Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 491576, 5 pages
doi:10.1155/2009/491576

Research Article

On an Extension of Shapiro's Cyclic Inequality

Nguyen Minh Tuan[1] and Le Quy Thuong[2]

[1] *Department of Mathematical Analysis, University of Hanoi, 334 Nguyen Trai Street, Hanoi, Vietnam*
[2] *Department of Mathematics, University of Hanoi, 334 Nguyen Trai Street, Hanoi, Vietnam*

Correspondence should be addressed to Nguyen Minh Tuan, tuannm@hus.edu.vn

Received 21 August 2009; Accepted 13 October 2009

Recommended by Kunquan Lan

We prove an interesting extension of the Shapiro's cyclic inequality for four and five variables and formulate a generalization of the well-known Shapiro's cyclic inequality. The method used in the proofs of the theorems in the paper concerns the positive quadratic forms.

1. Introduction

In 1954, Harold Seymour Shapiro proposed the inequality for a cyclic sum in n variables as follows:

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geq\frac{n}{2}, \tag{1.1}$$

where $x_i \geq 0$, $x_i + x_{i+1} > 0$, and $x_{i+n} = x_i$ for $i \in \mathbb{N}$. Although (1.1) was settled in 1989 by Troesch [1], the history of long year proofs of this inequality was interesting, and the certain problems remain (see [1–8]). Motivated by the directions of generalizations and proofs of (1.1), we consider the following inequality:

$$\begin{aligned} P(n,p,q) &:= \frac{x_1}{px_2+qx_3}+\frac{x_2}{px_3+qx_4}+\cdots+\frac{x_{n-1}}{px_n+qx_1}+\frac{x_n}{px_1+qx_2} \\ &\geq \frac{n}{p+q}, \end{aligned} \tag{1.2}$$

where $p, q \geq 0$ and $p + q > 0$. It is clear that (1.2) is true for $n = 3$. Indeed, by the Cauchy inequality, we have

$$(x_1 + x_2 + x_3)^2 = \left(\sqrt{\frac{x_1}{px_2 + qx_3}}\sqrt{x_1(px_2 + qx_3)} + \sqrt{\frac{x_2}{px_3 + qx_1}}\sqrt{x_2(px_3 + qx_1)} + \sqrt{\frac{x_3}{px_1 + qx_2}}\sqrt{x_3(px_1 + qx_2)}\right)^2 \leq P(3, p, q)(p + q)(x_1x_2 + x_2x_3 + x_3x_1). \tag{1.3}$$

It follows that

$$P(3, p, q) \geq \frac{(x_1 + x_2 + x_3)^2}{(p + q)(x_1x_2 + x_2x_3 + x_3x_1)} \geq \frac{3}{p + q}. \tag{1.4}$$

Obviously, (1.2) is true for every $n \geq 4$ if $p = 0$ or $q = 0$.

In this note, by studying (1.2) in the case $n = 4$, we show that it is true when $p \geq q$, and false when $p < q$. Moreover, we give a sufficient condition of p, q under which (1.2) is true in the case $n = 5$. It is worth saying that if $p < q$, then (1.2) is false for every even $n \geq 4$. Two open questions are discussed at the end of this paper.

2. Main Result

Without loss generality of (1.2), we assume that $p + q = 1$. However, (1.2) for $n = 4$ now is of the form

$$P(4, p, q) = \frac{x_1}{px_2 + qx_3} + \frac{x_2}{px_3 + qx_4} + \frac{x_3}{px_4 + qx_1} + \frac{x_4}{px_1 + qx_2} \geq 4. \tag{2.1}$$

Theorem 2.1. *It holds that* (2.1) *is true for $p \geq q$, and it is false for $p < q$.*

Proof. By the Cauchy inequality, we have

$$(x_1 + x_2 + x_3 + x_4)^2 \leq P(4, p, q)\left[x_1(px_2 + qx_3) + x_2(px_3 + qx_4) + x_3(px_4 + qx_1) + x_4(px_1 + qx_2)\right]. \tag{2.2}$$

Hence

$$P(4, p, q) \geq \frac{(x_1 + x_2 + x_3 + x_4)^2}{px_1x_2 + 2qx_1x_3 + px_1x_4 + px_2x_3 + 2qx_2x_4 + px_3x_4}. \tag{2.3}$$

It is an equality if and only if

$$px_2 + qx_3 = px_3 + qx_4 = px_4 + qx_1 = px_1 + qx_2. \tag{2.4}$$

Consider the following quadratic form:

$$\omega(x_1,x_2,x_3,x_4) = (x_1+x_2+x_3+x_4)^2 - 4(px_1x_2+2qx_1x_3+px_1x_4+px_2x_3+2qx_2x_4+px_3x_4). \tag{2.5}$$

By a simple calculation we obtain the canonical quadratic form ω as follows:

$$\omega(t_1,t_2,t_3,t_4) = t_1^2 + 4pqt_2^2 + \frac{4q(2p-1)}{p}t_3^2, \tag{2.6}$$

where

$$\begin{aligned} t_1 &= x_1 + (1-2p)x_2 + (1-4q)x_3 + (1-2p)x_4, \\ t_2 &= x_2 + \frac{1-2p}{p}x_3 - \frac{q}{p}x_4, \\ t_3 &= x_3 - x_4. \end{aligned} \tag{2.7}$$

It is easily seen that if $p \geq q$, that is, $p \geq 1/2$, then $\omega \geq 0$ for all $t_1, t_2, t_3 \in \mathbb{R}$. This implies that ω is positive. We thus have $P(4,p,q) \geq 4$.

Now let us consider the cases when ω vanishes. This depends considerably on the comparison of p with q. If $p = q$, that is, $p = 1/2$, then the quadratic form ω attains 0 at $t_1 = x_1 - x_3 = 0$ and $t_2 = x_2 - x_4 = 0$. By (2.4) we assert that $P(4,p,q) = 4$ whenever $x_1 = x_3$ and $x_2 = x_4$. Also, if $p > 1/2$, then ω vanishes if and only if

$$\begin{aligned} t_1 &= x_1 + (1-2p)x_2 + (1-4q)x_3 + (1-2p)x_4 = 0, \\ t_2 &= x_2 + \frac{1-2p}{p}x_3 - \frac{q}{p}x_4 = 0, \\ t_3 &= x_3 - x_4 = 0. \end{aligned} \tag{2.8}$$

Combining these facts with (2.4) we conclude that $P(4,p,q) = 4$ when $x_1 = x_2 = x_3 = x_4$.

Now we give a counter-example to (2.1) in the case $p < q$, that is, $p < 1/2$. Let $x_1 = x_3 = a$, $x_2 = x_4 = b$, and $a \neq b$. We will prove that

$$\frac{a}{pb+qa} + \frac{b}{pa+qb} + \frac{a}{pb+qa} + \frac{b}{pa+qb} = 2\left(\frac{a}{pb+qa} + \frac{b}{pa+qb}\right) < 4. \tag{2.9}$$

It is obvious that

$$(2.9) \Longleftrightarrow p(2q-1)\left(a^2+b^2\right) + 2\left(p^2+q^2-q\right)ab > 0 \Longleftrightarrow p(1-2p)(a-b)^2 > 0. \tag{2.10}$$

The last inequality is evident as $a \neq b$ and $p < 1/2$, so (2.9) follows.

The theorem is proved. □

Remark 2.2. Let A denote the matrix of the quadratic form ω in the canonical base of the real vector space $\mathbb{R}^4$. Namely,

$$A = \begin{pmatrix} 1 & 1-2p & 1-4q & 1-2p \\ 1-2p & 1 & 1-2p & 1-4q \\ 1-4q & 1-2p & 1 & 1-2p \\ 1-2p & 1-4q & 1-2p & 1 \end{pmatrix}. \tag{2.11}$$

Let D_1, D_2, D_3, and D_4 be the principal minors of orders 1, 2, 3, and 4, respectively, of A. By direct calculation we obtain

$$D_1 = 1, \qquad D_2 = 4pq, \qquad D_3 = 16q^2(2p-1), \qquad D_4 = 0. \tag{2.12}$$

Then ω is positive if and only if $D_i \geq 0$ for every $i = 1,2,3,4$. We find the first part of Theorem 2.1.

Thanks to the idea of using positive quadratic form we now study (1.2) in the case $n = 5$. It is sufficient to consider the case $p + q = 1$. By the Cauchy inequality, we reduce our work to the following inequality

$$\begin{aligned} \varphi(x_1,\dots,x_5) = \sum_{i=1}^{5} x_i^2 &+ (2-5p)x_1x_2 + (2-5q)x_1x_3 + (2-5q)x_1x_4 \\ &+ (2-5p)x_1x_5 + (2-5p)x_2x_3 + (2-5q)x_2x_4 + (2-5q)x_2x_5 \\ &+ (2-5p)x_3x_4 + (2-5q)x_3x_5 + (2-5p)x_4x_5 \geq 0. \end{aligned} \tag{2.13}$$

The matrix of φ in an appropriate system of basic vectors is of the form

$$B = \frac{1}{2}\begin{pmatrix} 2 & 2-5p & 2-5q & 2-5q & 2-5p \\ 2-5p & 2 & 2-5p & 2-5q & 2-5q \\ 2-5q & 2-5p & 2 & 2-5p & 2-5q \\ 2-5q & 2-5q & 2-5p & 2 & 2-5p \\ 2-5p & 2-5q & 2-5q & 2-5p & 2 \end{pmatrix}, \tag{2.14}$$

which has the principal minors

$$D_1 = 1, \qquad D_2 = \frac{5p(4-5p)}{4}, \qquad D_3 = \frac{25q(5pq-1)}{4}, \qquad D_4 = \frac{125(1-5pq)^2}{16}, \qquad D_5 = 0. \tag{2.15}$$

This implies that the necessary and sufficient condition for the positivity of the quadratic form φ is

$$\frac{5-\sqrt{5}}{10} \leq p \leq \frac{5+\sqrt{5}}{10}. \tag{2.16}$$

We thus obtain a sufficient condition under which (1.2) holds for $n = 5$.

Theorem 2.3. *If* $(5-\sqrt{5})/10 \leq p \leq (5+\sqrt{5})/10$, *then* (1.2) *is true for* $n = 5$.

Remark 2.4. Consider (1.2) in the case $n \geq 4$, n is even, and $p < q$. According to the proof of the second part of Theorem 2.1, this inequality is false. Indeed, we choose $x_1 = x_3 = \cdots = a$, $x_2 = x_4 = \cdots = b$. By the above counter-example, we conclude $P(n, p, q) < n/(p+q)$.

Open Questions. (a) Find pairs of nonnegative numbers p, q so that (1.2) is true for every $n \geq 4$.

(b) For certain $n \geq 5$, which is sufficient condition of the pair p, q so that (1.2) is true.

Acknowledgment

This work is supported partially by Vietnam National Foundation for Science and Technology Development.

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