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Research Article

Subnormal Solutions of Second-Order Nonhomogeneous Linear Differential Equations with Periodic Coefficients

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We obtain the representations of the subnormal solutions of nonhomogeneous linear differential equation $f'' + [P_1(e^z) + Q_1(e^{-z})]f' + [P_2(e^z) + Q_2(e^{-z})]f = R_1(e^z) + R_2(e^{-z})$, where $P_1(z), P_2(z), Q_1(z), Q_2(z), R_1(z)$, and $R_2(z)$ are polynomials in z such that $P_1(z), P_2(z), Q_1(z)$, and $Q_2(z)$ are not all constants, $\deg P_1 > \deg P_2$. We partly resolve the question raised by G. G. Gundersen and E. M. Steinbart in 1994.

1. Introduction

We use the standard notations from Nevanlinna theory in this paper (see [1–3]).

The study of the properties of the solutions of a linear differential equation with periodic coefficients is one of the difficult aspects in the complex oscillation theory of differential equations. However, it is also one of the important aspects since it relates to many special functions. Some important researches were done by different authors; see, for instance, [4–9].

Now, we firstly consider the second-order homogeneous linear differential equations

$$f'' + P_1(e^z)f' + P_2(e^z)f = 0, \tag{1.1}$$

where $P_1(z)$ and $P_2(z)$ are polynomials in z and are not both constants. It is well known that every solution $f(z)$ of (1.1) is an entire function.

Let $f(z)$ be an entire function. We define

$$\rho_e(f) = \overline{\lim_{r\to+\infty}}\frac{\log T(r,f)}{r}, \tag{1.2}$$

to be the e-type order of $f(z)$.

If $f(z)\not\equiv 0$ is a solution of (1.1) and if $f(z)$ satisfies $\rho_e(f) = 0$, then we say that $f(z)$ is a subnormal solution of (1.1). For convenience, we also say that $f(z) \equiv 0$ is a subnormal solution of (1.1).

H. Wittich has given the general forms of all subnormal solutions of (1.1) that are shown in the following theorem.

Theorem A (see [9]). *If $f(z)\not\equiv 0$ is a subnormal solution of* (1.1)*, where $P_1(z)$ and $P_2(z)$ are polynomials in z and are not both constants, then $f(z)$ must have the form*

$$f(z) = e^{cz}\left(a_0 + a_1 e^z + \cdots + a_m e^{mz}\right), \tag{1.3}$$

where $m \geq 0$ is an integer and $c, a_0, a_1, \ldots, a_m$ are constants with $a_0 a_m \neq 0$.

G. G. Gundersen and E. M. Steinbart refined Theorem A and obtained the exact forms of subnormal solutions of (1.1) as follows.

Theorem B (see [6]). *In addition to the statement of Theorem A, the following statements hold with regard to the subnormal solutions $f(z)$ of* (1.1).

(i) *If $\deg P_1 > \deg P_2$ and $P_2\not\equiv 0$, then any subnormal solution $f(z)\not\equiv 0$ of* (1.1) *must have the form*

$$f(z) = \sum_{k=0}^{m} a_k e^{-kz}, \tag{1.4}$$

where $m \geq 1$ is an integer and $a_0, a_1, \ldots, a_m$ are constants with $a_0 a_m \neq 0$.

(ii) *If $P_2 \equiv 0$ and $\deg P_1 \geq 1$, then any subnormal solution $f(z)$ of* (1.1) *must be a constant.*

(iii) *If $\deg P_1 < \deg P_2$, then the only subnormal solution $f(z)$ of* (1.1) *is $f(z) \equiv 0$.*

Whether the conclusions of Theorem A and Theorem B can be generalized or not, Gundersen and Steinbart considered the second-order nonhomogeneous linear differential equations

$$f'' + P_1(e^z)f' + P_2(e^z)f = R_1(e^z) + R_2(e^{-z}), \tag{1.5}$$

where $P_1(z), P_2(z), R_1(z)$, and $R_2(z)$ are polynomials in z such that $P_1(z), P_2(z)$ are not both constants. They found the exact forms of all subnormal solutions of (1.5), that is, what is mentioned in [6, Theorem 2.2, Theorem 2.3 and Theorem 2.4].

In [6], they also have raised the following problem, that is, what about the forms of the subnormal solutions of the equation

$$f'' + \left[P_1(e^z) + Q_1(e^{-z})\right]f' + \left[P_2(e^z) + Q_2(e^{-z})\right]f = R_1(e^z) + R_2(e^{-z}), \tag{1.6}$$

where $P_1(z), P_2(z), Q_1(z), Q_2(z), R_1(z)$, and $R_2(z)$ are polynomials in z such that $P_1(z)$, $P_2(z)$, $Q_1(z)$, and $Q_2(z)$ are not all constants?

In [7], we have obtained the exact forms of all subnormal solutions of homogeneous equation

$$f'' + \left[P_1(e^z) + Q_1(e^{-z})\right]f' + \left[P_2(e^z) + Q_2(e^{-z})\right]f = 0, \tag{1.7}$$

where $P_1(z), P_2(z), Q_1(z)$, and $Q_2(z)$ are polynomials in z and are not all constants.

In this paper, we obtain the forms of subnormal solutions of nonhomogeneous linear differential equation (1.6) when $\deg P_1 > \deg P_2$. We have the following theorem.

Theorem 1.1. *Suppose that $f(z)$ is a subnormal solution of* (1.6), *where $P_1(z), P_2(z)$, $Q_1(z), Q_2(z), R_1(z)$, and $R_2(z)$ are polynomials in z such that $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are not all constants.*

(i) *If $\deg P_1 > \deg P_2$ and $\deg P_1 > \deg R_1$, then $f(z)$ must have the form*

$$f(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \tag{1.8}$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

(ii) *If $\deg P_1 > \deg P_2$ and $\deg P_1 \leq \deg R_1$, then $f(z)$ must have the form*

$$f(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}) + g_0(e^z), \tag{1.9}$$

where β is a constant, c_1 and c_2 are constants that may or may not be equal to zero, $g_0(z)$ may be equal to zero or may be a polynomial in z, $g_1(z), g_2(z), g_3(z)$, and $g_4(z)$ are polynomials in z with $\deg\{g_3\} \geq 1$.

2. Lemmas for the Proof

In order to prove Theorem 1.1, we need some lemmas.

Lemma 2.1 (see [7]). *Suppose that $f(z)$ is a subnormal solution of* (1.7), *where $P_1(z)$, $P_2(z)$, $Q_1(z)$ and $Q_2(z)$ are polynomials in z and are not all constants.*

(i) *If $\deg P_1 > \deg P_2$ and $P_2 + Q_2 \equiv 0$, then any subnormal solution $f(z)$ must be a constant.*

(ii) *If $\deg P_1 > \deg P_2$ and $P_2 + Q_2 \not\equiv 0$, then $f(z) \not\equiv 0$ must have the form*

$$f(z) = g_2(e^{-z}), \tag{2.1}$$

where $g_2(z)$ is a polynomial in z with $\deg\{g_2\} \geq 1$.

Lemma 2.2 (see [10]). *Let $f(z)$ be a transcendental meromorphic function, let $\alpha > 1$ be a given real constant, and let $k > j \geq 0$. Then there exists a constant $C = C(\alpha) > 0$ such that the following two statements hold (where $r = |z|$).*

(i) *There exists a set $E_1 \subset [-\pi, \pi)$ that has linear measure zero such that if $\psi \in [-\pi, \pi) \setminus E_1$, then there is a constant $R = R(\psi) > 0$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R$, one has*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left(\frac{T(\alpha r, f)}{r}\log^{\alpha} r \log T(\alpha r, f)\right)^{k-j}. \tag{2.2}$$

(ii) *There exists a set $E_2 \subset (0, \infty)$ that has finite logarithmic measure such that* (2.2) *holds for all z satisfying $|z| = r \notin E_2 \cup [0,1]$.*

Lemma 2.3 (see [6]). *Let $0 < r_1 < r_2 < \cdots < r_j < \cdots$ with $r_j \to \infty$ as $j \to \infty$, and let $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Let*

$$\begin{gathered} W_1 = \{z : \arg z = \theta_1\}, \qquad W_2 = \{z : \arg z = \theta_2\}, \\ W_3 = \{z : |z| = r_j \text{ for some } j \text{ and } \theta_1 \leq \arg z \leq \theta_2\}, \end{gathered} \tag{2.3}$$

and set

$$W = W_1 \cup W_2 \cup W_3. \tag{2.4}$$

Let $f(z)$ be analytic on the set W. Suppose that $f'(z)$ is unbounded on the set W. Then there exists an infinite sequence of points $z_j \in W$ with $|z_j| \to \infty$ as $j \to \infty$ such that

$$\begin{gathered} f'(z_j) \longrightarrow \infty, \\ \left|\frac{f'(z_j)}{f(z_j)}\right| \geq (1 + o(1))\frac{1}{8|z_j|}. \end{gathered} \tag{2.5}$$

Lemma 2.4 (see [8]). *Consider the nth- order differential equation of the form*

$$P_0(e^z, e^{-z})f^{(n)} + P_1(e^z, e^{-z})f^{(n-1)} + \cdots + P_n(e^z, e^{-z})f = P_{n+1}(e^z, e^{-z}), \tag{2.6}$$

where $P_j(e^z, e^{-z})$ $(j = 0, 1, 2, \ldots, n+1)$ are polynomials in e^z and e^{-z} with $P_0(e^z, e^{-z}) \not\equiv 0$. Suppose that $f(z)$ is an entire and subnormal solution of (2.6) *and that $f(z)$ can be expressed as $f(z) = e^{\beta z}G(e^z)$, where β is a constant and $G(\zeta)$ is analytic on $0 < |\zeta| < \infty$. Then $f(z)$ has the form*

$$f(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})], \tag{2.7}$$

where β is a constant and $g_1(z)$ and $g_2(z)$ are polynomials in z.

As an application of Lemma 2.4, one has the following lemma.

Lemma 2.5. *Suppose that $f(z)$ is an entire subnormal solution of* (2.6)*, where $P_j(e^z, e^{-z})(j = 0, 1, 2, \ldots, n+1)$ are polynomials in e^z and e^{-z} with $P_0(e^z, e^{-z}) \not\equiv 0$, and that $f(z)$ and $f(z+2\pi i)$ are linearly dependent. Then $f(z)$ has the form*

$$f(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \tag{2.8}$$

where β is a constant and $g_1(z)$ and $g_2(z)$ are polynomials in z.

Proof. Since $f(z)$ is entire and is linearly dependent with $f(z+2\pi i)$, $f(z)$ can be written as $f(z) = e^{\beta z}G(e^z)$ (see [11, page 382]), where β is a constant and $G(\zeta)$ is analytic on $0 < |\zeta| < \infty$. Then we have the representation from Lemma 2.4. □

Lemma 2.6. *Suppose that $f(z)$ is a solution of* (1.6)*, where $P_1(z)$, $P_2(z)$, $Q_1(z)$, $Q_2(z)$, $R_1(z)$, and $R_2(z)$ are polynomials in z such that $P_1(z)$, $P_2(z)$, $Q_1(z)$ and $Q_2(z)$ are not all constants. If*

$$\deg P_2 < \deg P_1 < \deg R_1, \tag{2.9}$$

then there exists a polynomial $g_0(z)$ such that

$$f(z) = g(z) + g_0(e^z), \tag{2.10}$$

where $g(z)$ is a solution of

$$g'' + \left[P_1(e^z) + Q_1(e^{-z})\right]g' + \left[P_2(e^z) + Q_2(e^{-z})\right]g = R_3(e^z) + R_4(e^{-z}), \tag{2.11}$$

where $R_3(z)$ and $R_4(z)$ are polynomials in z with $\deg R_3 \leq \deg P_1$.

Proof. Let $n = \deg R_1 - \deg P_1 \geq 1$, and set

$$g(z) = f(z) - ae^{nz}, \tag{2.12}$$

where a is the constant such that

$$\deg\{R_1(z) - anz^n P_1(z)\} < \deg R_1. \tag{2.13}$$

It follows from (1.6) and (2.12) that

$$g'' + \left[P_1(e^z) + Q_1(e^{-z})\right]g' + \left[P_2(e^z) + Q_2(e^{-z})\right]g = T_1(e^z) + T_2(e^{-z}), \tag{2.14}$$

where

$$\begin{aligned} T_1(e^z) + T_2(e^{-z}) = R_1(e^z) + R_2(e^{-z}) &- ane^{nz}\left[P_1(e^z) + Q_1(e^{-z})\right] \\ &- ae^{nz}\left[P_2(e^z) + Q_2(e^{-z})\right] - an^2e^{nz}. \end{aligned} \tag{2.15}$$

So $T_1(e^z)$ and $T_2(e^{-z})$ are polynomials in e^z and e^{-z}, respectively, and $\deg T_1 < \deg R_1$ by (2.13), but $T_1(e^z)$ and $T_2(e^{-z})$ have the exact representations that depend on the relations of $n, \deg Q_1$, and $\deg Q_2$. If $\deg T_1 \leq \deg P_1$, then (2.14) is of the form (2.11), and (2.12) gives (2.10). If $\deg T_1 > \deg P_1$, then we repeat the above process finite times until we obtain (2.10) and (2.11). This completes the proof of Lemma 2.6. □

3. Proof of Theorem

In this section, we will prove Theorem 1.1.

Proof. (i) Suppose that $f(z)$ is a subnormal solution of (1.6) with $\deg P_1 > \deg P_2$ and $\deg P_1 > \deg R_1$. If $f(z)$ is a polynomial solution of (1.6), then $f(z)$ must be a constant, which is of the form (1.8). Thus we suppose that $f(z)$ is transcendental. It follows from Lemma 2.2(i) that there exists a set $E_1 \subset [-\pi, \pi)$ that has linear measure zero such that if $\psi \in [-\pi, \pi) \setminus E_1$, then there is a constant $R = R(\psi) > 0$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R$, we have

$$\left|\frac{f''}{f'}\right| \leq C\frac{T(2r, f)}{r}\log^2 r \log T(2r, f), \tag{3.1}$$

where $C > 0$ is a constant and $r = |z|$. It also follows from Lemma 2.2(ii) that there exists a set $E_2 \subset (0, \infty)$ that has finite logarithmic measure such that (3.1) holds for all z satisfying $|z| \notin E_2 \cup [0, 1]$.

Now let $r_1, r_2, \ldots, r_j$, be an infinite sequence satisfying $1 < r_1 < r_2 < \cdots < r_j < \cdots$ such that $r_j \notin E_2$ for all j and $r_j \to \infty$ as $j \to \infty$. Let ε_0 be a small constant such that $-(\pi/2) + \varepsilon_0 \notin E_1$ and $(\pi/2) - \varepsilon_0 \notin E_1$. Set

$$\begin{gathered} W_1 = \left\{z : \arg z = -\frac{\pi}{2} + \varepsilon_0\right\}, \qquad W_2 = \left\{z : \arg z = \frac{\pi}{2} - \varepsilon_0\right\}, \\ W_3 = \left\{z : |z| = r_j \text{ for some } j \text{ and } -\frac{\pi}{2} + \varepsilon_0 \leq \arg z \leq \frac{\pi}{2} - \varepsilon_0\right\}, \end{gathered} \tag{3.2}$$

and set

$$W = W_1 \cup W_2 \cup W_3. \tag{3.3}$$

From above, we have that (3.1) holds on the set W.

We now assert that $f'(z)$ is bounded on the set W. On the contrary, it follows from Lemma 2.3 that there exists a sequence of points $z_j \in W$ with $|z_j| \to \infty$ as $j \to \infty$ such that

$$|f'(z_j)| \longrightarrow \infty, \tag{3.4}$$

$$\left|\frac{f'(z_j)}{f(z_j)}\right| \geq (1 + o(1))\frac{1}{8|z_j|}. \tag{3.5}$$

By (1.6), we have for all $z_j \in W$,

$$\frac{f''(z_j)}{f'(z_j)} \cdot \frac{1}{P_1(e^{z_j}) + Q_1(e^{-z_j})} + 1 + \frac{P_2(e^{z_j}) + Q_2(e^{-z_j})}{P_1(e^{z_j}) + Q_1(e^{-z_j})} \cdot \frac{f(z_j)}{f'(z_j)} = \frac{1}{f'(z_j)} \cdot \frac{R_1(e^{z_j}) + R_2(e^{-z_j})}{P_1(e^{z_j}) + Q_1(e^{-z_j})}. \tag{3.6}$$

It follows from (3.4)–(3.6) and $\rho_e(f) = 0$ that (3.6) yields $1 \equiv 0$ as $|z_j| \to \infty$ on the set W. This is a contradiction.

By the maximum modulus principle, $f'(z)$ is bounded in the angular domain

$$D = \left\{ z : -\frac{\pi}{2} + \varepsilon_0 \leq \arg z \leq \frac{\pi}{2} - \varepsilon_0 \right\}. \tag{3.7}$$

However, we know

$$f(z) = f(z_0) + \int_{z_0}^{z} f'(t)dt, \tag{3.8}$$

where the integral of $f'(t)$ is defined on the simple contour C, extending from a point z_0 to a point z in the complex domain.

So we obtain

$$\left| f(z) \right| = O(|z|), \tag{3.9}$$

as $z \to \infty$ in the angular domain D.

Thus , from the Cauchy integral formula, we obtain

$$\left| f''(z) \right| = O(1), \tag{3.10}$$

as $z \to \infty$ in the angular domain D. By (1.6), (3.8), and (3.9), we have for some constant $A > 0$

$$\left| f'(z) \right| \leq \exp\{-(1 + o(1))A|z|\}, \tag{3.11}$$

as $z \to \infty$ in the angular domain D. Together with (3.8) and (3.11), $f(z)$ is bounded in the angular domain D.

If $f(z) \equiv f(z + 2\pi i)$, it follows from Lemma 2.5 that $f(z)$ must have the form (1.8).

If $f(z) \not\equiv f(z + 2\pi i)$, since $f(z)$ is a subnormal solution of (1.6), so is $f(z + 2\pi i)$. Thus,

$$h(z) = f(z) - f(z + 2\pi i) \tag{3.12}$$

will be a subnormal solution of (1.7). Since we suppose that $\deg P_1 > \deg P_2$, we will discuss the following two cases.

Case 1. If $P_2 + Q_2 \equiv 0$, we have, by Lemma 2.1(i),

$$h(z) = C, \tag{3.13}$$

where C is a constant. Hence $h'(z) \equiv 0$, that is,

$$f'(z) \equiv f'(z + 2\pi i). \tag{3.14}$$

From this, $f'(z)$ can be written as $f'(z) = e^{\beta_1 z} G_1(e^z)$(see [11, page 382]), where β_1 is a constant and $G_1(z)$ is analytic on $0 < |z| < \infty$. Thus, $f(z)$ can be written as $f(z) = e^{\beta_2 z} G_2(e^z)$, where β_2 is a constant and $G_2(z)$ is analytic on $0 < |z| < \infty$. It follows from Lemma 2.4 that

$$f(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \tag{3.15}$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z. Thus, $f(z)$ has the form of (1.8).

Case 2. If $P_2 + Q_2 \not\equiv 0$, we obtain from Lemma 2.1(ii) that

$$h(z) = f(z) - f(z + 2\pi i) = g_3(e^{-z}), \tag{3.16}$$

where $g_3(z)$ is a polynomial in z with $\deg\{g_3\} \geq 1$.

However, we can assert that $h(z) = g_3(e^{-z}) \equiv 0$ in (3.16). Otherwise, there exists $z_0 \in D$ such that

$$g_3(e^{-z}) = a \neq 0. \tag{3.17}$$

By (3.16), we have

$$f(z + 2\pi i) - f(z + 4\pi i) = g_3(e^{-z}). \tag{3.18}$$

Thus from (3.16) and (3.18), we have

$$f(z) - f(z + 4\pi i) = 2g_3(e^{-z}). \tag{3.19}$$

By repeating this process finite times, we obtain that for any integer $n \geq 1$,

$$f(z) - f(z + n \cdot 2\pi i) = ng_3(e^{-z}). \tag{3.20}$$

We have, by (3.17) and (3.20),

$$f(z) - f(z + n \cdot 2\pi i) = na \longrightarrow \infty, \tag{3.21}$$

as $n \to +\infty$. This is a contradiction to the fact that $f(z)$ is bounded in the angular domain D. This shows that $P_2 + Q_2 \not\equiv 0$ is not possible when $f(z) \not\equiv f(z+2\pi i)$ under the hypotheses. This completes the proof of part (i).

(ii) We firstly suppose that $\deg P_1 = \deg R_1$. Since $f(z)$ is a subnormal solution of (1.6), so is $f(z+2\pi \mathrm{i})$. Set

$$h(z) = f(z) - f(z+2\pi i). \tag{3.22}$$

Then $h(z)$ is a subnormal solution of (1.7). Now if $h(z) \equiv 0$, this shows that $f(z) \equiv f(z+2\pi i)$ and $f(z)$ has the form of (1.9) by Lemma 2.5. Thus, we suppose that $h(z) \not\equiv 0$ in the following.

Now, assume that $\deg P_1 > \deg P_2$.

If $P_2 + Q_2 \equiv 0$, it follows from the proof of Case 1 of Theorem 1.1(i) that $f(z)$ has the form of (1.9).

If $P_2 + Q_2 \not\equiv 0$, we obtain from Lemma 2.1(ii) that

$$h(z) = f(z) - f(z+2\pi i) = g_3(e^{-z}), \tag{3.23}$$

where $g_3(\mathrm{z})$ is a polynomial in z with $\deg\{g_3\} \geq 1$.

Set

$$g_3(e^{-z}) = \sum_{k=0}^{m} a_k e^{-kz}, \tag{3.24}$$

where $m \geq 1$ is an integer and $a_k (k = 0, 1, \ldots, m)$ are constants with $a_m \neq 0$.

Let $n = \deg P_1 = \deg R_1$ and set

$$P_1(e^z) = \sum_{k=0}^{n} p_k e^{kz}, \qquad R_1(e^z) = \sum_{k=0}^{n} r_k e^{kz}, \tag{3.25}$$

where $p_n r_n \neq 0$.

Now, we will discuss the following two cases.

Case A. We consider $a_0 \neq 0$ in (3.24). Let c_1 be a constant defined by

$$c_1 a_0 p_n = r_n, \tag{3.26}$$

and set

$$H_1(z) = c_1 z h(z). \tag{3.27}$$

Since $h(z)$ is a subnormal solution of (1.7), it follows from (3.27) that $H_1(z)$ satisfies

$$H_1'' + \left[P_1(e^z) + Q_1(e^{-z})\right] H_1' + \left[P_2(e^z) + Q_2(e^{-z})\right] H_1 = 2c_1 h'(z) + c_1 h(z)\left[P_1(e^z) + Q_1(e^{-z})\right]. \tag{3.28}$$

We obtain from (3.23)–(3.28) that

$$H_1'' + \left[P_1(e^z) + Q_1(e^{-z})\right]H_1' + \left[P_2(e^z) + Q_2(e^{-z})\right]H_1 = r_n e^{nz} + T_3(e^z) + T_4(e^{-z}), \qquad (3.29)$$

where $T_3(z)$ and $T_4(z)$ are polynomials in z with $\deg T_3 \le n-1$.

Set

$$\phi_1(z) = H_1(z) - f(z). \qquad (3.30)$$

It follows from (1.6), (3.25), (3.29) and (3.30) that $\phi_1'' + [P_1(e^z) + Q_1(e^{-z})]\phi_1' + [P_2(e^z) + Q_2(e^{-z})]\phi_1 = T_3(e^z) + T_4(e^{-z}) - \sum_{k=0}^{n-1} r_k e^{kz} - R_2(e^{-z})$.

Set

$$S_1(e^z) = T_3(e^z) - \sum_{k=0}^{n-1} r_k e^{kz}, \qquad S_2(e^{-z}) = T_4(e^{-z}) - R_2(e^{-z}). \qquad (3.31)$$

So $\deg S_1 \le n-1 < \deg P_1$, and $\phi_1(z)$ satisfies

$$\phi_1'' + \left[P_1(e^z) + Q_1(e^{-z})\right]\phi_1' + \left[P_2(e^z) + Q_2(e^{-z})\right]\phi_1 = S_1(e^z) + S_2(e^{-z}). \qquad (3.32)$$

We have by (3.27) that $h(z)$ is a subnormal solution of (1.7), $H_1(z)$ is a subnormal solution of (3.29). Moreover, $\phi_1(z)$ is also a subnormal solution of (3.32) by (3.30) and $f(z)$ is a subnormal solution of (1.6). Thus, we deduce from Theorem 1.1(i) and (3.32) with $\deg P_1 > \deg S_1$ that $\phi_1(z)$ has the form

$$\phi_1(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \qquad (3.33)$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z. Hence (3.23), (3.24), (3.27), (3.30), and (3.33) yield

$$f(z) = e^{\beta z}\left[-g_1(e^z) - g_2(e^{-z})\right] + c_1 z g_3(e^{-z}), \qquad (3.34)$$

where $c_1 \ne 0$ and β are constants, $g_1(z), g_2(z)$, and $g_3(z)$ are polynomials in z with $\deg g_3 \ge 1$. This is the form of (1.9).

Case B. We consider $a_0 = 0$ in (3.24). Let c_2 be a constant defined by

$$c_2 s a_s p_n = r_n, \qquad (3.35)$$

where $s \in \{0, 1, \ldots, m\}$ is a number such that a_s is the first coefficient $a_0, a_1, \ldots, a_m$ in (3.24) which is not equal to zero. Set

$$H_2(z) = c_2 e^{sz} h(z). \qquad (3.36)$$

Similar to the proof of Case A of Theorem 1.1(ii), we have

$$f(z) = e^{\beta z}\left[-g_1(e^z) - g_2(e^{-z})\right] + c_2\left[e^{sz}g_3(e^{-z})\right], \tag{3.37}$$

where $c_2 \neq 0$ and β are constants, $g_1(z), g_2(z)$ and, $g_3(z)$ are polynomials in z with $\deg g_3 \geq 1$. Set $g_4(e^{-z}) = c_2 e^{sz} g_3(e^{-z})$. Then $g_4(e^{-z}) \not\equiv 0$ is a polynomial in e^{-z} by the hypotheses of s in (3.36). This is the form of (1.9). We have proved Theorem 1.1(ii) when $\deg P_1 = \deg R_1$.

Now we suppose that $\deg P_1 < \deg R_1$. By Lemma 2.6, there exists a polynomial $g_0(z)$ in z, satisfies (2.10) and (2.11).

Since $\deg R_3 \leq \deg P_1$ and since we have proved Theorem 1.1 holds in the cases when $\deg P_1 \geq \deg R_1$ holds, we can apply this result to (2.11).

If $\deg P_1 > \deg R_3$, it follows from Theorem 1.1(i) that

$$g(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \tag{3.38}$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z. By (2.10) and (3.38), we obtain that

$$f(z) = g_0(e^z) + e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right], \tag{3.39}$$

where β is a constant, $g_0(z), g_1(z)$ and $g_2(z)$ are polynomials in z. This is a form of (1.9).

If $\deg P_1 = \deg R_3$, it follows from the proof of Theorem 1.1(ii) when $\deg P_1 = \deg R_1$ that

$$g(z) = e^{\beta z}\left[g_1(e^z) + g_2(e^{-z})\right] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}), \tag{3.40}$$

where $g_1(z), g_2(z), g_3(z)$ and $g_4(z)$ are polynomials in z with $\deg g_3 \geq 1$, c_1 and c_2 are constants that may or may not be equal to zero. By (2.10) and (3.40), we obtain that $f(z)$ has the form of (1.9). Theorem 1.1(ii) is completed. □

Now, we give some examples to show that Theorem 1.1 is correct.

Example 3.1. Let $f(z) = e^{-z} + e^{-2z}$, then $f(z)$ satisfies

$$f'' + \left(e^{2z} + e^{-z} + 1\right)f' - \left(e^z + 2e^{-z} - 1\right)f = -e^z - 3 + 2e^{-2z}. \tag{3.41}$$

This is an example of Theorem 1.1(i).

Example 3.2. Let $f(z) = z(1 + e^{-z}) + e^{-2z}$, then $f(z)$ satisfies

$$f'' + \left(e^{2z} + 2e^z + e^{-z} + 3\right)f' + \left(e^z + e^{-z} + 1\right)f = e^{2z} + 3e^z + 3 - e^{-z} + e^{-3z}. \tag{3.42}$$

This is an example of Theorem 1.1(ii) with $\deg P_1 > \deg P_2$ and $\deg P_1 = \deg R_1$.

Example 3.3. Let $f(z) = z(e^{-z}) + e^z + e^{-z}$, then $f(z)$ satisfies

$$f'' + \left(e^{2z} + e^z + e^{-z}\right)f' + \left(e^z + e^{-z}\right)f = e^{3z} + 2e^{2z} + e^z - e^{-z} + e^{-2z} + 3. \tag{3.43}$$

This is an example of Theorem 1.1 (ii) with $\deg P_1 > \deg P_2$ and $\deg P_1 < \deg R_1$.

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