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Research Article

Numerical Radius and Operator Norm Inequalities

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A general inequality involving powers of the numerical radius for sums and products of Hilbert space operators is given. This inequality generalizes several recent inequalities for the numerical radius, and includes that if A and B are operators on a complex Hilbert space H, then $w^r(A^*B) \leq (1/2)\||A|^{2r} + |B|^{2r}\|$ for $r \geq 1$. It is also shown that if X_i is normal ($i = 1, 2, \ldots, n$), then $\|\sum_{i=1}^n X_i\|^r \leq n^{r-1}\|\sum_{i=1}^n |X_i|^r\|$. Related numerical radius and usual operator norm inequalities for sums and products of operators are also presented.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. For $A \in B(H)$, the usual operator norm of an operator A is defined by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \tag{1.1}$$

where $\|x\| = \langle x, x \rangle^{1/2}$.

The numerical range of A, known also as the field of values of A, is defined as the set of complex numbers given by

$$W(A) = \{\langle Ax, x \rangle : x \in H, \|x\| = 1\}. \tag{1.2}$$

The most important properties of the numerical range are that it is convex and its closure contains the spectrum of the operator.

A unitarily invariant norm $|||\cdot|||$ on H is a norm on the ideal $C_{|||\cdot|||}$ of $B(H)$, satisfying $|||UAV||| = |||A|||$ for all $A \in B(H)$ and all unitary operators U and V in $B(H)$. It is called weakly unitarily invariant norm (or invariant under similarities) if $|||UAU^*||| = |||A|||$ for all $A \in B(H)$ and all unitary operators $U \in B(H)$.

The most familiar example of weakly unitarily invariant norm is the numerical radius $w(A)$, defined by

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}. \tag{1.3}$$

It is well known that $w(A)$ defines a norm on $B(H)$ and for every $A \in B(H)$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.4}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. The inequalities in (1.4) are sharp: if $A^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if A is normal. In fact, for a nilpotant operator A with $A^n = 0$, Haagerup and Harpe [1] show that $w(A) \leq \|A\| \cos(\pi/(n+1))$. In particular, when $n = 2$, we get the reverse inequality of the first inequality in (1.4). For a comprehensive account on the theory of the numerical range and numerical radius, the reader is referred to [2, 3]. A detailed study for the field of values of a matrix is given in [4].

The inequalities in (1.4) have been improved considerably by Kittaneh in [5, 6]. It has been shown that if $A \in B(H)$, then

$$w(A) \leq \frac{1}{2}\left\||A| + |A^*|\right\| \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right), \tag{1.5}$$

$$\frac{1}{4}\left\|A^*A + AA^*\right\| \leq w^2(A) \leq \frac{1}{2}\left\|A^*A + AA^*\right\|, \tag{1.6}$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of A. The second inequality in (1.5) refines the second inequality in (1.4). For diverse applications of these inequalities we refer to [5, 7].

Considerable generalizations of the first inequality in (1.5) and the second inequality in (1.6) have been established in [8] for the numerical radius of one operator and for the sum of two operators. It has been shown that if $A,B \in B(H)$, then

$$w^r(A) \leq \frac{1}{2}\left\||A|^{2r\alpha} + |A^*|^{2r(1-\alpha)}\right\|, \tag{1.7}$$

$$w^r(A+B) \leq 2^{r-2}\left\||A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} + |B|^{2r\alpha} + |B^*|^{2r(1-\alpha)}\right\| \tag{1.8}$$

for $0 < \alpha < 1$ and $r \geq 1$. Other recent inequalities have been obtained in [9, 10], which are related to the Euclidean radius of two Hilbert space operators and (α, β)-normal operators in Hilbert spaces, respectively.

A general numerical radius inequality has been proved by Kittaneh, it has been shown in [6] that if $A, B, C, D, S, T \in B(H)$, then

$$w(ATB + CSD) \leq \frac{1}{2}\left\| A\left|T^*\right|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C\left|S^*\right|^{2(1-\alpha)}C^* + D^*\left|S\right|^{2\alpha}D \right\| \tag{1.9}$$

for all $\alpha \in (0,1)$. In particular,

$$w(AB \pm BA) \leq \frac{1}{2}\left\| A^*A + AA^* + B^*B + BB^* \right\|. \tag{1.10}$$

Usual operator norm inequalities for sums of operators have attracted the attention of several mathematicians. Some of these inequalities have been introduced in [3, 11]. It has been shown in [6] that if A and B are normal and $r \geq 1$, then

$$\|A + B\|^r \leq 2^{r-1}\left\| |A|^r + |B|^r \right\|. \tag{1.11}$$

Another important norm inequalities for unitarily invariant norms, which are related to (1.11) assert that if $A_1, A_2, \ldots, A_n \in B(H)$ are positive and $r \geq 1$, then

$$\left|\left|\left| \sum_{i=1}^{n} A_i^r \right|\right|\right| \leq \left|\left|\left| \left(\sum_{i=1}^{n} A_i \right)^r \right|\right|\right|, \tag{1.12}$$

$$\left|\left|\left| \left(\sum_{i=1}^{n} A_i \right)^r \right|\right|\right| \leq n^{r-1} \left|\left|\left| \sum_{i=1}^{n} A_i^r \right|\right|\right| \tag{1.13}$$

(see, e.g., [12]).

In Section 2 of this paper, we establish a general numerical radius inequality that generalizes (1.6), (1.7), (1.8), and (1.9), from which numerical radius inequalities for sums, products, and commutators of operators are obtained. Usual operator norm inequalities that generalize (1.11) and related to (1.13) are presented in Section 3.

2. A General Numerical Radius Inequality

In this section, we establish a general numerical radius inequality for Hilbert space operators which yields well known and new numerical radius inequalities as special cases. To prove our generalized inequality, we need the following basic lemmas. The first lemma is a generalized form of the mixed Schwarz inequality, which has been proved by Kittaneh [13].

Lemma 2.1. *Let A be an operator in $B(H)$, and let f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$\left|\langle Ax, y\rangle\right| \leq \|f(|A|)x\| \left\|g\left(\left|A^*\right|\right)y\right\| \tag{2.1}$$

for all x and y in H.

The second lemma, which is called Hölder-McCarthy inequality, is a well-known result that follows from the spectral theorem for positive operators and Jensen's inequality (see [13]).

Lemma 2.2. *Let A be a positive operator in $B(H)$ and let $x \in H$ be any unit vector. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad \forall r \geq 1. \tag{2.2}$$

The third lemma concerned with positive real numbers, and it is a consequence of the convexity of the function $f(t) = t^r$, $r \geq 1$.

Lemma 2.3. *Let a_i be a positive real number $(i = 1, 2, \ldots, n)$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r \quad \forall r \geq 1. \tag{2.3}$$

The fourth lemma is a norm inequality for the sum of two operators, which can be found in [14].

Lemma 2.4. *If A and B are positive operators in $B(H)$, then*

$$\|A + B\| \leq \max\left(\|A\|, \|B\|\right) + \left\|A^{1/2} B^{1/2}\right\|. \tag{2.4}$$

Another important usual operator norm inequality which will be used in this section says that for any positive operators $A,B \in B(H)$ we have (see [11])

$$\|A^r B^r\| \leq \|AB\|^r \quad \forall 0 \leq r \leq 1. \tag{2.5}$$

Our main result of this paper, which leads to a generalization of (1.6), (1.7), (1.8), and (1.9), can be stated as follows.

Theorem 2.5. *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left(\left[A_i^* g^2\left(\left|X_i^*\right|\right) A_i\right]^r + \left[B_i^* f^2\left(\left|X_i\right|\right) B_i\right]^r \right) \right\| \tag{2.6}$$

for all $r \geq 1$.

Proof. For every unit vector $x \in H$, we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)x, x\right\rangle\right|^r \\
&\quad= \left|\sum_{i=1}^{n}\left\langle A_i^* X_i B_i x, x\right\rangle\right|^r \\
&\quad\le \left(\sum_{i=1}^{n}\left|\left\langle A_i^* X_i B_i x, x\right\rangle\right|\right)^r \\
&\quad= \left(\sum_{i=1}^{n}\left|\left\langle X_i B_i x, A_i x\right\rangle\right|\right)^r \\
&\quad\le \left(\sum_{i=1}^{n}\left\langle f^2(|X_i|) B_i x, B_i x\right\rangle^{1/2}\left\langle g^2(|X_i^*|) A_i x, A_i x\right\rangle^{1/2}\right)^r \text{ by (2.1)} \\
&\quad\le n^{r-1}\sum_{i=1}^{n}\left\langle B_i^* f^2(|X_i|) B_i x, x\right\rangle^{r/2}\left\langle A_i^* g^2(|X_i^*|) A_i x, x\right\rangle^{r/2} \text{ by (2.3)} \\
&\quad\le n^{r-1}\sum_{i=1}^{n}\left\langle\left(B_i^* f^2(|X_i|) B_i\right)^r x, x\right\rangle^{1/2}\left\langle\left(A_i^* g^2(|X_i^*|) A_i\right)^r x, x\right\rangle^{1/2} \text{ by (2.2)} \\
&\quad\le \frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle\left[B_i^* f^2(|X_i|) B_i\right]^r x, x\right\rangle + \left\langle\left[A_i^* g^2(|X_i^*|) A_i\right]^r x, x\right\rangle\right) \\
&\qquad\qquad\text{by the arithmetic-geometric mean inequality} \\
&\quad= \frac{n^{r-1}}{2}\left\langle\sum_{i=1}^{n}\left(\left[B_i^* f^2(|X_i|) B_i\right]^r + \left[A_i^* g^2(|X_i^*|) A_i\right]^r\right)x, x\right\rangle.
\end{aligned} \tag{2.7}$$

Now the result follows by taking the supremum over all unit vectors in H. □

Inequality (2.6) includes several numerical radius inequalities as special cases. Samples of inequalities are demonstrated in what follows.

For $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0,1)$, in inequality (2.6), we get the following inequality that generalizes (1.9).

Corollary 2.6. *Let* $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, $r \ge 1$, *and* $0 < \alpha < 1$. *Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \le \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(\left(A_i^* |X_i^*|^{2(1-\alpha)} A_i\right)^r + \left(B_i^* |X_i|^{2\alpha} B_i\right)^r\right)\right\|. \tag{2.8}$$

In particular,

$$w\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \le \frac{1}{2}\left\|\sum_{i=1}^{n}\left(A_i^* |X_i^*| A_i + B_i^* |X_i| B_i\right)\right\|. \tag{2.9}$$

For $A_i = B_i = I$ $(i = 1, 2, \ldots, n)$ in inequality (2.6), we get the following numerical radius inequalities for sums of operators that generalizes (1.8).

Corollary 2.7. *Let $X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, and let f and g be as in Lemma 2.1. Then*

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(f^{2r}(|X_i|) + g^{2r}(|X_i^*|)\right)\right\| \quad \forall\, r \geq 1. \tag{2.10}$$

In particular,

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(|X_i|^{2r\alpha} + |X_i^*|^{2r(1-\alpha)}\right)\right\| \quad \forall\, \alpha \in (0, 1). \tag{2.11}$$

It should be mentioned here that the inequality in (2.11) generalizes (1.7) in the case $X_1 = X_2 = \cdots = X_n$.

Remark 2.8. The case $\alpha = 1/2$ in (2.11) gives

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(|X_i|^{r} + |X_i^*|^{r}\right)\right\| \quad \forall\, r \geq 1, \tag{2.12}$$

which generalizes the second inequality in (1.6), while the choice $n = 1$ will give a generalization of the first inequality in (1.5) and can be stated as

$$w^r(X) \leq \frac{1}{2}\left\||X|^r + |X^*|^r\right\| \quad \forall\, r \geq 1. \tag{2.13}$$

Note that using (2.4) and (2.5), a related inequality can be derived from (2.13). Indeed,

$$\begin{aligned} w^r(X) &\leq \frac{1}{2}\left\||X|^r + |X^*|^r\right\| \\ &\leq \frac{1}{2}\left(\max\left(\||X|^r\|, \left\||X^*|^r\right\|\right) + \left\||X|^{r/2}|X^*|^{r/2}\right\|\right) \\ &= \frac{1}{2}\left(\|X\|^r + \left\||X|^r|X^*|^r\right\|^{1/2}\right). \end{aligned} \tag{2.14}$$

The above inequality generalizes the second inequality in (1.5). In fact, for $1 \leq r \leq 2$, we have

$$\begin{aligned} w^r(X) &\leq \frac{1}{2}\left(\|X\|^r + \left\||X|^{r/2}|X^*|^{r/2}\right\|\right) \\ &\leq \frac{1}{2}\left(\|X\|^r + \||X||X^*|\|^{r/2}\right) \\ &= \frac{1}{2}\left(\|X\|^r + \|X^2\|^{r/2}\right). \end{aligned} \tag{2.15}$$

The last equality can be proved using the polar decomposition. In fact, if $X = U|X|$ and $X^* = V|X^*|$ are the polar decompositions of X and X^*, respectively, then $\||X||X^*|\| = \|U^*XXV\| = \|X^2\|$.

It is known that $w(A+B) \leq w(A) + w(B)$. However, the numerical radius is not submultiplicative, even for commuting operators. On the other hand, we have the power inequality, which asserts that if $A \in B(H)$, then

$$w(A^n) \leq w^n(A) \quad \text{for } n = 1, 2, \ldots. \tag{2.16}$$

It is evident from the first inequality in (1.4) that if $A, B \in B(H)$, then

$$w(AB) \leq 4w(A)w(B). \tag{2.17}$$

Moreover, if $AB = BA$, then

$$w(AB) \leq 2w(A)w(B). \tag{2.18}$$

These inequalities, among other related ones, can be found in [2].

For $X_i = I$ $(i = 1, 2, \ldots, n)$ in inequality (2.6), we get the following numerical radius inequalities for products of operators that are related to the above inequalities.

Corollary 2.9. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ and $r \geq 1$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(|A_i|^{2r} + |B_i|^{2r}\right)\right\|. \tag{2.19}$$

In particular,

$$w\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}(A_i^* A_i + B_i^* B_i)\right\|. \tag{2.20}$$

Remark 2.10. The case $n = 1$ in (2.19), provides the following inequality

$$w^r(A^*B) \leq \frac{1}{2}\|(A^*A)^r + (B^*B)^r\|, \tag{2.21}$$

which is a numerical radius inequality for the product of operators and is related to the arithmetic-geometric mean inequality for operators. Note that a more general inequality can be obtained by letting $\alpha = 1/2$ and $n = 1$ in (2.8). In fact, we have

$$w^r(A^*XB) \leq \frac{1}{2}\|(A^*|X^*|A)^r + (B^*|X|B)^r\|. \tag{2.22}$$

For $r = 1$ in (2.22), we obtain the inequality

$$w(A^*XB) \leq \frac{1}{2}\left\| A^*\left|X^*\right|A + B^*|X|B \right\| \tag{2.23}$$

as well as

$$w(A^*XB) \leq \frac{1}{2}\| AA^*X + XBB^* \| \tag{2.24}$$

which follows from the the arithmetic-geometric mean inequality for operators (see [15]). Inequalities (2.23) and (2.24) are not equivalent. This can be seen from the example $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $X = I$, $B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

The inequality in (2.22) can be used to give an upper bound for the numerical radius of A^2 and A^3. In fact, we have

$$w^r(A^2) \leq \frac{1}{2}\| (AA^*)^r + (A^*A)^r \|, \tag{2.25}$$

$$w^r(A^3) \leq \frac{1}{2}\| A|A^*|A^* + A^*|A|A \|. \tag{2.26}$$

The commutator of A and B is the operator $AB - BA$. Commutators play an important role in operator theory. It follows by the triangle inequality that if $A, B \in B(H)$, then $\|AB - BA\| \leq 2\|A\|\|B\|$.

For $n = 2$ in inequality (2.6), we get the following numerical radius inequalities that generalize (1.9), and give an estimate for the numerical radius of commutators.

Corollary 2.11. *Let $A, B, C, D, S, T \in B(H)$, and let f and g be as in Theorem 2.5. Then*

$$\begin{aligned} &w^r(ATB + CSD) \\ &\quad \leq 2^{r-2}\left\| \left(Ag^2(\left|T^*\right|)A^*\right)^r + \left(B^*f^2(|T|)B\right)^r + \left(Cg^2(\left|S^*\right|)C^*\right)^r + \left(D^*f^2(|S|)D\right)^r \right\| \end{aligned} \tag{2.27}$$

for $r \geq 1$. In particular,

$$w^r(ATB + CSD) \leq 2^{r-2}\left\| \left(A\left|T^*\right|A^*\right)^r + \left(B^*|T|B\right)^r + \left(C\left|S^*\right|C^*\right)^r + \left(D^*|S|D\right)^r \right\|. \tag{2.28}$$

We end this section by the following remark.

Remark 2.12. Inequality (2.28) gives a numerical radius inequality for commutators of operators that generalizes (1.10). If $D = A$, $C = B$, and $S = \pm T = X$, then

$$w^r(AXB \pm BXA) \leq 2^{r-2}\left\| \left(A^*\left|X^*\right|A\right)^r + \left(A|X|A^*\right)^r + \left(B^*|X|B\right)^r + \left(B\left|X^*\right|B^*\right)^r \right\|. \tag{2.29}$$

In particular,

$$w^r(AB \pm BA) \le 2^{r-2} \left\| |A|^{2r} + \left|A^*\right|^{2r} + |B|^{2r} + \left|B^*\right|^{2r} \right\|. \tag{2.30}$$

In fact, by letting $B = A^*$ in (2.29) and (2.30), respectively, we get the following inequalities for the generalized commutator and the self commutator

$$w^r(A^*XA \pm AXA^*) \le 2^{r-1} \left\| (A^*|X|A)^r + (A|X|A^*)^r \right\|, \tag{2.31}$$

$$w^r(A^*A \pm AA^*) \le 2^{r-1} \left\| |A|^{2r} + \left|A^*\right|^{2r} \right\|. \tag{2.32}$$

3. A General Norm Inequality

In this section, we introduce a general norm inequality for Hilbert space operators, from which new inequalities for operators and generalizations of earlier results can be derived. The proof of this general inequality is similar to that of Theorem 2.5 under slight modification.

Theorem 3.1. *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, and let f and g be as in Theorem 2.5. Then*

$$\left\| \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \le \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} \left(A_i^* g^2(\left|X_i^*\right|) A\right)^r \right\| + \left\| \sum_{i=1}^{n} \left(B_i^* f^2(|X_i|) B_i\right)^r \right\| \right) \tag{3.1}$$

for $r \ge 1$.

Inequality (3.1) yields several norm inequalities as special cases. Samples of these inequalities are demonstrated below.

Corollary 3.2. *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, $r \ge 1$, and $\alpha \in (0, 1)$. Then*

$$\left\| \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \le \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} \left(A_i^* \left|X_i^*\right|^{2(1-\alpha)} A\right)^r \right\| + \left\| \sum_{i=1}^{n} \left(B_i^* |X_i|^{2\alpha} B_i\right)^r \right\| \right). \tag{3.2}$$

In particular,

$$\left\| A^*XB \right\|^r \le \frac{1}{2} \left(\left\| \left(A^*\left|X^*\right|A\right)^r \right\| + \left\| \left(B^*|X|B\right)^r \right\| \right). \tag{3.3}$$

For $A_i = B_i = I$ $(i = 1, 2, \ldots, n)$ in inequality (3.2), we get the following operator inequalities for sums of operators.

Corollary 3.3. *Let $X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, $r \ge 1$, and $\alpha \in (0, 1)$. Then*

$$\left\| \sum_{i=1}^{n} X_i \right\|^r \le \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} |X_i|^{2\alpha r} \right\| + \left\| \sum_{i=1}^{n} \left|X_i^*\right|^{2(1-\alpha) r} \right\| \right). \tag{3.4}$$

In particular, if X_i *is normal* $(i = 1, 2, \ldots, n)$, *then*

$$\left\| \sum_{i=1}^{n} X_i \right\|^r \leq n^{r-1} \left\| \sum_{i=1}^{n} |X_i|^r \right\|. \tag{3.5}$$

Remark 3.4. The inequality (3.5) is a generalized form of (1.11). The normality of X_i is necessary, this inequality is not true for arbitrary operators X_i, as may be seen for $n = 2, X_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $X_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

For $X_i = I \ (i = 1, 2, \ldots, n)$ in inequality (3.2), we get norm inequalities for products of operators.

Corollary 3.5. *Let* $A_i, B_i \in B(H) \ (i = 1, 2, \ldots, n)$ *and* $r \geq 1$. *Then*

$$\left\| \sum_{i=1}^{n} A_i^* B_i \right\|^r \leq \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} |A_i|^{2r} \right\| + \left\| \sum_{i=1}^{n} |B_i|^{2r} \right\| \right). \tag{3.6}$$

In particular,

$$\left\| \sum_{i=1}^{n} A_i^* B_i \right\| \leq \frac{1}{2} \left(\left\| \sum_{i=1}^{n} A_i^* A_i \right\| + \left\| \sum_{i=1}^{n} B_i^* B_i \right\| \right). \tag{3.7}$$

For $n = 2$ in inequality (3.2), we get the following norm inequalities that give an estimate for the usual norm of commutators.

Corollary 3.6. *Let* $A, B, C, D, S, T \in B(H)$, *and let* $r \geq 1$. *Then*

$$\|ATB + CSD\|^r \leq 2^{r-2} \left(\left\| \left(A|T^*|A^*\right)^r + \left(C|S^*|C^*\right)^r \right\| + \left\| \left(B^*|T|B\right)^r + \left(D^*|S|D\right)^r \right\| \right). \tag{3.8}$$

Finally, we end this paper by the following remark.

Remark 3.7. Inequality (3.8) gives a norm inequality for commutators of operators. If $D = A$, $C = B$, and $T = \pm S = X$, then we get

$$\|AXB \pm BXA\|^r \leq 2^{r-2} \left(\left\| \left(A|X^*|A^*\right)^r + \left(B|X^*|B^*\right)^r \right\| + \left\| \left(B^*|X|B\right)^r + \left(A^*|X|A\right)^r \right\| \right). \tag{3.9}$$

In particular,

$$\|AB \pm BA\|^r \leq 2^{r-2} \left\| |A|^{2r} + |A^*|^{2r} + |B|^{2r} + |B^*|^{2r} \right\|. \tag{3.10}$$

In fact, by letting $B = A^*$ in (3.10), we get the following inequality for self commutator

$$\|A^*A \pm AA^*\|^r \leq 2^{r-1} \left\| |A|^{2r} + |A^*|^{2r} \right\|. \tag{3.11}$$

Moreover a related inequality to (3.11) can be derived from (1.12) and (1.13). Indeed,

$$\left|\left|\left|\left(A^*A\right)^r+\left(AA^*\right)^r\right|\right|\right| \leq \left|\left|\left|\left(A^*A+AA^*\right)^r\right|\right|\right| \leq 2^{r-1}\left|\left|\left|\left(A^*A\right)^r+\left(AA^*\right)^r\right|\right|\right|. \tag{3.12}$$

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