Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 423792, 8 pages
doi:10.1155/2009/423792

Research Article

On Double Statistical P-Convergence of Fuzzy Numbers

Ekrem Savaş[1] and Richard F. Patterson[2]

[1] *Department of Mathematics, Istanbul Commerce University, 34672 Uskudar, Istanbul, Turkey*
[2] *Department of Mathematics and Statistics, University of North Florida, Building 11, Jacksonville, FL 32224, USA*

Correspondence should be addressed to Ekrem Savaş, ekremsavas@yahoo.com

Received 1 September 2009; Accepted 2 October 2009

Recommended by Andrei Volodin

Savas and Mursaleen defined the notions of statistically convergent and statistically Cauchy for double sequences of fuzzy numbers. In this paper, we continue the study of statistical convergence by proving some theorems.

1. Introduction

For sequences of fuzzy numbers, Nanda [1] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Kwon [2] introduced the definition of strongly p-Cesàro summability of sequences of fuzzy numbers. Savaş [3] introduced and discussed double convergent sequences of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. Savaş [4] studied some equivalent alternative conditions for a sequence of fuzzy numbers to be statistically Cauchy and he continue to study in [5, 6]. Recently Mursaleen and Başarir [7] introduced and studied some new sequence spaces of fuzzy numbers generated by nonnegative regular matrix. Quite recently, Savaş and Mursaleen [8] defined statistically convergent and statistically Cauchy for double sequences of fuzzy numbers. In this paper, we continue the study of double statistical convergence and introduce the definition of double strongly p-Cesàro summabilty of sequences of fuzzy numbers.

2. Definitions and Preliminary Results

Since the theory of fuzzy numbers has been widely studied, it is impossible to find either a commonly accepted definition or a fixed notation. We therefore being by introducing some notations and definitions which will be used throughout.

Let $C(R^n) = \{A \subset R^n : A \text{ compact and convex}\}$. The spaces $C(R^n)$ has a linear structure induced by the operations

$$A + B = \{a + b,\ a \in A,\ b \in B\},$$
$$\lambda A = \{\lambda a,\ \lambda \in A\} \tag{2.1}$$

for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\right\}. \tag{2.2}$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete (not separable) metric space.

A fuzzy number is a function X from R^n to $[0, 1]$ satisfying

(1) X is normal, that is, there exists an $x_0 \in R^n$ such that $X(x_0) = 1$;

(2) X is fuzzy convex, that is, for any $x, y \in R^n$ and $0 \le \lambda \le 1$,

$$X(\lambda x + (1 - \lambda)y) \ge \min\{X(x), X(y)\}; \tag{2.3}$$

(3) X is upper semicontinuous;

(4) the closure of $\{x \in R^n : X(x) > 0\}$; denoted by X^0, is compact.

These properties imply that for each $0 < \alpha \le 1$, the α-level set

$$X^\alpha = \{x \in R^n : X(x) \ge \alpha\} \tag{2.4}$$

is a nonempty compact convex, subset of R^n, as is the support X^0. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication λX, $\lambda \in R$, in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha,$$
$$[\lambda X]^\alpha = \lambda[X]^\alpha \tag{2.5}$$

for each $0 \le \alpha \le 1$.

Define for each $1 \le q < \infty$,

$$d_q(X, Y) = \left\{\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right\}^{1/q}, \tag{2.6}$$

and $d_\infty = \sup_{0\le\alpha\le1}\delta_\infty(X^\alpha, Y^\alpha)$ clearly $d_\infty(X, Y) = \lim_{q\to\infty} d_q(X, Y)$ with $d_q \le d_r$ if $q \le r$. Moreover d_q is a complete, separable, and locally compact metric space [9].

Throughout this paper, d will denote d^q with $1 \le q \le \infty$. We will need the following definitions (see [8]).

Definition 2.1. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in Pringsheim's sense or P-convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $N \in \mathcal{N}$ such that

$$d(X_{kl}, X_0) < \epsilon \quad \text{for } k, l > N, \tag{2.7}$$

and we denote by $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of X_{kl}.

More exactly we say that a double sequence (X_{kl}) converges to a finite number X_0 if X_{kl} tend to X_0 as both k and l tend to ∞ independently of one another.

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 2.2. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be P-Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$d\left(X_{pq}, X_{kl}\right) < \epsilon \quad \text{for } p \geq k \geq N,\ q \geq l \geq N. \tag{2.8}$$

Let $C^2(F)$ denote the set of all double Cauchy sequences of fuzzy numbers.

Definition 2.3. A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number M such that $d(X_{kl}, X_0) < M$ for all k and l,

$$\|X\|_{(\infty,2)} = \sup_{k,l} d(X_{kl}, X_0) < \infty. \tag{2.9}$$

We will denote the set of all bounded double sequences by $l^2_\infty(F)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.10}$$

In the case when the sequence $(K_{m,n}/mn)_{m,n=1,1}^{\infty,\infty}$ has a limit, then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.11}$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0 \tag{2.12}$$

(i.e., the set K has double natural density zero).

Definition 2.4. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be *statistically convergent* to X_0 provided that for each $\epsilon > 0$,

$$P - \lim_{m,n} \frac{1}{nm} |\{(k,l); k \leq m,\ l \leq n : d(X_{kl}, X_0) \geq \epsilon\}| = 0. \tag{2.13}$$

In this case we write $st_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy numbers by $st^2(F)$.

Definition 2.5. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be statistically P-Cauchy if for every $\varepsilon > 0$, there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that

$$P - \lim_{m,n} \frac{1}{nm} |\{(k,l); k \leq m,\ l \leq n : d(X_{kl}, X_{NM}) \geq \epsilon\}| = 0. \tag{2.14}$$

That is, $d(X_{kl}, X_{NM}) < \varepsilon$, a.a.$(k,l)$.

Let $C^2(F)$ denote the set of all double Cauchy sequences of fuzzy numbers.

Definition 2.6. A double sequence $X = (X_{kl})$ of fuzzy and let p be a positive real numbers. The sequence X is said to be strongly double p-Cesaro summable if there is a fuzzy number X_0 such that

$$P - \lim_{nm} \frac{1}{nm} \sum_{k,l=1,1}^{mn} d(X_{kl}, X_0)^p = 0. \tag{2.15}$$

In which case we say that X is strongly p-Cesaro summable to X_0.

It is quite clear that if a sequence $X = (X_{kl})$ is statistically P-convergent, then it is a statistically P-Cauchy sequence [8]. It is also easy to see that if there is a convergent sequence $Y = (Y_{kl})$ such that $X_{kl} = Y_{kl}$ a.a.(k,l), then $X = (X_{kl})$ is statistically convergent.

3. Main Result

Theorem 3.1. *A double sequence $X = (X_{kl})$ of fuzzy numbers is statistically P-Cauchy then there is a P-convergent double sequence $Y = (Y_{kl})$ such that $X_{kl} = Y_{kl}$ a.a.(k,l).*

Proof. Let us begin with the assumption that $X = (X_{kl})$ is statistically P-Cauchy this grant us a closed ball $B = \overline{B}(X_{M_1,N_1}, 1)$ that contains X_{kl} a.a.(k,l) for some positive numbers M_1 and N_1. Clearly we can choose M and N such that $B' = \overline{B}(X_{M,N}, 1/(2 \cdot 2))$ contains $X_{K,L}$ a.a.(k,l). It is also clear that $X_{k,l} \in B_{1,1} = B \cap \overline{B}$ a.a.(k,l); for

$$\begin{aligned} &\left\{(k,l)k \leq m;\ l \leq n : X_{k,l} \notin B \cap \overline{B}\right\} \\ &\quad = \left\{(k,l)k \leq m;\ l \leq n : X_{k,l} \notin \overline{B}\right\} \cup \{(k,l)k \leq m;\ l \leq n : X_{k,l} \notin B\}, \end{aligned} \tag{3.1}$$

we have

$$\begin{aligned} P-\lim_{m,n}\frac{1}{mn}\left|\left\{k\le m;\ l\le n: X_{k,l}\notin B\cap\overline{B}\right\}\right| &\le P-\lim_{m,n}\frac{1}{mn}\left|\left\{k\le m;\ l\le n: X_{k,l}\notin\overline{B}\right\}\right| \\ &\quad + P-\lim_{m,n}\frac{1}{mn}|\{k\le m;\ l\le n: X_{k,l}\notin B\}| \\ &= 0. \end{aligned} \tag{3.2}$$

Thus $B_{1,1}$ is a closed ball of diameter less than or equal to 1 that contains $X_{k,l}$ a.a.(k,l). Now we let us consider the second stage to this end we choose M_2 and N_2 such that $x_{k,l}\in B''=\overline{B}(X_{M_2,N_2},1/(2^2\cdot 2^2))$. In a manner similar to the first stage we have $X_{k,l}\in B_{2,2}=B_1\cap B''$, a.a.$(k,l)$. Note the diameter of $B_{2,2}$ is less than or equal $2^{1-2}\cdot 2^{1-2}$. If we now consider the (m,n)th general stage we obtain the following. First a sequence $\{B_{m,n}\}_{m,n=1,1}^{\infty,\infty}$ of closed balls such that for each (m,n), $B_{m,n}\supset\ B_{m+1,n+1}$, the diameter of $B_{m,n}$ is not greater than $2^{1-m}\cdot 2^{1-m}$ with $X_{k,l}\in\ B_{m,n}$, a.a.(k,l). By the nested closed set theorem of a complete metric space we have $\bigcap_{m,n=1,1}^{\infty,\infty}\ B_{mn}\neq\emptyset$. So there exists a fuzzy number $A\in\bigcap_{m,n=1,1}^{\infty,\infty}B_{m,n}$. Using the fact that $X_{k,l}\notin\ B_{m,n}$, a.a.(k,l), we can choose an increasing sequence $T_{m,n}$ of positive integers such that

$$\frac{1}{mn}|\{k\le m;\ l\le n: X_{k,l}\notin B_{m,n}\}|<\frac{1}{pq}\quad\text{if } m,n>T_{m,n}. \tag{3.3}$$

Now define a double subsequence $Z_{k,l}$ of $X_{k,l}$ consisting of all terms $X_{k,l}$ such that $k,l>T_{1,1}$ and if

$$T_{m,n}<k,l\le T_{m+1,n+1},\quad\text{then } X_{k,l}\notin B_{m,n}. \tag{3.4}$$

Next we define the sequence $(Y_{k,l})$ by

$$Y_{k,l}:=\begin{cases} A, & \text{if } X_{k,l} \text{ is a term of } Z,\\ X_{k,l}, & \text{otherwise.}\end{cases} \tag{3.5}$$

Then $P-\lim_{k,l}Y_{k,l}=A$ indeed if $\epsilon>1/m,n>0$, and $k,l>T_{m,n}$, then either $X_{k,l}$ is a term of Z. Which means $Y_{k,l}=A$ or $Y_{k,l}=X_{k,l}\in B_{m,n}$ and $d(Y_{k,l}-A)\le|B_{m,n}|\le$ diameter of $B_{m,n}\le 2^{1-m}\cdot 2^{1-n}$. We will now show that $X_{k,l}=Y_{k,l}$ a.a.(k,l). Note that if $T_{m,n}<m,n<T_{m+1,n+1}$, then

$$\{k\le m,\ l\le n: Y_{k,l}\neq X_{k,l}\}\subset\{k\le m,\ l\le n: X_{k,l}\notin B_{m,n}\}, \tag{3.6}$$

and by (3.3)

$$\frac{1}{mn}|\{k\le m,\ l\le n: Y_{k,l}\neq X_{k,l}\}|\le\frac{1}{mn}|\{k\le m,\ l\le n: X_{k,l}\notin B_{m,n}\}|<\frac{1}{mn}. \tag{3.7}$$

Hence the limit as (m,n) is 0 and $X_{k,l}=Y_{k,l}$ a.a.(k,l). This completes the proof. □

Theorem 3.2. *If $X = (X_{k,l})$ is strongly p-Cesaro summable or statistically P-convergent to X_0, then there is a P-convergent double sequences Y and a statistically P-null sequence Z such that $P-\lim_{k,l} Y_{k,l} = X_0$ and $st_2\lim_{k,l} Z_{k,l} = 0$.*

Proof. Note that if $X = [X_{k,l}]$ is strongly p-Cesaro summable to X_0, then X is statistically P-convergent to X_0. Let $N_0 = 0$ and $M_0 = 0$ and select two increasing index sequences of positive integers $N_1 < N_2 < \cdots$ and $M_1 < M_2 < \cdots$ such that $m > M_i$ and $n > N_j$, we have

$$\frac{1}{mn}\left|\left\{k \le m,\ l \le n : d(X_{k,l}, X_0) \ge \frac{1}{ij}\right\}\right| < \frac{1}{ij}. \tag{3.8}$$

Define Y and Z as follows: if $N_0 < k < N_1$ and $M_0 < l < M_1$, set $Z_{k,l} = 0$ and $Y_{k,l} = X_{k,l}$. Suppose that $i, j > 1$ and $N_i < k < N_{i+1}$, $M_j < l < M_{j+1}$, then

$$\begin{gathered} Y := \begin{cases} X_{k,l}, & d(X_{k,l}, X_0) < \dfrac{1}{ij}, \\ X_0, & \text{otherwise}, \end{cases} \\ Y_{k,l} := \begin{cases} 0, & d(X_{k,l}, X_0) < \dfrac{1}{ij}, \\ X_{k,l}, & \text{otherwise}. \end{cases} \end{gathered} \tag{3.9}$$

We now show that $P-\lim_{k,l} Y_{k,l} = x_0$. Let $\epsilon > 0$ be given, pick (i, j) be given, and pick i and j such that $\epsilon > 1/ij$, thus for $k, l > M_i, N_j$, since $d(Y_{k,l}, X_0) < d(X_{k,l}, X_0) < \epsilon$ if $d(X_{k,l}, X_0) < 1/ij$ and $d(Y_{k,l}, X_0) = 0$ if $d(X_{k,l}, X_0) > 1/ij$, we have $d(Y_{k,l}, X_0) < \epsilon$.

Next we show that Z is a statistically P-null double sequence, that is, we need to show that $P-\lim_{m,n}(1/mn)|\{k \le ml \le n : Z_{k,l} \ne 0\}| = 0$. Let $\delta > 0$ if $(i, j) \in NxN$ such that $1/ij < \delta$, then $|\{k \le ml \le n : Z_{k,l} \ne 0\}| < \delta$ for all $m, n > M_i, N_j$. From the construction of (M_i, N_j), if $M_i < k \le M_{i+1}$ and $N_j < l \le N_{j+1}$, then $Z_{k,l} \ne 0$ only if $d(X_{k,l}, X_0) > 1/ij$. It follows that if $M_i < k \le M_{i+1}$ and $N_j < l \le N_{j+1}$, then

$$\{k \le m; l \le n : Z_{k,l} \ne 0\} \subset \left\{k \le m; l \le n : d(X_{k,l}, X_0) < \frac{1}{pq}\right\}. \tag{3.10}$$

Thus for $M_i < m \le M_{i+1}$ and $N_j < n \le N_{j+1}$ and $p, q > i, j$, then

$$\frac{1}{mn}|\{k \le m; l \le n : Z_{k,l} \ne 0\}| \le \frac{1}{mn}\left|\left\{k \le m; l \le n : d(X_{k,l}, X_0) > \frac{1}{p,q}\right\}\right| < \frac{1}{mn} < \frac{1}{ij} < \delta. \tag{3.11}$$

this completes the proof. □

Corollary 3.3. *If $X = (X_{k,l})$ is a strongly p-Cesaro summable to X_0 or statistically P-convergent to X_0, then X has a double subsequence which is P-converges to X_0.*

4. Conclusion

In recent years the statistical convergence has been adapted to the sequences of fuzzy numbers. Double statistical convergence of sequences of fuzzy numbers was first deduced in similar form by Savas and Mursaleen as we explain now: a double sequences $X = \{X_{k,l}\}$ is said to be P-statistically convergent to X_0 provided that for each $\epsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn} \{\text{numbers of } (k,l) : k \leq m,\ l \leq n, d(X_{k,l}, X_0) \geq \epsilon\}, \tag{4.1}$$

Since the set of real numbers can be embedded in the set of fuzzy numbers, statistical convergence in reals can be considered as a special case of those fuzzy numbers. However, since the set of fuzzy numbers is partially ordered and does not carry a group structure, most of the results known for the sequences of real numbers may not be valid in fuzzy setting. Therefore, this theory should not be considered as a trivial extension of what has been known in real case. In this paper, we continue the study of double statistical convergence and also some important theorems are proved.

References

[1] S. Nanda, "On sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 123–126, 1989.

[2] J.-S. Kwon, "On statistical and p-Cesaro convergence of fuzzy numbers," *Journal of Applied Mathematics and Computing*, vol. 7, no. 1, pp. 195–203, 2000.

[3] E. Savaş, "A note on double sequences of fuzzy numbers," *Turkish Journal of Mathematics*, vol. 20, no. 2, pp. 175–178, 1996.

[4] E. Savaş, "On statistically convergent sequences of fuzzy numbers," *Information Sciences*, vol. 137, no. 1–4, pp. 277–282, 2001.

[5] E. Savaş, "A note on sequence of fuzzy numbers," *Information Sciences*, vol. 124, no. 1–4, pp. 297–300, 2000.

[6] E. Savaş, "On lacunary statistically convergent double sequences of fuzzy numbers," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 134–141, 2008.

[7] M. Mursaleen and M. Başarir, "On some new sequence spaces of fuzzy numbers," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 9, pp. 1351–1357, 2003.

[8] E. Savaş and Mursaleen, "On statistically convergent double sequences of fuzzy numbers," *Information Sciences*, vol. 162, no. 3-4, pp. 183–192, 2004.

[9] P. Diamond and P. Kloeden, "Metric spaces of fuzzy sets," *Fuzzy Sets and Systems*, vol. 35, no. 2, pp. 241–249, 1990.