by the argument functions. The classes $\mathcal{STC}(\alpha)$ and $\mathcal{STC}(\alpha)$ were introduced by Brannan and Kirwan [1] and studied by Mocanu [2] and Nunokawa [3, 4]. Also, we note that if $\alpha = 1$, then $\mathcal{STC}(\alpha)$ coincides with $\mathcal{S}^*$, the well-known class of starlike (univalent) functions with respect to origin, and if $0 < \alpha < 1$, then $\mathcal{STC}(\alpha)$ consists only of bounded starlike functions [1], and hence the inclusion relation $\mathcal{STC}(\alpha) \subset \mathcal{S}^*$ is proper. Furthermore, Nunokawa and Thomas [4] (see also [5]) found the value $\beta(\alpha)$ such that $\mathcal{STC}(\beta(\alpha)) \subset \mathcal{STC}(\alpha)$.

In the present paper, we consider general forms which cover the results by Mocanu [6] and Nunokawa and Thomas [4]. An application of a certain integral operator is also considered. Moreover, we give some sufficient conditions for univalent (close-to-convex) and (strongly) starlike functions (of order $\beta$) as special cases of main results.

2. Main Results

To prove our results, we need the following lemma due to Nunokawa [3].

**Lemma 2.1.** Let $p$ be analytic in $\mathbb{U}$, $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

\[
\begin{align*}
|\arg p(z)| &< \frac{\pi}{2} \alpha \quad \text{for } |z| < |z_0|, \\
|\arg p(z_0)| &> \frac{\pi}{2} \alpha \quad (0 < \alpha \leq 1).
\end{align*}
\]

Then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \alpha k,
\]

where

\[
\begin{align*}
k &\geq \frac{1}{2} \left( x + \frac{1}{x} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2} \alpha, \\
k &\leq -\frac{1}{2} \left( x + \frac{1}{x} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2} \alpha,
\end{align*}
\]

\[
\{p(z_0)\}^{1/\alpha} = \pm ix \quad (x > 0).
\]

With the help of Lemma 2.1, we now derive the following theorem.

**Theorem 2.2.** Let $p$ be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$ and let $p$ satisfy the differential equation

\[
\etazp'(z) + B(z)p(z) = a + ibA(z),
\]

where $\eta > 0$, $a \in \mathbb{R}^+$, $0 \leq b \leq a \tan(\pi/2)\alpha$, $0 < \alpha < 1$, $A(z) = \text{sign} (\text{Im } p(z))$ and $B(z)$ is analytic in $\mathbb{U}$ with $B(0) = a$. If

\[
|\arg B(z)| < \frac{\pi}{2} \beta(\eta, \alpha, a, b) \quad (z \in \mathbb{U}),
\]
where

\[
\beta(\eta, a, b) = \tan^{-1}\left\{ \frac{S(\alpha)T(\alpha)(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta a}{S(\alpha)T(\alpha)(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\},
\]

\[
S(\alpha) = (1 + a)^{(1-a)/2}, \quad T(\alpha) = (1 - a)^{(1-a)/2},
\]

then

\[
|\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

**Proof.** If there exists a point \(z_0 \in U\) such that the conditions (2.1) are satisfied, then (by Lemma 2.1) we obtain (2.2) under the restrictions (2.3). Then we obtain

\[
A(z_0) = \begin{cases} 
1, & \text{if } p(z_0) = (ix)^a, \\
-1, & \text{if } p(z_0) = (-ix)^a,
\end{cases}
\]

\[
B(z_0) = \frac{a + ibA(z_0)}{p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} = (a + ibA(z_0))(\pm ix)^{-a} - \eta \alpha k
\]

\[
= \left(\frac{a}{x^a} \cos \frac{\pi}{2} \alpha + \frac{b}{x^a} A(z_0) \sin \left(\pm \frac{\pi}{2} \alpha \right)\right) + i \left(\frac{b}{x^a} A(z_0) \cos \frac{\pi}{2} \alpha - \frac{a}{x^a} \sin \left(\pm \frac{\pi}{2} \alpha \right) - \eta \alpha k\right).
\]

Now we suppose that

\[
[p(z_0)]^{1/a} = ix \quad (x > 0).
\]

Then we have

\[
\arg B(z_0) = -\tan^{-1}\left\{ \frac{a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha + \eta \alpha x^a k}{a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha} \right\},
\]

where

\[
kx^a \geq \frac{1}{2}(x^{a+1} + x^{a-1}) \equiv g(x) \quad (x > 0).
\]
Then, by a simple calculation, we see that the function \( g(x) \) takes the minimum value at \( x = \sqrt{(1 - \alpha)/(1 + \alpha)} \). Hence, we have

\[
\arg B(z_0) \leq -\tan^{-1}\left\{ \frac{(1 + \alpha)^{(1+a)/2}(1 - \alpha)^{(1-a)/2}(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta\alpha}{(1 + \alpha)^{(1+a)/2}(1 - \alpha)^{(1-a)/2}(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\}
\]

\[
= -\frac{\pi}{2} \beta(\eta, \alpha, a, b),
\]

where \( \beta(\eta, \alpha, a, b) \) is given by (2.6). This evidently contradicts the assumption of Theorem 2.2.

Next, we suppose that

\[
\{p(z_0)\}^{1/\alpha} = -ix \quad (x > 0).
\]

Applying the same method as the above, we have

\[
\arg B(z_0) \geq \tan^{-1}\left\{ \frac{(1 + \alpha)^{(1+a)/2}(1 - \alpha)^{(1-a)/2}(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta\alpha}{(1 + \alpha)^{(1+a)/2}(1 - \alpha)^{(1-a)/2}(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\}
\]

\[
= \frac{\pi}{2} \beta(\eta, \alpha, a, b),
\]

where \( \beta(\eta, \alpha, a, b) \) is given by (2.6), which is a contradiction to the assumption of Theorem 2.2. Therefore, we complete the proof of Theorem 2.2. \( \square \)

**Corollary 2.3.** Let \( f \in \mathcal{A} \) and \( \eta > 0 \), \( 0 < \alpha < 1 \). If

\[
\left| \arg \left\{ (1 - \eta) \frac{zf'(z)}{f(z)} + \eta\left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{\pi}{2} \beta(\eta, \alpha) \quad (z \in U),
\]

where \( \beta(\eta, \alpha) \) is given by (2.6) with \( a = 1 \) and \( b = 0 \), then \( f \in \mathcal{S}\mathcal{T}\mathcal{S}(\alpha) \).

**Proof.** Taking

\[
p(z) = \frac{f(z)}{zf'(z)}, \quad B(z) = (1 - \eta) \frac{zf'(z)}{f(z)} + \eta\left( 1 + \frac{zf''(z)}{f'(z)} \right)
\]

in Theorem 2.2, we can see that (2.4) is satisfied. Therefore, the result follows from Theorem 2.2. \( \square \)

**Corollary 2.4.** Let \( f \in \mathcal{A} \) and \( 0 < \alpha < 1 \). Then \( \mathcal{S}\mathcal{T}\mathcal{C}(\beta(\alpha)) \subset \mathcal{S}\mathcal{T}\mathcal{S}(\alpha) \), where \( \beta(\alpha) \) is given by (2.6) with \( \eta = a = 1 \) and \( b = 0 \).
By a similar method of the proof in Theorem 2.2, we have the following theorem.

**Theorem 2.5.** Let \( p \) be nonzero analytic in \( U \) with \( p(0) = 1 \) and let \( p \) satisfy the differential equation

\[
\frac{zp'(z)}{p(z)} + B(z) = a + ibA(z), \tag{2.18}
\]

where \( a \in \mathbb{R}^+, b \in \mathbb{R}^- \cup \{0\} \), \( A(z) = \text{sign}(\text{Im } p(z)) \), and \( B(z) \) is analytic in \( U \) with \( B(0) = a \). If

\[
|\arg B(z)| < \frac{\pi}{2}a(\delta, a, b) \quad (z \in \mathbb{U}), \tag{2.19}
\]

where

\[
a(\delta) := a(\delta, a, b) = \frac{2}{\pi} \tan^{-1} \frac{\delta - b}{a} \quad (\delta > 0), \tag{2.20}
\]

then

\[
|\arg p(z)| < \frac{\pi}{2} \delta \quad (z \in \mathbb{U}). \tag{2.21}
\]

**Corollary 2.6.** Let \( f \in TS(\alpha(\delta)) \), where \( \alpha(\delta) \) is given by (2.20) with \( a = 1 \) and \( b = 0 \). Then

\[
|\arg \frac{f(z)}{z}| < \frac{\pi}{2} \delta \quad (z \in \mathbb{U}). \tag{2.22}
\]

**Proof.** Letting

\[
p(z) = \frac{z}{f(z)}, \quad B(z) = \frac{zf'(z)}{f(z)} \tag{2.23}
\]

in Theorem 2.5, we have Corollary 2.6 immediately. \( \square \)

If we combine Corollaries 2.4 and 2.6, then we obtain the following result obtained by Nunokawa and Thomas [4].

**Corollary 2.7.** Let \( f \in STC(\beta(\delta)) \), where

\[
\beta(\delta) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2}a(\delta) + \frac{\alpha(\delta)}{(1 + \alpha(\delta))(1 + \alpha(\delta))/2(1 - \alpha(\delta))(1 - \alpha(\delta))/2} \cos \left( \frac{\pi}{2} \alpha(\delta) \right) \right\} \tag{2.24}
\]

and \( \alpha(\delta) \) is given by (2.20). Then

\[
|\arg \frac{f(z)}{z}| < \frac{\pi}{2} \delta \quad (z \in \mathbb{U}). \tag{2.25}
\]
Corollary 2.8. Let \( f \in \mathcal{A} \), \( 0 < \alpha < 1 \) and \( \beta, \gamma \) be real numbers with \( \beta \neq 0 \) and \( \beta + \gamma > 0 \). If

\[
\left| \arg \left( \frac{\beta zf'(z)}{f(z)} + \gamma \right) \right| < \frac{\pi}{2} \delta(\alpha, \beta, \gamma) \quad (z \in U),
\]

where

\[
\delta(\alpha, \beta, \gamma) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} \alpha + \frac{\alpha}{(\beta + \gamma)(1 + \alpha)(1 - \alpha)(1 - \alpha)^{1/2} \cos(\pi/2)\alpha} \right\},
\]

then

\[
\left| \arg \left( \frac{\beta zF'(z)}{F(z)} + \gamma \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U),
\]

where \( F \) is the integral operator defined by

\[
F(z) = \left( \frac{\beta + \gamma}{z^\beta} \int_0^z f^\beta(t)t^{\beta-1} dt \right)^{1/\beta} \quad (z \in U).
\]

Proof. Let

\[
B(z) = \frac{1}{\beta + \gamma} \left( \beta zf'(z) + \gamma f(z) \right),
\]

\[
p(z) = \frac{\beta + \gamma}{z^{\beta} f^\beta(z)} \int_0^z f^\beta(t)t^{\beta-1} dt.
\]

Then \( B(z) \) and \( p(z) \) are analytic in \( U \) with \( B(0) = p(0) = 1 \). By a simple calculation, we have

\[
\frac{1}{\beta + \gamma} zp'(z) + B(z)p(z) = 1.
\]

Using a similar method of the proof in Theorem 2.2, we can obtain that

\[
\left| \arg p(z) \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

From (2.29) and (3.1), we easily see that

\[
F(z) = f(z) \{ p(z) \}^{1/\beta}.
\]
Since
\[ \beta \frac{zF'(z)}{F(z)} + \gamma = \frac{\beta + \gamma}{p(z)}, \tag{2.35} \]
the conclusion of Corollary 2.8 immediately follows.

Remark 2.9. Letting \( \alpha \to 1 \) in Corollary 2.8, we have the result obtained by Miller and Mocanu [7].

The proof of the following theorem below is much akin to that of Theorem 2.2 and so we omit for details involved.

**Theorem 2.10.** Let \( p \) be nonzero analytic in \( U \) with \( p(0) = 1 \) and let \( p \) satisfy the differential equation
\[ \frac{zp'(z)}{p(z)} + B(z)p(z) = a + ibA(z), \tag{2.36} \]
where \( a \in \mathbb{R}^+, b \in \mathbb{R}^- \cup \{0\} \), \( A(z) = \text{sign}(\text{Im} \ p(z)) \) and \( B(z) \) is analytic in \( U \) with \( B(0) = a \). If
\[ |\arg B(z)| < \frac{\pi}{2} \beta(\alpha, a, b) \quad (z \in U), \tag{2.37} \]
where
\[ \beta(\alpha, a, b) = \alpha + \frac{2}{\pi} \tan^{-1} \frac{a - b}{a} \quad (0 < \alpha \leq 1), \tag{2.38} \]
then
\[ |\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in U). \tag{2.39} \]

**Corollary 2.11.** Let \( f \in \mathcal{A} \) with \( f'(z) \neq 0 \) in \( U \) and \( 0 < \alpha \leq 1 \). If
\[ |\arg (f'(z) + zf''(z))| < \frac{\pi}{2} \beta(\alpha) \quad (z \in U), \tag{2.40} \]
where \( \beta(\alpha) \) is given by (2.38) with \( a = 1 \) and \( b = 0 \), then
\[ |\arg f'(z)| < \frac{\pi}{2} \alpha \quad (z \in U), \tag{2.41} \]
that is, \( f \) is univalent (close-to-convex) in \( U \).
Proof. Let

\[ p(z) = \frac{1}{f'(z)}, \quad B(z) = f'(z) + zf''(z) \]  \tag{2.42}

in Theorem 2.10. Then (2.36) is satisfied and so the result follows. \qed

By applying Theorem 2.10, we have the following result obtained by Mocanu [6].

**Corollary 2.12.** Let \( f \in \mathcal{A} \) with \( f(z)/z \neq 0 \) and \( a_0 \) be the solution of the equation given by

\[ 2a + \frac{2}{\pi} \tan^{-1} a = 1 \quad (0 < a < 1). \]  \tag{2.43}

If

\[ |\arg f'(z)| < \frac{\pi}{2} (1 - a_0) \quad (z \in U), \]  \tag{2.44}

then \( f \in S^* \).

**Proof.** Let

\[ p(z) = \frac{z}{f(z)}, \quad B(z) = f'(z). \]  \tag{2.45}

Then, by Theorem 2.10, condition (2.44) implies that

\[ \left| \arg \frac{z}{f(z)} \right| < \frac{\pi}{2} a_0. \]  \tag{2.46}

Therefore, we have

\[ \left| \arg \frac{zf'(z)}{f(z)} \right| \leq \left| \arg f'(z) \right| + \left| \arg \frac{z}{f(z)} \right| < \frac{\pi}{2}, \]  \tag{2.47}

which completes the proof of Corollary 2.12. \qed

**Corollary 2.13.** Let \( f \in \mathcal{A} \) with \( f(z)f'(z)/z \neq 0 \) in \( U \) and \( 0 < a \leq 1 \). If

\[ \left| \arg \frac{zf''(z)}{f(z)} - \left( 2 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta(a) \quad (z \in U), \]  \tag{2.48}

where \( \beta(a) \) is given by (2.38), then \( f \in S^* S(a) \).
Finally, we have the following result.

**Theorem 2.14.** Let $p$ be nonzero analytic in $U$ with $p(0) = 1$. If

$$|\arg((1 - \lambda)p(z) + \lambda zp'(z))| < \frac{\pi}{2} \beta(\lambda, \alpha),$$

$$\beta(\lambda, \alpha) = \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \quad (0 \leq \lambda < 1; \ 0 < \alpha < 1),$$

then

$$|\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in U).$$

**Proof.** If there exists a point $z_0 \in U$ satisfying the conditions of Lemma 2.1, then we have

$$(1 - \lambda)p(z_0) + \lambda z_0 p'(z_0) = (\pm ix)^a(1 - \lambda + i\lambda k).$$

Now we suppose that

$$\{p(z_0)\}^{1/a} = ix \quad (x > 0).$$

Then we have

$$\arg((1 - \lambda)p(z_0) + \lambda z_0 p'(z_0)) = \frac{\pi}{2} \alpha + \tan^{-1} \frac{\lambda \alpha}{1 - \lambda}$$

$$\geq \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \right)$$

$$= \frac{\pi}{2} \beta(\lambda, \alpha),$$

where $\beta(\lambda, \alpha)$ is given by (2.50). Also, for the case

$$\{p(z_0)\}^{1/a} = -ix \quad (x > 0),$$

we obtain

$$\arg((1 - \lambda)p(z_0) + \lambda z_0 p'(z_0)) \leq -\frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \right)$$

$$= -\frac{\pi}{2} \beta(\lambda, \alpha),$$

where $\beta(\lambda, \alpha)$ is given by (2.50). These contradict the assumption of Theorem 2.14 and so we complete the proof of Theorem 2.14. \qed
Corollary 2.15. Let $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ in $U$ and $0 < \alpha < 1$. If

$$\left|\arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right\} \right| < \frac{\pi}{2} (\alpha + 1) \quad (z \in U),$$

then $f \in \mathcal{S}_\mathcal{T}_S(\alpha)$.

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