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*Research Article*

# Derivatives of Integrating Functions for Orthonormal Polynomials with Exponential-Type Weights

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Let $w_\rho(x) := |x|^\rho \exp(-Q(x))$, $\rho > -1/2$, where $Q \in C^2 : (-\infty, \infty) \to [0, \infty)$ is an even function. In 2008 we have a relation of the orthonormal polynomial $p_n(w_\rho^2; x)$ with respect to the weight $w_\rho^2(x)$; $p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n p_n(x)/x$, where $A_n(x)$ and $B_n(x)$ are some integrating functions for orthonormal polynomials $p_n(w_\rho^2; x)$. In this paper, we get estimates of the higher derivatives of $A_n(x)$ and $B_n(x)$, which are important for estimates of the higher derivatives of $p_n(w_\rho^2; x)$.



## 1. Introduction and Results

Let $\mathbb{R} = (-\infty, \infty)$. Let $Q \in C^2 : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ be an even function, and let $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. For $\rho > -1/2$, we set

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree $n$ with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x)p_{m,\rho}(x)w_\rho^2(x)dx = \delta_{mn} \quad \text{(Kronecker's delta)},$$
$$p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0. \tag{1.2}$$

A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of $n$ (or $x$) such that $b_n \leq Cc_n$ for $n$ large enough. We write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote the class of polynomials of degree at most $n$ by $\mathcal{P}_n$.

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of $n$, $x$, $t$, and polynomials of degree at most $n$. The same symbol does not necessarily denote the same constant in different occurrences.

We will be interested in the following subclass of weights from [1].

*Definition 1.1.* Let $Q : \mathbb{R} \rightarrow \mathbb{R}^+$ be even and satisfy the following properties.

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.

(c)

$$\lim_{x \rightarrow \infty} Q(x) = \infty. \tag{1.3}$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.4}$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}. \tag{1.5}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{\left|Q'(x)\right|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{1.6}$$

Then we write $w \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{\left|Q'(x)\right|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \tag{1.7}$$

then we write $w \in \mathcal{F}(C^2+)$.

In the following we introduce useful notations.

(a) Mhaskar-Rahmanov-Saff (MRS) numbers $a_x$ are defined as the positive roots of the following equations:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1 - u^2)^{1/2}} du, \quad x > 0. \tag{1.8}$$

(b) Let

$$\eta_x = (xT(a_x))^{-2/3}, \quad x > 0. \tag{1.9}$$

(c) The function $\varphi_u(x)$ is defined as follows:

$$\varphi_u(x) = \begin{cases} \dfrac{a_{2u}^2 - x^2}{u[(a_u + x + a_u\eta_u)(a_u - x + a_u\eta_u)]^{1/2}}, & |x| \le a_u, \\ \varphi_u(a_u), & a_u < |x|. \end{cases} \tag{1.10}$$

In the rest of this paper we often denote $p_{n,\rho}(x)$ simply by $p_n(x)$. Let $\rho_n = \rho$ if $n$ is odd, $\rho_n = 0$ otherwise and define the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$ as follows:

$$\begin{aligned} A_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n^2(u)\overline{Q(x,u)}w_\rho^2(u)du, \\ B_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n(u)p_{n-1}(u)\overline{Q(x,u)}w_\rho^2(u)du, \end{aligned} \tag{1.11}$$

where $\overline{Q(x,u)} = (Q'(x) - Q'(u))/(x-u)$ and $b_n = \gamma_{n-1}/\gamma_n$. Then in [2, Theorem 4.1] we have a relation of the orthonormal polynomial $p_n(x)$ with respect to the weight $w_\rho^2(x)$:

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n\frac{p_n(x)}{x}, \quad \rho_n = \begin{cases} \rho, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases} \tag{1.12}$$

and in [2, Theorem 4.2] we already have the estimates of the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$. So, in this paper we will estimate the higher derivatives of $A_n(x)$ and $B_n(x)$ for the estimates of the higher derivatives of $p_n(w_\rho^2; x)$, because the higher derivatives of $p_{n,\rho}(x)$ play an important role in approximation theory such as investigating convergence of Hermite-Fejér and Hermite interpolation based on the zeros of $p_n(w_\rho^2; x)$ (see [3, 4]).

To estimate of the higher derivatives of $A_n(x)$ and $B_n(x)$ we need further assumptions for $Q(x)$ as follows.

*Definition 1.2.* Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, and let $\nu$ be a positive integer. Assume that $Q(x)$ is $\nu-$times continuously differentiable on $\mathbb{R}$ and satisfies the followings.

(a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $i = 0, 1, \ldots, \nu+1$ are nonnegative for $x > 0$.

(b) There exist positive constants $C_i > 0$ such that for $x \in \mathbb{R} \setminus \{0\}$

$$\left|Q^{(i+1)}(x)\right| \le C_i\left|Q^{(i)}(x)\right|\frac{\left|Q'(x)\right|}{Q(x)}, \quad i = 1, \ldots \nu. \tag{1.13}$$

(c) There exist constants $0 \le \delta < 1$ and $c_1 > 0$ such that on $(0, c_1]$

$$Q^{(\nu+1)}(x) \le C\left(\frac{1}{x}\right)^{\delta}. \tag{1.14}$$

Then we write $w(x) \in \mathcal{F}_\nu(C^2+)$.

Let $\nu$ be a positive integer. Define for $m + \alpha - \nu > 0$, $m \ge 0$, $l \ge 1$, and $\alpha \ge 0$,

$$Q_{l,\alpha,m}(x) := |x|^m \left(\exp_l\left(|x|^\alpha\right) - \alpha^* \exp_l(0)\right), \tag{1.15}$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$ and define

$$Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1. \tag{1.16}$$

Here we let $\exp_0(x) := x$ and for $l \ge 1$, $\exp_l(x) := \exp(\exp(\cdots(\exp(x))\cdots))$ denotes the $l$th iterated exponential. In particular, $\exp_l(x) = \exp(\exp_{l-1}(x))$. Then $\exp(-Q_{l,\alpha,m}(x))$ and $\exp(-Q_\alpha(x))$ are typical examples of $\mathcal{F}_\nu(C^2+)$ (see [5]).

In the following we improve the inequality (4.3) in [2, Theorem 4.2].

**Theorem 1.3.** *Let $\rho > -1/2$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. Additionally assume that $Q''(x)$ is nondecreasing. Then for $|x| \le \varepsilon a_n$ with $0 < \varepsilon < 1/2$ one has*

$$|B_n(x)| < \lambda(\varepsilon, n) A_n(x), \tag{1.17}$$

*where*

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lambda(\varepsilon, n) = 0. \tag{1.18}$$

In this paper our main theorem is as follows.

**Theorem 1.4.** *Let $\rho > -1/2$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}_\nu(C^2+)$ for positive integer $\nu \ge 2$. Assume that $1 + 2\rho - \delta \ge 0$ for $\rho < 0$ and*

$$a_n \lesssim n^{1/(1+\nu-\delta)}, \tag{1.19}$$

*where $0 \le \delta < 1$ is defined in (1.14).*

(a) *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_2, \infty)$, then one has for $|x| \le a_n(1 + \eta_n)$ and $j = 0, \ldots, \nu - 1$*

$$\left|A_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j. \tag{1.20}$$

*Moreover, for any* $0 < \varepsilon < 1/2$ *there exists* $\varepsilon^*(\varepsilon, n) > 0$ *such that for* $|x| \le \varepsilon a_n$ *and* $j = 1, \ldots, \nu - 1$*,*

$$\left|A_n^{(j)}(x)\right| \le \varepsilon^*(\varepsilon, n) A_n(x) \left(\frac{n}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \le \varepsilon^*(\varepsilon, n) A_n(x) \left(\frac{n}{a_n}\right)^j, \tag{1.21}$$

*with* $\varepsilon^*(\varepsilon, n) \to 0$ *as* $n \to \infty$*.*

(b) *If* $Q^{(\nu+1)}(x)$ *is non-decreasing on* $[c_2, \infty)$*, then one has* (1.20) *and* (1.21) *for the respective ranges of* $x$*.*

(c) *If there exists a constant* $0 \le \delta < 1$ *such that* $Q^{(\nu+1)}(x) \le C(1/x)^\delta$ *on* $[c_2, \infty)$*, then one has* (1.20) *and* (1.21) *for the respective ranges of* $x$*.*

The examples satisfying the conditions (a), (b), or (c) of Theorem 1.4 are given in [5].

*Remark 1.5.* Under the assumptions of Theorem 1.4, we have from [2, Theorem 4.2] that there exists $C, n_0 > 0$ such that for $n \ge n_0$ and $|x| \le a_n(1 + L\eta_n)$,

$$\frac{A_n(x)}{2b_n} \sim \varphi_n(x)^{-1}\left(a_n^2(1 + 2L\eta_n)^2 - x^2\right)^{-1/2}, \quad |B_n(x)| \lesssim A_n(x), \tag{1.22}$$

because $w(x) = \exp(-Q(x)) \in \mathcal{F}_\nu(C^2+)$ for positive integer $\nu \ge 1$ and $1 + 2\rho - \delta \ge 0$ for $\rho < 0$.

In addition, for our future work we estimate $a_t$ and $T(a_t)$ using $\lambda = C_1$ in (1.6) for the weight class $\mathcal{F}(C^2+)$.

**Theorem 1.6.** *Let* $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$*, and we assume*

$$\frac{Q''(x)}{|Q'(x)|} \le \lambda \frac{\left|Q'(x)\right|}{Q(x)}, \quad |x| \ge b > 0, \tag{1.23}$$

*where* $b > 0$ *is large enough.*

(a) *Assume that* $T(x)$ *is unbounded. Then for any* $\eta > 0$ *there exists* $C(\eta) > 0$ *such that for* $t \ge 1$*,*

$$a_t \le C(\eta) t^\eta. \tag{1.24}$$

(b) *Suppose that there exist constants* $\eta > 0$ *and* $C_2 > 0$ *such that* $a_t \le C_2 t^\eta$*. Then there exists a constant* $C$ *depending only on* $\lambda$*,* $\eta$*, and* $C_2$ *such that for* $a_t \ge 1$*, if* $\lambda > 1$

$$T(a_t) \le C t^{2(\eta+\lambda-1)/(\lambda+1)}, \tag{1.25}$$

*and if* $0 < \lambda \le 1$*,*

$$T(a_t) \le C t^\eta. \tag{1.26}$$

*Remark 1.7.* (a) Levin and Lubinsky showed the following [1, Lemma 3.7]: there exists $C > 0$ such that for some $\varepsilon > 0$, and for large enough $t$,

$$T(a_t) \le Ct^{2-\varepsilon}. \tag{1.27}$$

If from (1.25) and (1.26) we set for any $0 < \eta < 2$

$$\varepsilon = \begin{cases} 2-\eta, & 0 < \lambda \le 1, \\ \dfrac{2(2-\eta)}{(\lambda+1)}, & \lambda > 1, \end{cases} \tag{1.28}$$

then we have (1.27) in Levin and Lubinsky's lemma.

(b) If $T(x)$ is unbounded, then (1.19) is trivially satisfied by (1.24).

## 2. Proof of Theorems

In this section we will prove the theorems of Section 1.

**Lemma 2.1.** *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then uniformly for $n \ge 1$,*

(a)

$$\sup_{x\in\mathbb{R}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \left|x^2 - a_n^2\right|^{1/4} \sim 1. \tag{2.1}$$

(b)

$$\sup_{x\in\mathbb{R}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \sim a_n^{-1/2}(nT(a_n))^{1/6}. \tag{2.2}$$

(c) *Markov inequality. Let $0 < p \le \infty$. For any polynomial $P \in \mathcal{P}_n$*

$$\left\|(P'w)(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})} \lesssim \frac{nT(a_n)^{1/2}}{a_n} \left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})}. \tag{2.3}$$

(d) *Let $\beta \in \mathbb{R}$, $0 < p \le \infty$, and $r > 1$. Then there exist positive constants $L$, $\delta$, and $C_2$ such that for any polynomial $P \in \mathcal{P}_n$*

$$\begin{aligned} &\left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\beta}\right\|_{L_p(a_{rn}\le|x|)} \\ &\quad\lesssim \exp\left(-C_2 n^{\delta}\right) \left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\beta}\right\|_{L_p(La_n/n\le|x|\le a_n(1-L\eta_n))}. \end{aligned} \tag{2.4}$$

*Proof.* (a) follows from [2, Theorem 2.3]. (b) follows from [2, Theorem 2.4]. (c) follows form [6, Theorem 2.1(b)]. (d) follows form [6, Theorem 2.3]. □

**Lemma 2.2.** *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then one has for $c > 0$,*

$$\int_{0\le u\le c} (p_n w_\rho)^2(u)du \lesssim \frac{1}{a_n}. \tag{2.5}$$

*Proof.* For $\rho \ge 0$, the results are immediate from Lemma 2.1(a). So we assume $-1/2 < \rho < 0$. First we see

$$\begin{aligned}\int_{0\le u\le a_n/n} (p_n w_\rho)^2(u)du &= \int_{0\le u\le a_n/n} (p_n w)^2(u)\left(|u| + \frac{a_n}{n}\right)^{2\rho} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}}du \\ &\le C\frac{1}{a_n}\int_{0\le u\le a_n/n} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}}du \\ &\le C\frac{1}{a_n}\left(\frac{n}{a_n}\right)^{2\rho}\int_{0\le u\le a_n/n} |u|^{2\rho}du \\ &\le C\frac{1}{a_n}\left(\frac{n}{a_n}\right)^{2\rho}\left(\frac{a_n}{n}\right)^{1+2\rho} \\ &\le C\frac{1}{n},\end{aligned} \tag{2.6}$$

because we know that $a_n = o(n)$ from [1, Lemma 3.5(c)]. Next we see by Lemma 2.1(a)

$$\int_{a_n/n\le u\le c} (p_n w_\rho)^2(u)du \le C\frac{1}{a_n}. \tag{2.7}$$

Therefore, we have the result. □

**Lemma 2.3.** *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then*

(a) *one has*

$$\int_{0\le u\le\infty} (p_n w)^2(u)\left(|u| + \frac{a_n}{n}\right)^{2\rho} Q'(u)du \sim \frac{n}{a_n}, \tag{2.8}$$

(b) *for $x \in [0, a_n/2]$ one has*

$$Q'(x) \le C\frac{n}{a_n}\left(\frac{x}{a_n}\right)^{\Lambda-1}. \tag{2.9}$$

*Proof.* (a) It is from [2, Lemma 4.3(d)]. (b) It is from [1, Lemma 3.8 (3.42)]. □

*Proof of Theorem 1.3.* Since $B_n(x)$ is an odd function, we prove only for $0 \le x \le \varepsilon a_n$. Let $\theta := \varepsilon^{(\Lambda-1)/2\Lambda}$. Then we have the following two lemmas.

**Lemma 2.4.** *Uniformly for $\theta$ and $n$*

$$\left|\int_{|u|\le\theta a_n} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.10}$$

*Proof.* For $|u| \le \theta a_n$, we have by Lemma 2.1(a)

$$p_n^2(u)w_\rho^2(u) \lesssim \frac{1}{\sqrt{a_n^2-(\theta a_n)^2}}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}} \lesssim \frac{1}{a_n}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}}. \tag{2.11}$$

Since $Q''(x)$ is nondecreasing and $1-(1/2)^{(\Lambda+1)/2\Lambda} \le (\theta-\varepsilon)/\theta \le 1$, we have using Lemma 2.3(b):

$$\overline{Q(x,u)} \le \frac{Q'(\theta a_n)-Q'(x)}{\theta a_n - x} \lesssim \frac{Q'(\theta a_n)}{(\theta-\varepsilon)a_n} \lesssim \theta^{\Lambda-2}\frac{n}{a_n^2}. \tag{2.12}$$

Moreover we know that for $\rho > -1/2$,

$$\int_0^{\theta a_n}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}}dx = \int_{|u|\le a_n/n} + \int_{a_n/n\le|u|\le\theta a_n} \lesssim \frac{a_n}{n}+\theta a_n. \tag{2.13}$$

Therefore, we have

$$\left|\int_{|u|\le\theta a_n} p_n^2(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.14}$$

Consequently, we have the result using Cauchy-Schwartz inequality

$$\left|\int_{|u|\le\theta a_n} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.15}$$

□

**Lemma 2.5.** *Uniformly for $\theta = \varepsilon^{(\Lambda-1)/2\Lambda}$ and for $n$*

$$\left|\int_{\theta a_n\le|u|\le a_{2n}} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\varepsilon^{(1-1/\Lambda)(\Lambda-1)}+\varepsilon^{1/\Lambda}\right)\frac{n}{a_n^2}. \tag{2.16}$$

*Proof.* For $\theta a_n \leq |u| \leq a_{2n}$, we have similarly to [2, (4.6)]

$$\begin{aligned}
\left|\overline{Q(x,u)} - \overline{Q(x,-u)}\right| &= 2\left|\frac{uQ'(x) - xQ'(u)}{x^2 - u^2}\right| \\
&\lesssim \frac{a_n\left|Q'(\varepsilon a_n)\right| + \varepsilon a_n\left|Q'(u)\right|}{(\theta a_n)^2} \\
&\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2} + \frac{\varepsilon^{1/\Lambda}}{a_n}\left|Q'(u)\right|
\end{aligned} \tag{2.17}$$

(see Lemma 2.3(b)). Therefore, we have by Lemma 2.3(a),

$$\begin{aligned}
&\left|\int_{\theta a_n \leq |u| \leq a_{2n}} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \\
&\quad\leq \int_{\theta a_n \leq |u| \leq a_{2n}} \left|p_n(u)p_{n-1}(u)w_\rho^2(u)\right|\left|\overline{Q(x,u)} - \overline{Q(x,-u)}\right|du \\
&\quad\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2}\int_{\theta a_n \leq |u| \leq a_{2n}} \left|p_n(u)p_{n-1}(u)\right|w_\rho^2(u)du \\
&\qquad+ \frac{\varepsilon^{1/\Lambda}}{a_n}\int_{\theta a_n \leq |u| \leq a_{2n}} \left|p_n(u)p_{n-1}(u)\right|w_\rho^2(u)\left|Q'(u)\right|du \\
&\quad\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2} + \varepsilon^{1/\Lambda}\frac{n}{a_n^2}.
\end{aligned} \tag{2.18}$$

Here we used Lemma 2.1(b). □

Since for a constant $C > 0$

$$\left|\int_{a_{2n} \leq |u|} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim O\left(e^{-n^C}\right), \tag{2.19}$$

(see [2, page 233]), there exists $\lambda(n) > 0$ such that

$$\left|\int_{a_{2n} \leq |u|} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \lambda(n)\frac{n}{a_n^2}, \tag{2.20}$$

and $\lambda(n) \to 0$ as $n \to \infty$. We know from [2, Lemma 4.7] that $b_n = \gamma_{n-1}/\gamma_n \sim a_n$. From (1.22) we have $A_n(x)/b_n \sim n/a_n^2$ for $|x| \leq \varepsilon a_n$ and from the preceding considerations and the definition of $B_n(x)$ it follows that for $|x| \leq \varepsilon a_n$

$$\frac{|B_n(x)|}{b_n} \lesssim \frac{\lambda(\varepsilon,n)n}{a_n^2} \sim \frac{\lambda(\varepsilon,n)A_n(x)}{b_n}, \tag{2.21}$$

where for some positive constant $C > 0$

$$\lambda(\varepsilon, n) := C \cdot \max\left\{ \left( \frac{1}{n\theta} + 1 \right) \theta^{\Lambda-1}, \varepsilon^{(1-1/\Lambda)(\Lambda-1)}, \varepsilon^{1/\Lambda}, \lambda(n) \right\}. \tag{2.22}$$

Consequently, (1.17) is proved, and we can obtain that $\lim_{\varepsilon\to 0} \lim_{n\to\infty} \lambda(\varepsilon, n) = 0$. Now, we have for $|x| \le \varepsilon a_n$

$$A_n(x) \sim \frac{n}{a_n}, \qquad |B_n(x)| < \lambda(\varepsilon, n)\frac{n}{a_n}. \tag{2.23}$$

□

*Proof of Theorem 1.4.* First, we see that for $1 \le j \le \nu - 1$

$$A_n^{(j)}(x) = 2b_n \int_{-\infty}^{\infty} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)} du. \tag{2.24}$$

We split proof of (1.20) into some lemmas as follows:

(1) Lemma 2.6 is for $0 \le x \le a_n(1+\eta_n)$, $a_{4n} \le u$, and $1 \le j \le \nu - 1$;

(2) Lemma 2.9 is for $a_n/2 \le x \le a_n(1+\eta_n)$, $0 \le u \le a_{4n}$, and $j = \nu - 1$;

(3) Lemma 2.10 is for $0 \le x \le a_n/2$, $0 \le u \le a_{4n}$, and $j = \nu - 1$;

(4) Lemma 2.11 is for $0 \le x \le a_n(1+\eta_n)$, $0 \le u \le a_{4n}$, and $1 \le j \le \nu - 2$;

on the other hand, (1.21) will be proved by Lemmas 2.13 and 2.6.

For $1 \le j \le \nu - 1$ there exists $\eta$ between $u$ and $x$ such that

$$\begin{aligned} \frac{d^j}{dx^j} \overline{Q(x,u)} &= \frac{j!}{(x-u)^{j+1}} \left( \sum_{k=0}^{j} (-1)^k \frac{Q^{(j+1-k)}(x)}{(j-k)!} (x-u)^{j-k} + (-1)^{j+1} Q'(u) \right) \\ &= \frac{Q^{(j+1)}(x) - Q^{(j+1)}(\eta)}{x-u}. \end{aligned} \tag{2.25}$$

Then for $x \ge 0$ and $u \ge 0$, since $Q^{(j+1)}(u)$ is increasing for $1 \le j \le \nu - 1$, we have

$$0 \le \frac{d^j}{dx^j} \overline{Q(x,u)} \le \frac{Q^{(j+1)}(x) - Q^{(j+1)}(u)}{x-u}. \tag{2.26}$$

If $u < 0$ and $x > 0$, then since $|Q^{(j+1)}(\eta)| \le Q^{(j+1)}(-u)$ for $\eta < 0$,

$$\begin{aligned} \left| \frac{d^j}{dx^j} \overline{Q(x,u)} \right| &= \left| \frac{Q^{(j+1)}(x) - Q^{(j+1)}(\eta)}{x-u} \right| \\ &\le \frac{Q^{(j+1)}(x) + Q^{(j+1)}(-u)}{x+(-u)} \\ &\le \frac{Q^{(j+1)}(x) - Q^{(j+1)}(-u)}{x-(-u)} + 2\frac{Q^{(j+1)}(-u) - Q^{(j+1)}(0)}{-u-0}. \end{aligned} \tag{2.27}$$

So, for this case we can prove the result similarly to the case $x, u > 0$. For the other cases, we can prove it by the symmetry of $Q$, similarly. Therefore, we assume that $u$ and $x$ are nonnegative, and we will prove this theorem only for nonnegative $x$ and $u$. Moreover, for simplicity, we let $c_1 = c_2$ without loss of generality, because we know by (1.13) that $Q^{(\nu+1)}(u)$ is bounded for any $u$ between $c_1$ and $c_2$.

On the other hand, if $Q^{(j+2)}(u)$ is increasing, then

$$\frac{Q^{(j+1)}(u) - Q^{(j+1)}(x)}{u - x} \tag{2.28}$$

is also increasing for $u$ because there exists a point $\xi$ between $x$ and $u$ such that

$$\begin{aligned} \frac{d}{du}\left(\frac{Q^{(j+1)}(u) - Q^{(j+1)}(x)}{u - x}\right) &= \frac{Q^{(j+2)}(u) - \left(Q^{(j+1)}(u) - Q^{(j+1)}(x)\right)/(u - x)}{u - x} \\ &= \frac{Q^{(j+2)}(u) - Q^{(j+2)}(\xi)}{u - x} \geq 0. \end{aligned} \tag{2.29}$$

Moreover, if $Q^{(\nu+1)}(t) \leq C(1/t)^{\delta}$ for $t$ between $x$ and $u$, then we see

$$\frac{Q^{(\nu)}(u) - Q^{(\nu)}(x)}{u - x} = \frac{1}{u - x}\int_x^u Q^{(\nu+1)}(t)dt \leq \frac{C}{u - x}\left(u^{1-\delta} - x^{1-\delta}\right) \leq C\left(\frac{1}{u}\right)^{\delta}. \tag{2.30}$$

To complete the proof of Theorem 1.4 we prove a series of lemmas.

**Lemma 2.6.** *Let $0 \leq x \leq a_n(1 + \eta_n)$ and $1 \leq j \leq \nu - 1$:*

$$\int_{a_{4n}<u} (p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du \lesssim \left(\frac{T(a_n)}{a_n}\right)^j \frac{A_n(x)}{a_n}. \tag{2.31}$$

*Proof.* Since

$$\frac{A_n^{(j)}(x)}{2b_n} = \left(\int_{0\leq u\leq a_{4n}} + \int_{a_{4n}<u}\right)(p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du, \tag{2.32}$$

we have to estimate

$$\int_{a_{4n}<u} (p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du =: \int_{a_{4n}<u}. \tag{2.33}$$

First, we see for $x > 0$ large enough,

$$Q^{(j+1)}(x)e^{-Q(x)} \tag{2.34}$$

is decreasing because

$$\left(Q^{(j+1)}(x)e^{-Q(x)}\right)' = \left(Q^{(j+2)}(x) - Q^{(j+1)}(x)Q'(x)\right)e^{-Q(x)}, \tag{2.35}$$

and so from our assumption,

$$\begin{aligned} Q^{(j+2)}(x) - Q^{(j+1)}(x)Q'(x) &\le CQ^{(j+1)}(x)\frac{Q'(x)}{Q(x)} - Q^{(j+1)}(x)Q'(x) \\ &= Q^{(j+1)}(x)Q'(x)\left(\frac{C}{Q(x)} - 1\right) < 0, \end{aligned} \tag{2.36}$$

if $C < Q(x)$. We use this fact. Let $2\rho = \beta + i$ where $\beta < 0$, and let $i$ be a nonnegative integer, and let $P(u) = p_n^2(u)u^i$. Let $u > 0$. Then since

$$\frac{Q^{(j+1)}(a_{4n})}{Q(a_{4n})} \le C\left(\frac{T(a_n)}{a_n}\right)^j, \tag{2.37}$$

by (1.13), we have for some $\xi$ between $x$ and $u$

$$\begin{aligned} \int_{a_{4n}<u} &= \int_{a_{4n}<u} (p_n w_\rho)^2(u)Q^{(j+2)}(\xi)du \\ &\le \int_{a_{4n}<u} (p_n w_\rho)^2(u)Q^{(j+2)}(u)du \\ &\le C\frac{Q^{(j+1)}(a_{4n})w(a_{4n})}{Q(a_{4n})}a_{4n}^{\beta}\int_{a_{4n}<u} P(u)w(u)Q'(u)du \quad (\text{by}(2.34)) \\ &\le \left(\frac{T(a_n)}{a_n}\right)^j w(a_{4n})a_{4n}^{\beta}\int_{a_{4n}}^{\infty} -P(u)\frac{d}{du}w(u)du, \end{aligned} \tag{2.38}$$

$$\int_{a_{4n}}^{\infty} P(u)\frac{d}{du}w(u)du = (Pw)(a_{4n}) - \int_{a_{4n}}^{\infty} P'(t)w(u)du.$$

Applying Lemma 2.1(d) with $L_\infty$, $L_1$-norm and Lemma 2.1(c),

$$\begin{aligned} |(Pw)(a_{4n})| &\le \exp(-C_2 n^\alpha)\|(Pw)(x)\|_{L_\infty(La_n/n\le|x|\le a_n(1-L\eta_n))}, \\ \int_{a_{4n}}^{\infty}\left|P'(u)w(u)\right|du &\le \exp(-C_2 n^\alpha)\left\|\left(P'w\right)(x)\right\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))} \\ &\le \exp(-C_2 n^\alpha)\frac{nT(a_n)^{1/2}}{a_n}\|(Pw)(x)\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))}. \end{aligned} \tag{2.39}$$

Therefore,

$$\int_{a_{4n}}^{\infty}\left|P(u)w(u)Q'(u)\right|du \le \exp(-C_2 n^{\alpha})\|(Pw)(x)\|_{L_\infty(La_n/n\le|x|\le a_n(1-L\eta_n))} + \exp(-C_2 n^{\alpha})\frac{nT(a_n)^{1/2}}{a_n}\|(Pw)(x)\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))}. \tag{2.40}$$

Consequently we have

$$\begin{aligned}
&\left(\frac{T(a_n)}{a_n}\right)^j w(a_{4n})a_{4n}^{\beta}\int_{a_{4n}<u}\left|P(u)w(u)Q'(u)\right|du \\
&\le \left(\frac{T(a_n)}{a_n}\right)^j \exp(-C_2 n^{\alpha})\left\|p_n^2 w_\rho^2\right\|_{L_\infty(La_n/n\le|x|\le a_n(1-L\eta_n))} \\
&\quad + \left(\frac{T(a_n)}{a_n}\right)^j \frac{nT(a_n)^{1/2}}{a_n}\exp(-C_2 n^{\alpha})\left\|p_n^2 w_\rho^2\right\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))} \\
&\le O\left(e^{-n^{d_3}}\right)\left(\frac{T(a_n)}{a_n}\right)^j \\
&\lesssim \left(\frac{T(a_n)}{a_n}\right)^j \frac{A_n(x)}{a_n}.
\end{aligned} \tag{2.41}$$

□

**Lemma 2.7.** *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1,\infty)$ or if $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1,\infty)$, then one has*

$$\frac{Q^{(\nu)}(x)-Q^{(\nu)}(u)}{x-u} \lesssim \begin{cases} 1+\left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & 0\le u\le c_1,\ c_1\le x\le \dfrac{a_n}{2}, \\ 1+\left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & c_1\le u\le 2c_1,\ 0\le x\le c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & 2c_1\le u\le \dfrac{a_n}{3},\ 0\le x\le c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & c_1\le u\le \dfrac{a_n}{3},\ c_1\le x\le \dfrac{a_n}{2}. \end{cases} \tag{2.42}$$

*Proof.* Case (a-1). $0\le u\le c_1$ and $c_1\le x\le a_n/2$. Let

$$\frac{Q^{(\nu)}(u)-Q^{(\nu)}(x)}{u-x} \le \frac{Q^{(\nu)}(u)-Q^{(\nu)}(c_1)}{u-c_1} + \frac{Q^{(\nu)}(c_1)-Q^{(\nu)}(x)}{c_1-x} =: Q_1(u)+Q_2(x). \tag{2.43}$$

Then we have $Q_1(u) \lesssim 1$ from (2.30). Then if $Q'(x)/Q(x)$ is quasi-increasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$\begin{aligned} Q_2(x) &= \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\left|\frac{Q'(x) - Q'(c_1)}{x - c_1}\right| \\ &\lesssim \left(\frac{Q'(\xi)}{Q(\xi)}\right)^{\nu-1}\left|\frac{Q'(a_n/2) - Q'(c_1)}{a_n/2 - c_1}\right| \\ &\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \end{aligned} \tag{2.44}$$

If $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (2.28) and (1.13)

$$\begin{aligned} Q_2(x) &\le \frac{Q^{(\nu)}(a_n/2) - Q^{(\nu)}(c_1)}{a_n/2 - c_1} \\ &\lesssim \frac{Q^{(\nu)}(a_n/2)}{Q'(a_n/2)}\frac{Q'(a_n/2) - Q'(c_1)}{a_n/2 - c_1} \\ &\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \end{aligned} \tag{2.45}$$

Case (a-2). For $c_1 \le u \le 2c_1$ and $0 \le x \le c_1$, we have similarly to Case (a-1),

$$Q_2(x) \lesssim 1, \qquad Q_1(u) \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \tag{2.46}$$

Case (b). $2c_1 \le u \le a_n/3$ and $0 \le x \le c_1$. Using the method of Case (a-1), and similarly to Case (a-2),

$$\frac{Q^{(\nu)}(u) - Q^{(\nu)}(x)}{u - x} \sim \frac{Q^{(\nu)}(u) - Q^{(\nu)}(c_1)}{u - c_1} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \tag{2.47}$$

Case (c). $c_1 \le u \le a_n/3$ and $c_1 \le x \le a_n/2$. We can prove similarly to $Q_1(u)$ and $Q_2(x)$ of Case (a-1). If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &= \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\left|\frac{Q'(x) - Q'(u)}{x-u}\right| \\ &\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\sim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.\end{aligned} \tag{2.48}$$

If $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &\le \frac{Q^{(\nu)}(a_n/2) - Q^{(\nu)}(u)}{a_n/2 - u} \\ &\lesssim \left(\frac{|Q'(a_n/2)|}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\sim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.\end{aligned} \tag{2.49}$$

□

**Lemma 2.8.** *One has*

$$\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} \lesssim \begin{cases} \dfrac{1}{u^{\delta}}, & 0 \le u \le c_1,\ 0 \le x \le c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}, & 0 \le u \le a_{4n},\ \dfrac{a_n}{2} \le x \le a_n(1+\eta_n), \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)} & \dfrac{a_n}{3} \le u \le a_{4n},\ 0 \le x \le \dfrac{a_n}{2}. \end{cases} \tag{2.50}$$

*Proof.* Case (a). $0 \le u \le c_1$ and $0 \le x \le c_1$. From (2.30) and (1.14)

$$\frac{Q^{(\nu)}(u) - Q^{(\nu)}(x)}{u-x} \le C\left(\frac{1}{u}\right)^{\delta}. \tag{2.51}$$

Case (b-1). $0 \le u \le a_n/3$ and $a_n/2 \le x \le a_n(1+\eta_n)$. Since by [1, page 64, Lemma 3.2(a)]

$$\frac{Q'(a_n/2)}{Q'(a_n/3)} \ge \left(\frac{3}{2}\right)^{\Lambda-1}, \tag{2.52}$$

we have

$$Q'(x) - Q'(u) \geq Q'(x)\left(1 - \frac{Q'(a_n/3)}{Q'(a_n/2)}\right) \geq Q'(x)\left(1 - \left(\frac{2}{3}\right)^{\Lambda-1}\right). \tag{2.53}$$

Therefore, since for this case

$$\left(Q'(x) - Q'(u)\right) \sim Q'(x), \tag{2.54}$$

we have

$$\begin{aligned} \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &= \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{Q'(x) - Q'(u)}\overline{Q(x,u)} \\ &\lesssim \left|\frac{Q^{(\nu)}(x)}{Q'(x)}\right|\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}. \end{aligned} \tag{2.55}$$

Case (b-2). $a_n/3 \leq u \leq a_{4n}$ and $a_n/2 \leq x \leq a_n(1+\eta_n)$. There exists a point $\xi$ between $x$ and $u$ such that by (1.13)

$$\begin{aligned} \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &= \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{Q'(x) - Q'(u)}\overline{Q(x,u)} \\ &\lesssim \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(\xi)}{\xi}\right)^{\nu-1}\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}. \end{aligned} \tag{2.56}$$

Case (c). $a_n/3 \leq u \leq a_{4n}$ and $0 \leq x \leq a_n/4$. By the same method as Case (b), we have

$$\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}. \tag{2.57}$$

□

**Lemma 2.9.** *Let $a_n/2 \leq x \leq a_n(1+\eta_n)$. Then*

$$\int_{0\leq u\leq a_{4n}} (p_n w_\rho)^2(u)\frac{d^{\nu-1}}{dx^{\nu-1}}\overline{Q(x,u)}du \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{A_n(x)}{a_n}. \tag{2.58}$$

*Proof.* It is trivial from (2.26) and Lemma 2.8. □

**Lemma 2.10.** *Let $0 \leq x \leq a_n/2$.*

(a) *If $0 \leq x \leq c_1$, then*

$$\int_{0\leq u\leq c_1} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du \lesssim \frac{1}{a_n}. \tag{2.59}$$

*Moreover, one knows that*

$$\frac{1}{a_n} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}. \tag{2.60}$$

(b) *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1,\infty)$, or if $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1,\infty)$, then*

$$\int_{0\leq u\leq a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}. \tag{2.61}$$

(c) *If there exists a constant $0 \leq \delta < 1$ such that $Q^{(\nu+1)}(x) \leq C(1/x)^{\delta}$ on $(0,\infty)$, then one has* (2.61).

*Proof.* (a) For $0 \leq x \leq c_1$ we have from Lemmas 2.8, 2.1(a), and 2.2

$$\begin{aligned} \int_{0\leq u\leq c_1} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du &\lesssim \int_{0\leq u\leq c_1} (p_n w_\rho)^2(u) u^{-\delta} du \\ &\lesssim \begin{cases} \dfrac{1}{a_n}, & \rho \geq 0, \\ \dfrac{1}{a_n}\left(\dfrac{n}{a_n}\right)^{2\rho} \displaystyle\int_{0\leq u\leq c_1} u^{2\rho-\delta} \lesssim \dfrac{1}{a_n}\left(\dfrac{n}{a_n}\right)^{2\rho}, & \rho < 0 \end{cases} \\ &\lesssim \frac{1}{a_n}, \end{aligned} \tag{2.62}$$

because $1+2\rho-\delta \geq 0$ for $\rho < 0$. On the other hand, from (1.19) we see $a_n^{\nu} \leq n^{\nu/(1+\nu-\delta)} \leq n$, and from (1.22) we see $A_n(x) \sim n/a_n$ for $0 \leq x \leq c_1$. So we have

$$\frac{1}{a_n} \lesssim \frac{n}{a_n^{\nu+1}} \lesssim \frac{n}{a_n^2}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \sim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}. \tag{2.63}$$

(b) For $0 \le x \le c_1$, we have from (a), Lemmas 2.7, and 2.8

$$\begin{aligned}\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du &\lesssim \int_{0\le u\le c_1} + \int_{c_1\le u\le 2c_1} + \int_{2c_1\le u\le a_n/3} + \int_{a_n/3\le u\le a_{4n}} \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}.\end{aligned} \tag{2.64}$$

Similarly, for $c_1 \le x \le a_n/2$ we have from Lemmas 2.7 and 2.8

$$\begin{aligned}\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du &\lesssim \int_{0\le u\le c_1} + \int_{c_1\le u\le a_n/3} + \int_{a_n/3\le u\le a_{4n}} \\ &\lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}.\end{aligned} \tag{2.65}$$

(c) Then by (2.26) and Lemma 2.1(a)

$$\begin{aligned}\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du &\lesssim \int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) u^{-\delta} du \\ &\lesssim \int_{0\le u\le a_{4n}} \frac{u^{2\rho-\delta}}{(u+a_n/n)^{2\rho}\sqrt{u^2-a_n^2}} du \\ &\lesssim \int_{0<u<a_n/n} + \int_{a_n/n\le u\le a_n/2} + \int_{a_n/2\le u\le a_{4n}} \\ &\lesssim \frac{1}{a_n^\delta} \\ &\lesssim \frac{n}{a_n^2}\left(\frac{1}{a_n}\right)^{\nu-1} \\ &\lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}.\end{aligned} \tag{2.66}$$

Here, we use the fact $1/a_n^\delta < n/a_n^{\nu+1}$ from (1.19) for the last inequality. □

**Lemma 2.11.** *Let $0 \le x \le a_n(1+\eta_n)$. Then for $1 \le j \le \nu - 2$,*

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)}du \lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^j. \tag{2.67}$$

*Proof.* By the same reason as the proof of Lemma 2.10 when $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, it is proved. □

To prove (1.21) we need some lemmas.

**Lemma 2.12.** *Let $0 < \varepsilon < 1$ and $|x| \leq \varepsilon a_n$.*

(a) *For some $C > 0$ one has*

$$\frac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)} \leq C \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \frac{n}{a_n}. \tag{2.68}$$

(b) *For any $0 < \varepsilon < 1$, there exists $\varepsilon_1(\varepsilon, n) > 0$ such that for $2\varepsilon a_n \leq u$*

$$\frac{d^j}{dx^j} \overline{Q(x,u)} \leq \varepsilon_1(\varepsilon, n) \frac{A_n(x)}{a_n} \left(\frac{n}{a_n}\right)^j + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j}, \tag{2.69}$$

*and $\varepsilon_1(\varepsilon, n) \to 0$ as $n \to \infty$.*

*Proof.* (a) It follows from Lemma 2.3(b). (b) By (2.25), Lemma 2.3(b), and (a), we have

$$\begin{aligned}
\frac{d^j}{dx^j} \overline{Q(x,u)} &\leq C \sum_{k=0}^{j-1} \frac{Q^{(j+1-k)}(x)}{(\varepsilon a_n)^{k+1}} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\leq C \sum_{k=0}^{j-1} \frac{Q'(\varepsilon a_n)}{(\varepsilon a_n)^{k+1}} \left( \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \frac{n}{a_n} \right)^{j-k} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\leq C \sum_{k=0}^{j-1} \frac{\varepsilon^{\Lambda-1}}{(\varepsilon a_n)^{k+1}} \frac{n}{a_n} \left( \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \frac{n}{a_n} \right)^{j-k} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\leq C \frac{n}{a_n^2} \left(\frac{n}{a_n}\right)^j \sum_{k=0}^{j-1} \varepsilon^{\Lambda-k-2} \left( \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \right)^{j-k} \left(\frac{1}{n}\right)^k + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\leq \varepsilon_1(\varepsilon, n) \frac{A_n(x)}{a_n} \left(\frac{n}{a_n}\right)^j + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j},
\end{aligned} \tag{2.70}$$

where we let

$$\varepsilon_1(\varepsilon, n) := C \sum_{k=0}^{j-1} \varepsilon^{\Lambda-k-2} \left( \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \right)^{j-k} \left(\frac{1}{n}\right)^k \longrightarrow 0 \tag{2.71}$$

as $n \to \infty$. Therefore, this lemma is proved. □

**Lemma 2.13.** *Suppose that the one of the three conditions (a), (b), and (c) in Theorem 1.4 is satisfied. Then for any $0 < \varepsilon < 1/2$, there exists $\varepsilon_2(\varepsilon, n) > 0$ such that for $|x| \leq \varepsilon a_n$ and $j = 1, \ldots, \nu - 1$,*

$$\int_{0 \leq u \leq a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)} du \lesssim \varepsilon_2(\varepsilon, n) \frac{A_n(x)}{a_n} \left(\frac{n}{a_n}\right)^j, \tag{2.72}$$

*with $\varepsilon_2(\varepsilon, n) \to 0$ as $n \to \infty$.*

*Proof.* First, we consider the case of which (c) in Theorem 1.4 is satisfied. Then the lemma follows from (2.66) with $\varepsilon_2(\varepsilon,n) := (1/n)^{\nu-1}$. Now, we consider the other cases. If we consider only for $|x| \le \varepsilon a_n$ and $|u| \le 2\varepsilon a_n$ in proving Lemmas 2.7 and 2.8, then we know that for $|x| \le \varepsilon a_n$ and $j = 1,\ldots,\nu-1$

$$\frac{d^j}{dx^j}\overline{Q(x,u)} \lesssim \begin{cases} 1 + u^{-\delta} + \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & 0 \le u \le 2c_1, \\ \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & 2c_1 \le u \le \dfrac{\varepsilon}{2} a_n, \\ \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & \dfrac{\varepsilon}{2} a_n \le u \le 2\varepsilon a_n. \end{cases} \tag{2.73}$$

Then we have by Lemma 2.12(a)

$$\begin{aligned} \int_{0\le u\le 2\varepsilon a_n} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}\,du &\lesssim \int_{0\le u\le 2c_1} + \int_{2c_1\le u\le 2\varepsilon a_n} \\ &\lesssim \frac{1}{a_n^\delta} + \left(\frac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \frac{Q'(\varepsilon a_n)}{\varepsilon a_n} \\ &\lesssim \left(\frac{a_n^{2+j-\delta}}{n^{1+j}} + \varepsilon^{\Lambda-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^j\right) \frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j, \end{aligned} \tag{2.74}$$

and we can see that

$$\varepsilon_3(\varepsilon,n) := \frac{a_n^{2+j-\delta}}{n^{1+j}} + \varepsilon^{\Lambda-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^j \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.75}$$

Finally, we estimate $\int_{2\varepsilon a_n \le u \le a_{4n}}$. By Lemma 2.12(b) we have

$$\begin{aligned} &\int_{2\varepsilon a_n\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}\,du \\ &\quad\le \int_{2\varepsilon a_n\le u\le a_{4n}} \left(\varepsilon_1(\varepsilon,n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j + \frac{1}{(\varepsilon a_n)^j}\overline{Q(x,u)}\right)(p_n w_\rho)^2(u)\,du \\ &\quad\le \left(\varepsilon_1(\varepsilon,n) + \frac{1}{(\varepsilon n)^j}\right)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j. \end{aligned} \tag{2.76}$$

Therefore, if we let $\varepsilon_2(\varepsilon, n) := \varepsilon_3(\varepsilon, n) + \varepsilon_1(\varepsilon, n) + 1/(\varepsilon n)^j$, then

$$\int_{0 \le u \le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)} du \lesssim \varepsilon_2(\varepsilon, n) \frac{A_n(x)}{a_n} \left(\frac{n}{a_n}\right)^j, \tag{2.77}$$

and $\varepsilon_2(\varepsilon, n) \to 0$ as $n \to \infty$ by (2.75) and (2.76). □

From the proof of Lemma 2.6, we have the following. There exists $\varepsilon_4(n) > 0$ satisfying $\varepsilon_4(n) \to 0$ as $n \to \infty$ such that

$$\int_{u \ge a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)} du \le \varepsilon_4(n) \frac{A_n(x)}{a_n} \left(\frac{n}{a_n}\right)^j. \tag{2.78}$$

Therefore, from Lemmas 2.6, 2.9, 2.10, and 2.11 we obtain the estimate for $A_n^{(j)}(x)$ in (1.20), and from Lemma 2.13 and (2.78) we have the estimate for $A_n^{(j)}(x)$ in (1.21). Using Cauchy-Schwarz Inequality we also have the estimate for $B_n^{(j)}(x)$ in (1.20) and (1.21). Consequently, we proved Theorem 1.4, completely. □

*Proof of Theorem 1.6.* (a) (1.24) follows from [1, (3.45)] easily.

(b) Suppose that (1.23) is satisfied on $|x| \ge D$ for some $D > 0$ large enough. Let $x > D$. From (1.23) we have for large $x > D$

$$\ln\left(\frac{Q'(x)}{Q'(D)}\right) \le \ln\left(\frac{Q(x)}{Q(D)}\right)^\lambda, \tag{2.79}$$

and we have for large $x > D$

$$\frac{Q'(x)}{Q'(D)} \le \left(\frac{Q(x)}{Q(D)}\right)^\lambda. \tag{2.80}$$

Case $\lambda > 1$. Then we can see by [1, Lemma 3.4 (3.18)] and (2.80)

$$T(a_t) = \frac{a_t Q'(a_t)}{Q(a_t)} \le \frac{Q'(D)}{Q(D)^\lambda} a_t Q(a_t)^{\lambda-1} \le C a_t \left(\frac{t}{\sqrt{T(a_t)}}\right)^{\lambda-1}. \tag{2.81}$$

Therefore from the assumption $a_t \le C_2 t^\eta$ we have for any $\eta > 0$

$$T(a_t) \le C(\lambda, \eta) t^{2(\eta+\lambda-1)/(\lambda+1)}. \tag{2.82}$$

Case $0 < \lambda \le 1$. Then we have by (2.80)

$$T(x) = \frac{xQ'(x)}{Q(x)} \le x \frac{Q'(D)}{Q(D)^\lambda} Q(x)^{\lambda-1} \le x \frac{Q'(D)}{Q(D)}. \tag{2.83}$$

Therefore, from the assumption $a_t \leq C_2 t^\eta$ we have for any $\eta > 0$

$$T(a_t) \leq C(\lambda, \eta) t^\eta. \tag{2.84}$$

□

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