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Research Article

Some Limit Properties of Random Transition Probability for Second-Order Nonhomogeneous Markov Chains Indexed by a Tree

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We study some limit properties of the harmonic mean of random transition probability for a second-order nonhomogeneous Markov chain and a nonhomogeneous Markov chain indexed by a tree. As corollary, we obtain the property of the harmonic mean of random transition probability for a nonhomogeneous Markov chain.

1. Introduction

A tree is a graph $G = \{T, E\}$ which is connected and contains no circuits. Given any two vertices $\sigma, t(\sigma \neq t \in T)$, let $\overline{\sigma t}$ be the unique path connecting σ and t. Define the graph distance $d(\sigma, t)$ to be the number of edges contained in the path $\overline{\sigma t}$.

Let T_o be an arbitrary infinite tree that is partially finite (i.e., it has infinite vertices, and each vertex connects with finite vertices) and has a root o. Meanwhile, we consider another kind of double root tree T; that is, it is formed with the root o of T_o connecting with an arbitrary point denoted by the root -1. For a better explanation of the double root tree T, we take Cayley tree $T_{C,N}$ for example. It is a special case of the tree T_o, the root o of Cayley tree has N neighbors, and all the other vertices of it have $N+1$ neighbors each. The double root tree $T'_{C,N}$ (see Figure 1) is formed with root o of tree $T_{C,N}$ connecting with another root -1.

Let σ, t be vertices of the double root tree T. Write $t \leq \sigma(\sigma, t \neq -1)$ if t is on the unique path connecting o to σ, and $|\sigma|$ for the number of edges on this path. For any two vertices σ, $t(\sigma, t \neq -1)$ of the tree T, denote by $\sigma \wedge t$ the vertex farthest from o satisfying $\sigma \wedge t \leq \sigma$ and $\sigma \wedge t \leq t$.

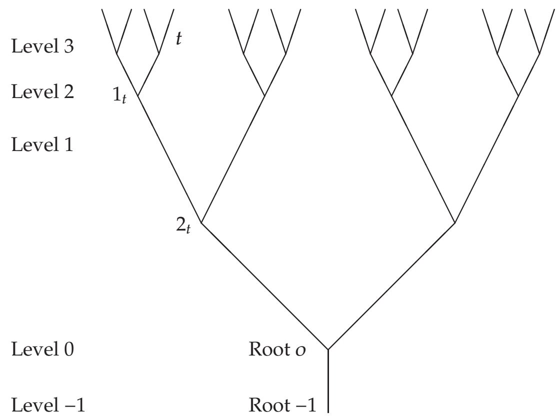

Figure 1: Double root tree $T'_{C,2}$.

The set of all vertices with distance n from root o is called the nth generation of T, which is denoted by L_n. We say that L_n is the set of all vertices on level n and especially root -1 is on the -1st level on tree T. We denote by $T^{(n)}$ the subtree of the tree T containing the vertices from level -1 (the root -1) to level n and denote by $T_o^{(n)}$ the subtree of the tree T_o containing the vertices from level 0 (the root o) to level n. Let $t(\neq o,-1)$ be a vertex of the tree T. We denote the first predecessor of t by 1_t, the second predecessor of t by 2_t, and denote by n_t the nth predecessor of t. Let $X^A = \{X_t, t \in A\}$, and let x^A be a realization of X^A and denote by $|A|$ the number of vertices of A.

Definition 1.1. Let $G = \{1, 2, \ldots, N\}$ and $P(z \mid y, x)$ be nonnegative functions on G^3. Let

$$P = \left(P(z \mid y, x)\right), \quad P(z \mid y, x) \geq 0, \; x, y, z \in G. \tag{1.1}$$

If

$$\sum_{z \in G} P(z \mid y, x) = 1, \tag{1.2}$$

then P is called a second-order transition matrix.

Definition 1.2. Let T be double root tree and let $G = \{1, 2, \ldots, N\}$ be a finite state space, and let $\{X_t, t \in T\}$ be a collection of G-valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Let

$$P = \left(p(x, y)\right), \quad x, y \in G \tag{1.3}$$

be a distribution on G^2, and

$$P_t = \left(P_t(z \mid y, x)\right), \quad x, y, z \in G, \; t \in T \setminus \{o\}\{-1\} \tag{1.4}$$

be a collection of second-order transition matrices. For any vertex t ($t \neq o, -1$), if

$$\begin{gathered}
P\left(X_t = z \mid X_{1_t} = y, X_{2_t} = x,\ X_\sigma \text{ for } \sigma \wedge t \leq 1_t\right) \\
= P\left(X_t = z \mid X_{1_t} = y, X_{2_t} = x\right) = P_t(z \mid y, x) \quad \forall x, y, z \in G, \\
P\left(X_{-1} = x, X_o = y\right) = p(x, y), \quad x, y \in G,
\end{gathered} \tag{1.5}$$

then $\{X_t, t \in T\}$ is called a G-value second-order nonhomogeneous Markov chain indexed by a tree T with the initial distribution (1.3) and second-order transition matrices (1.4), or called a T-indexed second-order nonhomogeneous Markov chain.

Remark 1.3. Benjamini and Peres [1] have given the definition of the tree-indexed homogeneous Markov chains. Here we improve their definition and give the definition of the tree-indexed second-order nonhomogeneous Markov chains in a similar way. We also give the following definition (Definition 2.3) of tree-indexed nonhomogeneous Markov chains.

There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres [1] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [2] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger (see [3, 4]), by using Pemantle's result [5] and a combinatorial approach, have studied the Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu [6] have studied a strong law of large numbers for the frequency of occurrence of states for Markov chains field on a homogeneous tree (a particular case of tree-indexed Markov chains field and PPG-invariant random fields). Yang (see [7]) has studied the strong law of large numbers for frequency of occurrence of state and Shannon-McMillan theorem for homogeneous Markov chains indexed by a homogeneous tree. Recently, Yang (see [8]) has studied the strong law of large numbers and Shannon-McMillan theorem for nonhomogeneous Markov chains indexed by a homogeneous tree. Huang and Yang (see [9]) have also studied the strong law of large numbers for Markov chains indexed by an infinite tree with uniformly bounded degree.

Let $P_t(x_t \mid x_{1_t}, x_{2_t}) = P_t(X_t = x_t \mid X_{1_t} = x_{1_t}, X_{2_t} = x_{2_t})$. Then $P_t(X_t \mid X_{1_t}, X_{2_t})$ is called the random transition probability of a T-indexed second-order nonhomogeneous Markov chain. Liu [10] has studied a strong limit theorem for the harmonic mean of the random transition probability of finite nonhomogeneous Markov chains. In this paper, we study some limit properties of the harmonic mean of random transition probability for a second-order nonhomogeneous Markov chain and a nonhomogeneous Markov chain indexed by a tree. As corollary, we obtain the results of [10, 11].

2. Main Results

Lemma 2.1. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space G defined as in Definition 1.2, and let $\{g_t(x, y, z), t \in T\}$ be a collection of functions defined on G^3. Let $L_{-1} = \{-1\}$, $L_0 = \{o\}$, and $\mathcal{F}_n = \sigma(X^{T^{(n)}})$. Set*

$$t_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} g_t(X_{2_t}, X_{1_t}, X_t)}}{\prod_{t \in T^{(n)} \setminus \{o\}\{-1\}} E\left[e^{\lambda g_t(X_{2_t}, X_{1_t}, X_t)} \mid X_{1_t}, X_{2_t}\right]}, \tag{2.1}$$

where λ is a real number. Then $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ is a nonnegative martingale.

Proof. Obviously, when $n \geq 1$, we have

$$P\left(x^{T^{(n)}}\right) = P\left(X^{T^{(n)}} = x^{T^{(n)}}\right) = P(X_{-1} = x_{-1}, X_o = x_o) \prod_{t \in T^{(n)} \setminus \{o\}\{-1\}} P_t(x_t \mid x_{1_t}, x_{2_t}). \tag{2.2}$$

Hence

$$P\left(X^{L_n} = x^{L_n} \mid X^{T^{(n-1)}} = x^{T^{(n-1)}}\right) = \frac{P\left(x^{T^{(n)}}\right)}{P\left(x^{T^{(n-1)}}\right)} = \prod_{t \in L_n} P_t(x_t \mid x_{1_t}, x_{2_t}). \tag{2.3}$$

Then

$$\begin{aligned}
&E\left[e^{\lambda \sum_{t \in L_n} g_t(X_{2_t}, X_{1_t}, X_t)} \mid \mathcal{F}_{n-1}\right] \\
&= \sum_{x^{L_n} \in G^{L_n}} e^{\lambda \sum_{t \in L_n} g_t(X_{2_t}, X_{1_t}, x_t)} P\left(X^{L_n} = x^{L_n} \mid X^{T^{(n-1)}}\right) \\
&= \sum_{x^{L_n} \in G^{L_n}} e^{\lambda \sum_{t \in L_n} g_t(X_{2_t}, X_{1_t}, x_t)} \prod_{t \in L_n} P_t(x_t \mid X_{1_t}, X_{2_t}) \\
&= \prod_{t \in L_n} \sum_{x_t \in G} e^{\lambda g_t(X_{2_t}, X_{1_t}, x_t)} P_t(x_t \mid X_{1_t}, X_{2_t}) \\
&= \prod_{t \in L_n} E\left[e^{\lambda g_t(X_{2_t}, X_{1_t}, X_t)} \mid X_{1_t}, X_{2_t}\right] \quad \text{a.e.}
\end{aligned} \tag{2.4}$$

On the other hand, we also have

$$t_n(\lambda, \omega) = t_{n-1}(\lambda, \omega) \frac{e^{\lambda \sum_{t \in L_n} g_t(X_{2_t}, X_{1_t}, X_t)}}{\prod_{t \in L_n} E\left(e^{\lambda g_t(X_{2_t}, X_{1_t}, X_t)} \mid X_{1_t}, X_{2_t}\right)}. \tag{2.5}$$

Combining (2.4) and (2.5), we arrive at

$$E[t_n(\lambda, \omega) \mid \mathcal{F}_{n-1}] = t_{n-1}(\lambda, \omega) \quad \text{a.e.}, \tag{2.6}$$

Thus the lemma is proved. □

Theorem 2.2. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space G defined as in Definition 1.2, and its initial distribution and probability transition collection satisfying*

$$\begin{aligned}
&P(X_{-1} = x_{-1}, X_o = x_o) = P(x, y) > 0, \quad \forall x, y \in G, \\
&P_t(z \mid y, x) > 0, \quad \forall x, y, z \in G, \ t \in T \setminus \{o\}\{-1\},
\end{aligned} \tag{2.7}$$

respectively. Let

$$b_t = \min\{P_t(z \mid y, x),\ x, y, z \in G\}, \quad t \in T \setminus \{o\}\{-1\}. \tag{2.8}$$

If there exists $a(>0)$ *such that*

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{a/b_t} = M < \infty, \tag{2.9}$$

then the harmonic mean of the random conditional probability $\{P_t(X_t \mid X_{1_t}, X_{2_t}), t \in T^{(n)} \setminus \{o\}\{-1\}\}$ *converges to* $1/N$ *a.e., that is,*

$$\lim_{n\to\infty} \frac{|T^{(n)}|}{\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t \mid X_{1_t}, X_{2_t})^{-1}} = \frac{1}{N} \quad a.e\ . \tag{2.10}$$

Proof. Let $g_t(x, y, z) = P_t(z \mid y, x)^{-1}$ in Lemma 2.1. Then it follows from Lemma 2.1 that

$$t_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t|X_{1_t}, X_{2_t})^{-1}}}{\prod_{t\in T^{(n)}\setminus\{o\}\{-1\}} E\left[e^{\lambda P_t(X_t|X_{1_t}, X_{2_t})^{-1}} \mid X_{1_t}, X_{2_t}\right]} \tag{2.11}$$

is a nonnegative martingale. According to Doob martingale convergence theorem, we have

$$\lim_{n\to\infty} t_n(\lambda, \omega) = t(\lambda, \omega) < \infty \quad \text{a.e .} \tag{2.12}$$

Thus

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \ln t_n(\lambda, \omega) \leq 0 \quad \text{a.e.} \tag{2.13}$$

It follows from (2.11) and (2.13) that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \Bigg\{ \lambda &\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} \\ &- \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \ln E\left[e^{\lambda P_t(X_t|X_{1_t}, X_{2_t})^{-1}} \mid X_{1_t}, X_{2_t}\right] \Bigg\} \leq 0 \quad \text{a.e.} \end{aligned} \tag{2.14}$$

By (2.14) and the inequalities $\ln x \le x - 1 (x > 0)$, and $0 \le e^x - 1 - x \le (x^2/2)e^{|x|}$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\left[\lambda P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} - \lambda N\right] \\
&\quad\le \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\left\{\ln E\left[e^{\lambda P_t(X_t|X_{1_t},X_{2_t})^{-1}} \mid X_{1_t}, X_{2_t}\right] - \lambda N\right\} \\
&\quad\le \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\left\{E\left[e^{\lambda P_t(X_t|X_{1_t},X_{2_t})^{-1}} \mid X_{1_t}, X_{2_t}\right] - 1 - \lambda N\right\} \\
&\quad= \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in G}P_t(x_t \mid X_{1_t}, X_{2_t})\left[e^{\lambda P_t(x_t|X_{1_t},X_{2_t})^{-1}} - 1 - \lambda P_t(x_t \mid X_{1_t}, X_{2_t})^{-1}\right] \\
&\quad\le \frac{\lambda^2}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in G}P_t(x_t \mid X_{1_t}, X_{2_t})^{-1}e^{|\lambda|P_t(x_t|X_{1_t},X_{2_t})^{-1}} \\
&\quad\le \frac{\lambda^2}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in G}\frac{1}{b_t}e^{|\lambda|/b_t} \\
&\quad\le \frac{\lambda^2 N}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{1}{b_t}e^{|\lambda|/b_t} \qquad \text{a.e.}
\end{aligned} \tag{2.15}$$

It is easy to see that

$$\max_{0<\lambda<1}\{x\lambda^x, x > 0\} = -\frac{e^{-1}}{\ln\lambda}. \tag{2.16}$$

Let $0 < \lambda < a$, by (2.15), (2.16), (2.8), and (2.9) we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\left[P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} - N\right] \\
&\quad\le \frac{\lambda N}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{1}{b_t}e^{\lambda/b_t} \\
&\quad= \frac{\lambda N}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{1}{b_t}\left(\frac{e^{\lambda}}{e^{a}}\right)^{1/b_t} e^{a/b_t} \\
&\quad\le \frac{\lambda N}{2(a-\lambda)e}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{a/b_t} \\
&\quad= \frac{\lambda N}{2(a-\lambda)e}M,
\end{aligned} \tag{2.17}$$

Letting $\lambda \to 0^+$, by (2.17), we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \left[P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} - N\right] \leq 0 \quad \text{a.e}\ . \tag{2.18}$$

Let $-a < \lambda < 0$, by (2.15),(2.8), and (2.9) we have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \left[P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} - N\right] \\
&\geq \frac{\lambda N}{2} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{1}{b_t} e^{-\lambda/b_t} \\
&= \frac{\lambda N}{2} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{1}{b_t} \left(\frac{e^{-\lambda}}{e^{a}}\right)^{1/b_t} e^{a/b_t} \\
&\geq \frac{\lambda N}{2(a+\lambda)e} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{a/b_t} \\
&= \frac{\lambda N}{2(a+\lambda)e} M.
\end{aligned} \tag{2.19}$$

Letting $\lambda \to 0^-$, by (2.19), we have

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \left[P_t(X_t \mid X_{1_t}, X_{2_t})^{-1} - N\right] \geq 0 \quad \text{a.e}\ . \tag{2.20}$$

Combining (2.18) and (2.20), we obtain (2.10) directly. □

From the definition above, we know that the difference between T_o and T lies in whether the root o is connected with another root -1. In the following, we will investigate some properties of the harmonic mean of the transition probability of nonhomogeneous Markov chains on the tree T_o. First, we give the definition of nonhomogeneous Markov chains on the tree T_o.

Definition 2.3. Let T_o be an arbitrary tree that is partly finite, let $G = \{1, 2, \ldots, N\}$ be a finite state space, and let $\{X_t, t \in T_o\}$ be a collection of G-valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Let

$$P = (p(x)), \quad x \in G \tag{2.21}$$

be a distribution on G, and

$$P_t = (P_t(y \mid x)), \quad x, y \in G,\ t \in T_o \setminus \{o\} \tag{2.22}$$

be a collection of transition matrices. For any vertex t $(t \neq o)$, if

$$P(X_t = y \mid X_{1_t} = x,\ X_\sigma \text{ for } \sigma \wedge t \leq 1_t) = P(X_t = y \mid X_{1_t} = x) = P_t(y \mid x), \quad \forall x, y \in G,$$
$$P(X_o = x) = p(x), \quad x \in G, \tag{2.23}$$

then $\{X_t, t \in T_o\}$ is called a G-value nonhomogeneous Markov chain indexed by a tree T_o with the initial distribution (2.21) and transition matrices (2.22), or called a T_o-indexed nonhomogeneous Markov chain.

Let $P_t(x_t \mid x_{1_t}) = P_t(X_t = x_t \mid X_{1_t} = x_{1_t})$. Then $P_t(X_t \mid X_{1_t})$ is called the random transition probability of a T_o-indexed nonhomogeneous Markov chain. Since a Markov chain is a special case of a second-order Markov chain, we may regard the nonhomogeneous Markov chain on T_o to be a special case of the second-order nonhomogeneous Markov chain on T when we do not take the difference of T_o and T on the root -1 into consideration. Thus for the nonhomogeneous Markov chain on the tree T_o, we can get the results similar to Lemma 2.1 and Theorem 2.2.

Lemma 2.4. *Let $\{X_t, t \in T_o\}$ be a T_o-indexed second-order nonhomogeneous Markov chain with state space G defined as in Definition 2.3, and let $\{g_t(x,y), t \in T_o\}$ be a collection of functions defined on G^2. Let $L_0 = \{o\}$ and $\mathcal{F}_n = \sigma(X^{T_o^{(n)}})$. Set*

$$t_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t \in T_o^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t)}}{\prod_{t \in T_o^{(n)} \setminus \{o\}} E\left[e^{\lambda g_t(X_{1_t}, X_t)} \mid X_{1_t}\right]}, \tag{2.24}$$

where λ is a real number. Then $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ is a nonnegative martingale.

Theorem 2.5. *Let $\{X_t, t \in T_o\}$ be a T_o-indexed nonhomogeneous Markov chain with state space G defined as in Definition 2.3, and its initial distribution and probability transition collection satisfying*

$$\begin{gathered} P(X_o = x) = p(x) > 0, \quad \forall x \in G, \\ P_t(y \mid x) > 0, \quad \forall x, y \in G,\ t \in T_o \setminus \{o\}, \end{gathered} \tag{2.25}$$

respectively. Let

$$b_t = \min\{P_t(y \mid x),\ x, y \in G\}, \quad t \in T_o \setminus \{o\}. \tag{2.26}$$

If there exists $a(> 0)$ such that

$$\limsup_{n \to \infty} \frac{1}{\left|T_o^{(n)}\right|} \sum_{t \in T_o^{(n)} \setminus \{o\}} e^{a/b_t} = M < \infty, \tag{2.27}$$

then the harmonic mean of the random conditional probability $\{P_t(X_t \mid X_{1_t}),\ t \in T_o^{(n)} \setminus \{o\}\}$ *converges to* $1/N$ *a.e., that is*

$$\lim_{n\to\infty} \frac{\left|T_o^{(n)}\right|}{\sum_{t\in T_o^{(n)}\setminus\{o\}} P_t(X_t \mid X_{1_t})^{-1}} = \frac{1}{N} \quad a.e. \tag{2.28}$$

If the successor of each vertex of the tree T_o has only one vertex, then the nonhomogeneous Markov chains on the tree T_o degenerate into the general nonhomogeneous Markov chains. Thus we obtain the results in [10, 11].

Corollary 2.6 (see [10, 11]). *Let* $\{X_n, n \geq 0\}$ *be a nonhomogeneous Markov chain with state space* G*, and its initial distribution and probability transition sequence satisfying*

$$\begin{gathered} p(i) > 0, \quad i \in G, \\ P_k(i,j) > 0, \quad i,j \in G,\ k = 1,2,\ldots, \end{gathered} \tag{2.29}$$

respectively. Let

$$a_k = \min\{P_k(i,j),\ i,j \in G\}, \quad k = 1,2,\ldots\ . \tag{2.30}$$

If there exists $a(>0)$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{a/a_k} = M < \infty, \tag{2.31}$$

then

$$\lim_{n\to\infty} \frac{n}{\sum_{k=1}^{n} P_k(X_k \mid X_{k-1})^{-1}} = \frac{1}{N} \quad a.e\ . \tag{2.32}$$

Proof. When the successor of each vertex of the tree T_o has only one vertex, the nonhomogeneous Markov chains on the tree T_o degenerate into the general nonhomogeneous Markov chains, the corollary follows directly from Theorem 2.5. □

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