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Research Article

Subordination Results on Subclasses Concerning Sakaguchi Functions

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We derive some subordination results for the subclasses $\mathcal{S}(\alpha,t)$, $\mathcal{T}(\alpha,t)$, $\mathcal{S}_0(\alpha,t)$, and $\mathcal{T}_0(\alpha,t)$ of analytic functions concerning with Sakaguchi functions. Several corollaries and consequences of the main results are also considered.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha,t)$, if it satisfies

$$\operatorname{Re}\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \alpha, \quad |t| \le 1,\ t \ne 1 \tag{1.2}$$

for some $0 \le \alpha < 1$ and for all $z \in \Delta$.

The class $\mathcal{S}(\alpha,t)$ was introduced and studied by Owa et al. [4], where the class $\mathcal{S}(0,-1)$ was introduced by Sakaguchi [5]. Therefore, a function $f(z) \in \mathcal{S}(\alpha,-1)$ is called Sakaguchi function of order α.

We also denote by $\mathcal{T}(\alpha, t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, t)$.

We note that $\mathcal{S}(\alpha, 0) \equiv \mathcal{S}^*(\alpha)$, the usual star-like function of order α and $\mathcal{T}(\alpha, 0) \equiv \mathcal{K}(\alpha)$ the usual convex function of order α.

We begin by recalling each of the following coefficient inequalities associated with the function classes $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$.

Theorem 1.1 (see [4]). *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{|n - u_n| + (1-\alpha)|u_n|\}|a_n| \le 1 - \alpha, \tag{1.3}$$

where $u_n = 1 + t + t + \cdots + t^{n-1}$ and $0 \le \alpha < 1$, then $f(z) \in \mathcal{S}(\alpha, t)$.

Theorem 1.2 (see [4]). *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n\{|n - u_n| + (1-\alpha)|u_n|\}|a_n| \le 1 - \alpha, \tag{1.4}$$

where $u_n = 1 + t + t + \cdots + t^{n-1}$ and $0 \le \alpha < 1$, then $f(z) \in \mathcal{T}(\alpha, t)$.

In view of Theorems 1.1 and 1.2, Owa et al. [4] defined the subclasses $\mathcal{S}_0(\alpha, t) \subset \mathcal{S}(\alpha, t)$ and $\mathcal{T}_0(\alpha, t) \subset \mathcal{T}(\alpha, t)$, where

$$\begin{aligned} \mathcal{S}_0(\alpha, t) &= \{f(z) \in \mathcal{A} : f(z) \text{ satisfies } (1.3)\}, \\ \mathcal{T}_0(\alpha, t) &= \{f(z) \in \mathcal{A} : f(z) \text{ satisfies } (1.4)\}. \end{aligned} \tag{1.5}$$

Before we state and prove our main results we need the following definitions and lemma.

Definition 1.3 (Hadamard product). Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) $f * g$ is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.6}$$

Definition 1.4 (subordination principle). Let $g(z)$ be analytic and univalent in Δ. If $f(z)$ is analytic in Δ, $f(0) = g(0)$, and $f(\Delta) \subset g(\Delta)$, then we see that the function $f(z)$ is subordinate to $g(z)$ in Δ, and we write $f(z) \prec g(z)$.

Definition 1.5 (subordinating factor sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in Δ, we have the subordination given by

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z) \quad (z \in \Delta,\ a_1 = 1). \tag{1.7}$$

Lemma 1.6 (see [6]). *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \quad (z \in \Delta). \tag{1.8}$$

In this paper, we obtain a sharp subordination results associated with the classes $\mathcal{S}(\alpha,t)$, $\mathcal{T}(\alpha,t)$, $\mathcal{S}_0(\alpha,t)$, and $\mathcal{T}_0(\alpha,t)$ by using the same techniques as in [1, 2, 7, 8].

2. Subordination Results for the Classes $\mathcal{S}_0(\alpha,t)$ and $\mathcal{S}(\alpha,t)$

Theorem 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}_0(\alpha,t)$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. If $\{n|n-u_n| + (1-\alpha)|u_n|\}_{n=2}^{\infty}$ is increasing sequence for all $n \geq 2$, then*

$$\frac{|1-t| + (1-\alpha)|1+t|}{2(|1-t| + (1-\alpha)(1+|1+t|))}(f * g)(z) \prec g(z) \quad (|t| \leq 1,\ t \neq 1;\ 0 \leq \alpha < 1;\ z \in \Delta;\ g \in \mathcal{K}), \tag{2.1}$$

$$\operatorname{Re}(f(z)) > -\frac{|1-t| + (1-\alpha)(1+|1+t|)}{|1-t| + (1-\alpha)|1+t|} \quad (z \in \Delta). \tag{2.2}$$

The constant $(|1-t| + (1-\alpha)|1+t|)/2(|1-t| + (1-\alpha)(1+|1+t|))$ is the best estimate.

Proof. Let $f(z) \in \mathcal{S}_0(\alpha,t)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\frac{|1-t| + (1-\alpha)|1+t|}{2(|1-t| + (1-\alpha)(1+|1+t|))}(f * g)(z) = \frac{|1-t| + (1-\alpha)|1+t|}{2(|1-t| + (1-\alpha)(1+|1+t|))}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right). \tag{2.3}$$

Thus, by Definition 1.5, the assertion of our theorem will hold if the sequence

$$\left\{\frac{|1-t| + (1-\alpha)|1+t|}{2(|1-t| + (1-\alpha)(1+|1+t|))} a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.6, this will be the case if and only if

$$\operatorname{Re}\left\{1+\sum_{n=1}^{\infty}\frac{|1-t|+(1-\alpha)|1+t|}{|1-t|+(1-\alpha)(1+|1+t|)}a_n z^n\right\}>0 \quad (z\in\Delta). \tag{2.5}$$

Now

$$\begin{aligned}
&\operatorname{Re}\left\{1+\frac{|1-t|+(1-\alpha)|1+t|}{|1-t|+(1-\alpha)(1+|1+t|)}\sum_{n=1}^{\infty}a_n z^n\right\}\\
&\quad=\operatorname{Re}\left\{1+\frac{|1-t|+(1-\alpha)|1+t|}{|1-t|+(1-\alpha)(1+|1+t|)}z+\frac{1}{|1-t|+(1-\alpha)(1+|1+t|)}\sum_{n=2}^{\infty}|1-t|\right.\\
&\qquad\left.+(1-\alpha)|1+t|a_n z^n\right\}\\
&\quad\geq 1-\frac{|1-t|+(1-\alpha)|1+t|}{|1-t|+(1-\alpha)(1+|1+t|)}r-\frac{1}{|1-t|+(1-\alpha)(1+|1+t|)}\sum_{n=2}^{\infty}|n-u_n|\\
&\qquad+(1-\alpha)|u_n||a_n|r^n\\
&\quad>1-\frac{|1-t|+(1-\alpha)|1+t|}{|1-t|+(1-\alpha)(1+|1+t|)}r-\frac{1-\alpha}{|1-t|+(1-\alpha)(1+|1+t|)}r\\
&\quad>0, \quad (|z|=r<1).
\end{aligned} \tag{2.6}$$

Thus (2.5) holds true in Δ. This proves inequality (2.1). Inequality (2.2) follows by taking the convex function $g(z)=z/(1-z)=z+\sum_{n=2}^{\infty}z^n$ in (2.1). To prove the sharpness of the constant $(|1-t|+(1-\alpha)|1+t|)/(2(|1-t|+(1-\alpha)(1+|1+t|)))$, we consider the function $f_0(z)\in\mathcal{S}_0(\alpha,t)$ given by

$$f_0(z)=z-\frac{1-\alpha}{|1-t|+(1-\alpha)|1+t|}z^2 \quad (0\leq\alpha<1). \tag{2.7}$$

Thus from (2.1), we have

$$\frac{|1-t|+(1-\alpha)|1+t|}{2(|1-t|+(1-\alpha)(1+|1+t|))}f_0(z)\prec\frac{z}{1-z}. \tag{2.8}$$

It can easily verified that

$$\min\left\{\operatorname{Re}\left(\frac{|1-t|+(1-\alpha)|1+t|}{2(|1-t|+(1-\alpha)(1+|1+t|))}f_0(z)\right)\right\}=-\frac{1}{2} \quad (z\in\Delta). \tag{2.9}$$

This shows that the constant $(|1-t|+(1-\alpha)|1+t|)/(2(|1-t|+(1-\alpha)(1+|1+t|)))$ is best possible. □

Corollary 2.2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}(\alpha, t)$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. If $\{|n - u_n| + (1-\alpha)|u_n|\}_{n=2}^{\infty}$ is increasing sequence for all $n \geq 2$, then* (2.1) *and* (2.2) *of Theorem 2.1 hold true. Furthermore, the constant $(|1-t| + (1-\alpha)|1+t|)/(2(|1-t| + (1-\alpha)(1+|1+t|)))$ is the best estimate.*

Letting $t = -1$ in Corollary 2.2, we have the following.

Corollary 2.3. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}(\alpha, -1)$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then*

$$\frac{1}{3-\alpha}(f * g)(z) \prec g(z) \quad (0 \leq \alpha < 1;\ z \in \Delta;\ g \in \mathcal{K}),$$
$$\operatorname{Re}(f(z)) > -\frac{3-\alpha}{2} \quad (z \in \Delta). \tag{2.10}$$

The constant $1/(3-\alpha)$ is the best estimate.

Letting $t = 0$ in Corollary 2.2, we have the following result obtained by Ali et al. [1] and Frasin [2].

Corollary 2.4 (see [1, 2]). *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}(\alpha)$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then*

$$\frac{2-\alpha}{2(3-2\alpha)}(f * g)(z) \prec g(z) \quad (0 \leq \alpha < 1;\ z \in \Delta;\ g \in \mathcal{K}),$$
$$\operatorname{Re}(f(z)) > -\frac{3-2\alpha}{2-\alpha} \quad (z \in \Delta). \tag{2.11}$$

The constant $(2-\alpha)/2(3-2\alpha)$ is the best estimate.

Letting $\alpha = 0$ in Corollary 2.4, we have the following result obtained by Singh [3].

Corollary 2.5 (see [3]). *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}^*$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then*

$$\frac{1}{3}(f * g)(z) \prec g(z) \quad (z \in \Delta;\ g \in \mathcal{K}),$$
$$\operatorname{Re}(f(z)) > -\frac{3}{2} \quad (z \in \Delta). \tag{2.12}$$

The constant $1/3$ is the best estimate.

3. Subordination Results for the Classes $\mathcal{T}_0(\alpha, t)$ and $\mathcal{T}(\alpha, t)$

By applying Theorem 1.2 instead of Theorem 1.1, the proof of the next theorem is much akin to that of Theorem 2.1.

Theorem 3.1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{T}_0(\alpha,t)$. *Also let* $\mathcal{K}$ *denote the familiar class of functions* $f(z) \in \mathcal{A}$ *which are also univalent and convex in* Δ. *If* $\{(|n-u_n|+(1-\alpha)|u_n|)\}_{n=2}^{\infty}$ *is increasing sequence for all* $n \geq 2$, *then*

$$\frac{|1-t|+(1-\alpha)|1+t|}{2|1-t|+(1-\alpha)(1+2|1+t|)}(f*g)(z) \prec g(z) \quad (|t|\leq 1,\ t\neq 1;\ 0\leq\alpha<1;\ z\in\Delta;\ g\in\mathcal{K}), \tag{3.1}$$

$$\operatorname{Re}(f(z)) > -\frac{2|1-t|+(1-\alpha)(1+2|1+t|)}{2(|1-t|+(1-\alpha)|1+t|)} \quad (z\in\Delta). \tag{3.2}$$

The constant $(|1-t|+(1-\alpha)|1+t|)/(2|1-t|+(1-\alpha)(1+2|1+t|))$ *is the best estimate.*

Corollary 3.2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{T}(\alpha,t)$. *Also let* $\mathcal{K}$ *denote the familiar class of functions* $f(z) \in \mathcal{A}$ *which are also univalent and convex in* Δ. *If* $\{n|n-u_n|+(1-\alpha)|u_n|\}_{n=2}^{\infty}$ *is increasing sequence for all* $n\geq 2$, *then* (3.1) *and* (3.2) *of Theorem 3.1 hold true. Furthermore, the constant* $(|1-t|+(1-\alpha)|1+t|)/(2|1-t|+(1-\alpha)(1+2|1+t|))$ *is the best estimate.*

Letting $t=-1$ in Corollary 3.2, we have the following.

Corollary 3.3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{T}(\alpha,-1)$. *Also let* $\mathcal{K}$ *denote the familiar class of functions* $f(z)\in\mathcal{A}$ *which are also univalent and convex in* Δ. *Then*

$$\begin{gathered}\frac{2}{5-\alpha}(f*g)(z)\prec g(z) \quad (0\leq\alpha<1;\ z\in\Delta;\ g\in\mathcal{K}),\\ \operatorname{Re}(f(z)) > -\frac{5-\alpha}{4} \quad (z\in\Delta).\end{gathered} \tag{3.3}$$

The constant $2/(5-\alpha)$ *is the best estimate.*

Letting $t=0$ in Corollary 3.2, we have the following result obtained by Ali et al. [1], andFrasin [2] (see also [9]).

Corollary 3.4 (see [1]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{T}(\alpha,0)$. *Also let* $\mathcal{K}$ *denote the familiar class of functions* $f(z)\in\mathcal{A}$ *which are also univalent and convex in* Δ. *Then*

$$\begin{gathered}\frac{2-\alpha}{5-3\alpha}(f*g)(z)\prec g(z) \quad (0\leq\alpha<1;\ z\in\Delta;\ g\in\mathcal{K})\\ \operatorname{Re}(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z\in\Delta).\end{gathered} \tag{3.4}$$

The constant $(2-\alpha)/(5-3\alpha)$ *is the best estimate.*

Letting $\alpha = 0$ in Corollary 3.4, we have the following result obtained by Özkan [9].

Corollary 3.5 (see [9]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{K}$. *Then*

$$\frac{2}{5}(f * g)(z) \prec g(z) \quad (z \in \Delta;\ g \in \mathcal{K}),$$
$$\operatorname{Re}(f(z)) > -\frac{5}{4} \quad (z \in \Delta). \tag{3.5}$$

The constant $2/5$ *is the best estimate.*

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