Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 409809, 20 pages
doi:10.1155/2009/409809

Research Article

Generalized Bihari Type Integral Inequalities and the Corresponding Integral Equations

László Horváth

Department of Mathematics, University of Pannonia, Egyetem u. 10, 8200 Veszprém, Hungary

Correspondence should be addressed to László Horváth, lhorvath@almos.vein.hu

Received 2 February 2009; Accepted 23 June 2009

Recommended by Alberto Cabada

We study some special nonlinear integral inequalities and the corresponding integral equations in measure spaces. They are significant generalizations of Bihari type integral inequalities and Volterra and Fredholm type integral equations. The kernels of the integral operators are determined by concave functions. Explicit upper bounds are given for the solutions of the integral inequalities. The integral equations are investigated with regard to the existence of a minimal and a maximal solution, extension of the solutions, and the generation of the solutions by successive approximations.

1. Introduction and the Main Results

In this paper we study integral inequalities of the form

$$y(x) \leq f(x) + g(x)\int_{S(x)} q \circ y \, d\mu, \quad x \in X, \tag{1.1}$$

and the corresponding integral equations

$$y(x) = f(x) + g(x)\int_{S(x)} q \circ y \, d\mu, \quad x \in X, \tag{1.2}$$

where

(A_1) $(X, \mathcal{A}, \mu)$ is a measure space;

(A_2) S is a function from X into $\mathcal{A}$ such that the following properties hold:

(A_2^1) $\mu(S(x)) < \infty$ for every $x \in X$,
(A_2^2) if $x_2 \in S(x_1)$, then $S(x_2) \subset S(x_1)$,
(A_2^3) $\{(x_1, x_2) \in X^2 \mid x_2 \in S(x_1)\}$ is μ^2-measurable;

(A_3) q is a function from $[0, \infty[$ into $[0, \infty[$ with the following conditions:

(A_3^1) q is concave,
(A_3^2) $\lim_{t \to \infty}(q(t)/t) = 0$;

(A_4) the functions f and g belong to

$$\mathcal{L}_{\mathrm{loc}}(X) := \{p : X \longrightarrow [0, \infty[\mid p \text{ is } \mu\text{-integrable over } S(x)\ \forall x \in X\}. \tag{1.3}$$

It may be noted that under the condition (A_3^1) the function q is increasing (see Lemma 2.1 for the justification).

$\mathcal{A}$ always represents a σ-algebra in X. The μ-integrable functions over a measurable set $A \in \mathcal{A}$ are considered to be μ-almost measurable on A. The product of the measure space $(X, \mathcal{A}, \mu)$ with itself is understood as in [1], and it is denoted by $(X^2, \mathcal{A}^2, \mu^2)$.

By $\mathbb{N}$ we designate the set of nonnegative integers.

Special cases of (1.1) seem first to have been investigated by Lasalle [2] and Bihari [3]. Bihari's classical result gives an explicit upper bound for the solutions of the integral inequality

$$u(t) \le a + \int_c^t k(s)h(u(s))ds, \quad t \in [c, d], \tag{1.4}$$

where $a \ge 0$, k and u are nonnegative continuous functions on $[c, d]$, and h is a positive continuous and increasing function on $[0, \infty[$. A group of inequalities is now associated with Bihari's name. Results for the various forms of such inequalities and references to different works in this topic can be found in [4–6]. Bihari type inequalities have been widely studied because they can be applied in the theory of difference, differential and integral equations. Riemann or classical Lebesgue integral is used in most of the theorems in this area. There are relatively few papers using other types of integral. For generalizations to abstract Lebesgue integral; see [7–9]. The linear version of (1.1) is given in [7]. The special case $q(x) = x^\alpha$, $(x \ge 0,\ 0 < \alpha < 1)$ of (1.1) is considered in [8], while the special case $q(x) = x^\alpha$, $(x \ge 0,\ 1 < \alpha)$ of (1.1) is discussed in [9]. It turns out to be useful to study Bihari type inequalities with abstract Lebesgue integral. It is motivated proceeding in this direction as follows. We can get new facts about the nature of Bihari type inequalities even in the finite dimensional environment; the results can be applied in the study of certain new classes of differential and integral equations (see [7–11]).

The traditional treatment assumes not only that $X \subset \mathbb{R}^n$, but also that the sets $S(x)$, $x \in X$ are intervals, while the present treatment (it should be emphasized that the methods employed to establish our results are not usual in this topic) makes it possible to consider more general sets (examples for functions satisfying (A_2^2) and (A_2^3) can be found in [11]). Such results are not quite so easy to find in literature, although they can be used as powerful tools in many fields of mathematics.

Besides $\mathcal{L}_{\mathrm{loc}}(X)$, the following function spaces will play an important role.

Definition 1.1. If A is a nonempty subset of X such that $S(x) \subset A$ for every $x \in A$, then let

$$\mathcal{L}_{\mathrm{loc}}(A) := \{p : A \longrightarrow [0, \infty[\mid p \text{ is } \mu\text{-integrable over } S(x)\ \forall x \in A\}. \tag{1.5}$$

Next, the basic concepts of the solutions of the inequalities (1.1) and

$$y(x) \geq f(x) + g(x) \int_{S(x)} q \circ y \, d\mu, \quad x \in X, \tag{1.6}$$

and the equation (1.2) are defined.

Definition 1.2. We say that a function $y : D_y \to \mathbb{R}$ is a solution of (1.1), (1.6), or (1.2) if

(i) D_y is a nonempty subset of X such that $S(x) \subset D_y$ for every $x \in D_y$,

(ii) $y \in \mathcal{L}_{\mathrm{loc}}(D_y)$,

(iii) y satisfies (1.1), (1.6), or (1.2) for each $x \in D_y$.

It is easily verified (see Lemma 2.5) that if $y : D_y \to \mathbb{R}$ is a solution of (1.1), (1.6), or (1.2), then $q \circ y$ is μ-integrable over $S(x)$ for all $x \in D_y$.

After these preparations we set ourselves the task of obtaining an upper bound for the solutions of (1.1). The following definition will be useful.

Definition 1.3. (a) For every $x \in X$ with $\mu(S(x)) > 0$, let

$$a_x := \frac{1}{\mu(S(x))} \int_{S(x)} f \, d\mu, \quad b_x := \int_{S(x)} g \, d\mu. \tag{1.7}$$

(b) Let

$$t(x) := \begin{cases} \max\{t \geq 0 \mid t = a_x + b_x q(t)\}, & \text{if } \mu(S(x)) > 0, \\ 0, & \text{if } \mu(S(x)) = 0, \end{cases} \quad x \in X. \tag{1.8}$$

By (A_3^2), $t(x)$ is a nonnegative real number for every $x \in X$.

Now we are in a position to formulate the first main result.

Theorem 1.4. *Assume the conditions (A_1)–(A_4).*

(a) *Every solution $y : D_y \to \mathbb{R}$ of* (1.1) *satisfies*

$$y(x) \leq f(x) + g(x)\mu(S(x))q(t(x)), \quad x \in D_y. \tag{1.9}$$

(b) *The function z defined on X by*

$$z(x) = f(x) + g(x)\mu(S(x))q(t(x)) \tag{1.10}$$

belongs to $\mathcal{L}_{\mathrm{loc}}(X)$.

In the second main result we test the scope of the previous theorem by applying it to prove the existence of a maximal and a minimal solution of the integral equation (1.2). At the same time, we show that every solution has maximal domain of existence X, and we apply the method of successive approximations to (1.2). Moreover, the behavior of the solutions is studied in a special case. The considered integral equations are in a very general form, there are classical Volterra and Fredholm type integral equations among them.

Theorem 1.5. *Suppose the conditions* (A_1)*–*(A_4)*.*

(a_1) *There exists a solution* $y_{\min} \in \mathcal{L}_{\mathrm{loc}}(X)$ *of* (1.2)*, which is minimal in the sense that* $y_{\min}(x) \leq y(x)$*,* $x \in D_y$ *whenever* $y : D_y \to \mathbb{R}$ *is a solution of* (1.6)*.*

(a_2) *There exists a solution* $y_{\max} \in \mathcal{L}_{\mathrm{loc}}(X)$ *of* (1.2)*, which is maximal in the sense that* $y_{\max}(x) \geq y(x)$*,* $x \in D_y$ *whenever* $y : D_y \to \mathbb{R}$ *is a solution of* (1.1)*.*

(b) *If* $y : D_y \to \mathbb{R}$ *is a solution of* (1.2)*, then* y *has an extension* $\widehat{y}$ *to* X *that is a solution of* (1.2) *on* X*.*

(c) *Let* $y : D_y \to \mathbb{R}$ *be a solution of* (1.1)*. Then the successive approximations determined by* y*,*

$$y_0 := y, \qquad y_{n+1}(x) := f(x) + g(x)\int_{S(x)} q \circ y_n \, d\mu, \quad x \in D_y, \; n \in \mathbb{N}, \tag{1.11}$$

are well defined, $y_n \in \mathcal{L}_{\mathrm{loc}}(D_y)$*,* $n \in \mathbb{N}$*; the sequence* $(y_n)_{n=0}^{\infty}$ *is increasing and converges pointwise on* D_y *to a solution of* (1.2)*.*

(d) *Let* $y : D_y \to \mathbb{R}$ *be a solution of* (1.6)*. Then the successive approximations* (1.11) *determined by* y *are well defined,* $y_n \in \mathcal{L}_{\mathrm{loc}}(D_y)$*,* $n \in \mathbb{N}$*, and the sequence* $(y_n)_{n=0}^{\infty}$ *is decreasing. Moreover, if either* q *is continuous (at* 0*) or* $q(0) > 0$*, then they converge pointwise on* D_y *to a solution of* (1.2)*.*

(e) *If in addition* f *and* g *are bounded on* $S(x)$ *for all* $x \in X$*, then every solution* $y : D_y \to \mathbb{R}$ *of* (1.2) *is bounded on* $S(x)$ *for all* $x \in D_y$*.*

We conclude this section with some remarks.

Remark 1.6. The next example shows that the concavity of q alone does not imply neither the existence of an upper bound for the solutions of the integral inequality (1.1) nor the existence of a solution of the integral equation (1.2). Consider the integral inequality

$$y(x) \leq 1 + \int_{[0,x]} q \circ y d\varepsilon_0 = 1 + q\big(y(0)\big), \quad x \in [0, \infty[, \tag{1.12}$$

and the corresponding integral equation

$$y(x) = 1 + \int_{[0,x]} q \circ y d\varepsilon_0 = 1 + q\big(y(0)\big), \quad x \in [0, \infty[, \tag{1.13}$$

where ε_0 is the unit mass at 0 defined on the σ-algebra of Borel subsets of $[0, \infty[$, and

$$q : [0, \infty[\longrightarrow [0, \infty[, \qquad q(t) := t. \tag{1.14}$$

Then the conditions (A_1)–(A_4) are satisfied without (A_3^2). It is obvious that the functions

$$y_n : [0,\infty[\longrightarrow [0,\infty[, \qquad y_n(x) := n, \quad n \in \mathbb{N} \tag{1.15}$$

are solutions of (1.12), showing that there are no either global or local upper bounds for the solutions of (1.12). It is easy to check that (1.13) has no solution.

Remark 1.7. It is illustrated by an example that under the conditions (A_1)–(A_4) the maximal domain of existence of a solution of (1.1) may be a proper subset of X. Let $X := [0,1]$, let $\mathcal{A}$ be the Lebesgue measurable subsets of X, and let μ be the Lebesgue measure on $\mathcal{A}$. The function S is defined on $\mathcal{A}$ by

$$S(x) := \begin{cases} \emptyset, & \text{if } x \in [0,1[, \\ [0,1], & \text{if } x = 1. \end{cases} \tag{1.16}$$

The functions f and g are defined on X by $f(x) = g(x) := 1$. Suppose $q : [0,\infty[\to \mathbb{R}$, $q(t) := t^{1/2}$. Then (A_1)–(A_4) are satisfied. Let $y : [0,1[\to [0,1]$ be a non (Lebesgue) measurable function. Then y is a solution of (1.1) which has no extension to X.

Remark 1.8. The following example makes it clear that some extra conditions for q are necessary in Theorem 1.5(d). Let

$$X := \left\{1 - \frac{1}{k} \mid k \in \mathbb{N} \setminus \{0\}\right\} \cup \{1,2\}, \tag{1.17}$$

let $\mathcal{A}$ be the power set of X, and let the measure μ be defined on $\mathcal{A}$ by

$$\mu := \sum_{k=1}^{\infty} \frac{1}{2^k} \varepsilon_{1-1/k} + \varepsilon_1 + \varepsilon_2, \tag{1.18}$$

where the measure $\varepsilon_x (x \in X)$ is the unit mass at x defined on $\mathcal{A}$. We consider the integral equation

$$y(x) = \int_{S(x)} q \circ y \, d\mu, \quad x \in X, \tag{1.19}$$

where $S : X \to \mathcal{A}$, $S(x) := \{s \in X \mid s < x\}$, and

$$q : [0,\infty[\longrightarrow [0,\infty[, \qquad q(t) := \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.20}$$

The conditions (A_1)–(A_4) can be immediately verified. Define the function $y_0 : X \to \mathbb{R}$ by $y_0(x) := 4$. Noting that $\mu(X) = 3$ leads to the inequality

$$y_0(x) \geq \int_{S(x)} q \circ y_0 \, d\mu, \quad x \in X. \tag{1.21}$$

A few easy calculations imply that, for every $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{gathered} y_n\left(1 - \frac{1}{k}\right) = \begin{cases} 0, & \text{if } 1 \leq k \leq n, \\ \displaystyle\sum_{i=n}^{k-1} \frac{1}{2^i}, & \text{if } k \geq n+1, \end{cases} \\ y_n(1) = \sum_{i=n}^{\infty} \frac{1}{2^i}, \qquad y_n(2) = 1 + \sum_{i=n}^{\infty} \frac{1}{2^i}, \end{gathered} \tag{1.22}$$

and therefore

$$\lim_{n \to \infty} y_n(x) = \begin{cases} 0, & \text{if } x = 1 - \dfrac{1}{k}, \ k \in \mathbb{N} \setminus \{0\} \text{ or } x = 1, \\ 1, & \text{if } x = 2. \end{cases} \tag{1.23}$$

Since (1.19) has the unique solution $y(x) = 0$, $x \in X$, then the successive approximations $(y_n)_{n=0}^{\infty}$ do not converge to the solution of (1.19).

2. Preliminaries

This section is devoted to some preparatory results. In the following three lemmas we establish some useful properties of concave functions.

Lemma 2.1. *If the function* $r : [0, \infty[\to [0, \infty[$ *is concave, then* r *is increasing.*

Proof. Suppose that there exist $0 \leq t_1 < t_2$ for which $r(t_1) > r(t_2)$. By the concavity of r, the points of the graph of r are below or on the ray from t_1 through t_2 for all $t \geq t_2$, and therefore

$$r(t) \leq \frac{r(t_2) - r(t_1)}{t_2 - t_1}(t - t_2) + r(t_2), \quad t \geq t_2. \tag{2.1}$$

Since

$$\frac{r(t_2) - r(t_1)}{t_2 - t_1} < 0, \tag{2.2}$$

it follows from (2.1) that $r(t) < 0$ if t is large enough. This contradicts the range of r. □

Lemma 2.2. *Suppose the function* $r : [0,\infty[\to \mathbb{R}$ *is concave,* $r(0) \geq 0$*, and* $\lim_{t\to\infty} r(t) = -\infty$*. Associate to* r *the nonnegative real number*

$$t_r := \max\{t \geq 0 \mid r(t) = 0\}. \tag{2.3}$$

Then

(a) r *is strictly decreasing on* $[t_r, \infty[$;

(b) $r(t) \geq 0$ *for all* $t \in [0, t_r]$;

(c) *If* $t_r > 0$*, and there is a* $t_1 \in]0, t_r[$ *such that* $r(t_1) = 0$*, then* $r(t) = 0$ *for all* $t \in [0, t_r]$.

Proof. The hypotheses on r (since r is concave, r is continuous on $]0,\infty[$) guarantee that exactly one of the following three cases holds:

(i) $r(0) = 0$ and $r(t) < 0$ for all $t \in]0,\infty[$;

(ii) there exists a $\hat{t} > 0$ such that $r(t) = 0$ for all $t \in [0, \hat{t}]$ and $r(t) < 0$ for all $t \in]\hat{t}, \infty[$;

(iii) there is a unique $\hat{t} > 0$ such that $r(\hat{t}) = 0$, $r(t) > 0$ for all $t \in]0, \hat{t}[$ and $r(t) < 0$ for all $t \in]\hat{t}, \infty[$.

It follows that $t_r = 0$ if (i) is satisfied, and $t_r = \hat{t}$ otherwise. At the same time (b) and (c) are proved.

It remains to show (a). If $t_r < t$, then (i)–(iii) show that $0 = r(t_r) > r(t)$. Assume $t_r < t_1 < t_2$. By the concavity of r,

$$\frac{r(t_1)}{t_1 - t_r} \geq \frac{r(t_2)}{t_2 - t_r}, \tag{2.4}$$

and hence

$$r(t_1) \geq \frac{t_1 - t_r}{t_2 - t_r} r(t_2) > r(t_2). \tag{2.5}$$

The proof is complete. □

Lemma 2.3. *Suppose the function* $r : [0,\infty[\to [0,\infty[$ *is concave, and* $\lim_{t\to\infty}(r(t)/t) = 0$*. Associate to* r *and to each of the nonnegative real numbers* a*,* b *the function*

$$r_{a,b} : [0,\infty[\longrightarrow \mathbb{R}, \qquad r_{a,b}(t) := a + br(t) - t. \tag{2.6}$$

Then

(a) $r_{a,b}$ *is concave, and* $\lim_{t\to\infty} r_{a,b}(t) = -\infty$;

(b) *If* $t \leq a + br(t)$, $t \geq 0$*, then* $t \in [0, t_{a,b}]$*, where*

$$t_{a,b} := \max\{t \geq 0 \mid r_{a,b}(t) = 0\}; \tag{2.7}$$

(c) *If* $(a_0, b_0) \in [0,\infty[\times[0,\infty[$ *and* $t_0 > 0$, *then* $r_{a,b} \to r_{a_0,b_0}$ *uniformly on* $[0, t_0]$ *as* $(a,b) \to (a_0, b_0)$ *in* $[0,\infty[\times[0,\infty[$;

(d) *the function* $(a,b) \to t_{a,b}$ *defined on* $[0,\infty[\times[0,\infty[$ *is upper semicontinuous.*

Proof. (a) It is obvious.

(b) By (a), Lemma 2.2(b) and (a) give the result.

(c) The triangle inequality insures that

$$|r_{a,b}(t) - r_{a_0,b_0}(t)| \le |a - a_0| + |b - b_0| r(t), \quad t \in [0, t_0]. \tag{2.8}$$

Then from Lemma 2.1

$$|r_{a,b}(t) - r_{a_0,b_0}(t)| \le |a - a_0| + |b - b_0| r(t_0), \quad t \in [0, t_0], \tag{2.9}$$

and this gives the result.

(d) To prove this, choose $(a_0, b_0) \in [0,\infty[\times[0,\infty[$, and $\epsilon > 0$. The definition of t_{a_0,b_0} and Lemma 2.2(a) imply that

$$\delta := -r_{a_0,b_0}(t_{a_0,b_0} + \epsilon) > 0. \tag{2.10}$$

By (b), $r_{a,b} \to r_{a_0,b_0}$ uniformly on $[0, t_{a_0,b_0} + \epsilon]$ as $(a,b) \to (a_0, b_0)$ in $[0,\infty[\times[0,\infty[$, and hence there exists a neighborhood U of (a_0, b_0) in $[0,\infty[\times[0,\infty[$ such that

$$|r_{a,b}(t) - r_{a_0,b_0}(t)| < \delta, \quad (a,b) \in U, \ t \in [0, t_{a_0,b_0} + \epsilon]. \tag{2.11}$$

It now follows from Lemma 2.2 (a) that

$$t_{a,b} \le t_{a_0,b_0} + \epsilon, \quad (a,b) \in U, \tag{2.12}$$

and the proof is complete. □

The next result was proved in [8, Lemma 5(b)].

Lemma 2.4. *Suppose that* (A_1) *and* (A_2^3) *hold. Let* $A \in \mathcal{A}$ *such that* $S(x) \subset \mathcal{A}$ *for every* $x \in A$. *Suppose* $u : A \to \mathbb{R}$ *is* μ*-integrable over* A, $v : A \to \mathbb{R}$ *is* μ*-almost measurable on* A, *and there exists a measurable subset* C *of* A *such that* $\mu(C)$ *is* σ*-finite and* $v(x) = 0$ *for all* $x \in A \setminus C$. *Then the function*

$$x \longrightarrow v(x) \int_{S(x)} u \, d\mu, \quad x \in A \tag{2.13}$$

is μ*-almost measurable on* A.

Lemma 2.5. *Assume the conditions* (A_1)–(A_3)*. If* A *is a nonempty subset of* X *such that* $S(x) \subset A$ *for every* $x \in A$ *and* $u \in \mathcal{L}_{\text{loc}}(A)$*, then* $q \circ u \in \mathcal{L}_{\text{loc}}(A)$*.*

Proof. Let $x \in A$ be fixed. Since q is increasing it is Borel measurable. Consequently, since u is μ-almost measurable on $S(x)$, $q \circ u$ is μ-almost measurable on $S(x)$. By (A_3^2), we can find $t_0 > 0$ such that $q(t) \le t$ for all $t \ge t_0$. Hence, note that q is increasing:

$$q(u(s)) \le \begin{cases} u(s), & \text{if } u(s) \ge t_0, \\ q(t_0), & \text{if } u(s) < t_0, \end{cases} \quad s \in S(x). \tag{2.14}$$

It now follows from the definition of $\mathcal{L}_{\text{loc}}(A)$ and from (A_2^1) that $q \circ u$ is μ-integrable over $S(x)$. The proof is complete. □

A consequence of the previous results that will be important later on is follows.

Lemma 2.6. *Suppose that* (A_1)–(A_4) *hold. If* A *is a nonempty subset of* X *such that* $S(x) \subset A$ *for every* $x \in A$ *and* $u \in \mathcal{L}_{\text{loc}}(A)$*, then the function*

$$x \longrightarrow f(x) + g(x) \int_{S(x)} q \circ u \, d\mu, \quad x \in A \tag{2.15}$$

belongs to $\mathcal{L}_{\text{loc}}(A)$*.*

Proof. Let $x \in X$.

By Lemma 2.4, the function

$$s \longrightarrow g(s) \int_{S(s)} q \circ u \, d\mu, \quad s \in A \tag{2.16}$$

is μ-almost measurable on $S(x)$. Hence it follows from the μ-integrability of g and $q \circ u$ over $S(x)$ (the latter can be seen from Lemma 2.5), combined with the inequality

$$g(s) \int_{S(s)} q \circ u \, d\mu \le g(s) \int_{S(x)} q \circ u \, d\mu, \quad s \in S(x), \tag{2.17}$$

that the function (2.16) is μ-integrable over $S(x)$. We conclude that the function (2.15) is μ-integrable over $S(x)$.

The proof is complete. □

We need the concept of AL-space, which is of fundamental significance in the proof of Theorem 1.5.

Definition 2.7. Suppose (A_1), and let A be a nonempty set from $\mathcal{A}$.

(a) Let

$$\mathcal{L}(A) := \{u : A \longrightarrow \mathbb{R} \mid u \text{ is } \mu\text{-integrable over } A\}. \tag{2.18}$$

For a given $u \in \mathcal{L}(A)$, the symbol $\|u\|$ is defined by $\|u\| := \int_A |u| \, d\mu$.

(b) Let $\mathcal{N} := \{u \in \mathcal{L}(A) \mid \|u\| = 0\}$, and let $L(A) := \mathcal{L}(A)/\mathcal{N}$. For every $u \in \mathcal{L}(A)$, let $\overline{u} \in L(A)$ be the equivalence class containing u ($\overline{u} = u + \mathcal{N}$), and we set $\|\overline{u}\| := \|u\|$.

(c) We introduce the canonical ordering on $L(A)$: for $\overline{u_1}, \overline{u_2} \in L(A)$, and $\overline{u_1} \preceq \overline{u_2}$ means that $u_1 \leq u_2$ μ-almost everywhere on A.

Remark 2.8. (a) $(\mathcal{L}(A), \|\cdot\|)$ is a complete pseudometric space.

(b) $(L(A), \|\cdot\|, \preceq)$ is an L-normed Banach lattice, briefly, AL-space (see [12]).

If u is a function and A is a subset of the domain of u, then the restriction of u to A is denoted by $u \mid A$.

Lemma 2.9. *Suppose (A_1), and let A and B be nonempty sets from $\mathcal{A}$ such that $B \subset A$. If $F := \{\overline{u_\lambda} \mid \lambda \in \Lambda\}$ and $G := \{\overline{v_\lambda} \mid \lambda \in \Lambda\}$ are nonempty majorized subsets of $L(A)$ such that $\overline{u_\lambda} \mid B = \overline{v_\lambda} \mid B$ $(\lambda \in \Lambda)$, then*

$$(\sup F) \mid B = (\sup G) \mid B. \tag{2.19}$$

Proof. Since $(L(A), \|\cdot\|, \preceq)$ is an AL-space, it is order complete (see [12]), and hence $\sup F$ and $\sup G$ exist. Let $u \in \sup F$ and $v \in \sup G$. Then $u \geq u_\lambda$ μ-almost everywhere on A $(\lambda \in \Lambda)$, thus the function

$$\overline{v_1} \in L(A), \qquad v_1(x) := \begin{cases} u(x), & \text{if } x \in B, \\ v(x), & \text{if } x \in A \setminus B \end{cases} \tag{2.20}$$

is an upper bound of G. It follows that $\sup G \preceq \overline{v_1}$, that is $v \leq u$ μ-almost everywhere on B. An argument entirely similar to the preceding part gives that $\sup F \preceq \overline{u_1}$, where

$$\overline{u_1} \in L(A), \qquad u_1(x) := \begin{cases} v(x), & \text{if } x \in B, \\ u(x), & \text{if } x \in A \setminus B, \end{cases} \tag{2.21}$$

and therefore $u \leq v$ μ-almost everywhere on B.

The proof is now complete. □

The next result can be found in [11, Lemma 16].

Lemma 2.10. *Assume that the hypotheses (A_1), (A_2^2), (A_2^3), and (A_4) are satisfied. Let $L := \{x \in X \mid S(x) \neq \emptyset\}$. Suppose we are given solutions $s_x \in L(S(x))$, $x \in L$ of* (1.2) *such that $s_{x_2} \mid S(x_1) = s_{x_1}$ for each $x_1 \in L$, $x_2 \in X$ with $x_1 \in S(x_2)$. Then there exists exactly one solution $s : X \to \mathbb{R}$ of* (1.2) *for which $s \mid S(x) = s_x$, $x \in L$.*

3. Proofs of the Main Results

Consider now the proof of Theorem 1.4.

Proof. (a) If $x \in D_y$ such that $\mu(S(x)) = 0$, then (1.9) follows directly from (1.1).

Now, fix a point $x \in D_y$ with $\mu(S(x)) > 0$. To estimate the second term on the right of (1.1), we can apply Jensen's inequality (see [13]), by (A_2^1):

$$\begin{aligned} y(s) &\leq f(s) + g(s)\int_{S(s)} q \circ y \, d\mu \\ &\leq f(s) + g(s)\int_{S(x)} q \circ y \, d\mu \\ &\leq f(s) + g(s)\mu(S(x))q\left(\frac{1}{\mu(S(x))}\int_{S(x)} y \, d\mu\right), \quad s \in S(x), \end{aligned} \tag{3.1}$$

and therefore

$$\frac{1}{\mu(S(x))}\int_{S(x)} y \, d\mu \leq \frac{1}{\mu(S(x))}\int_{S(x)} f \, d\mu + \int_{S(x)} g \, d\mu \cdot q\left(\frac{1}{\mu(S(x))}\int_{S(x)} y \, d\mu\right). \tag{3.2}$$

This inequality, together with Definition 1.3(a), implies that the expression

$$\frac{1}{\mu(S(x))}\int_{S(x)} y \, d\mu \tag{3.3}$$

satisfies the inequality

$$t \leq a_x + b_x q(t), \quad t \geq 0. \tag{3.4}$$

By Lemma 2.3 (b) and Definition 1.3 (b), $t \in [0, t(x)]$, and hence (1.9) can be deduced from

$$y(x) \leq f(x) + g(x)\int_{S(x)} q \circ y \, d\mu \leq f(x) + g(x)\mu(S(x))q\left(\frac{1}{\mu(S(x))}\int_{S(x)} y \, d\mu\right). \tag{3.5}$$

(b) The properties defining $\mathcal{L}_{\mathrm{loc}}(\mathrm{X})$ are trivial if $x \in \mathrm{X}$ with $\int_{S(x)} g \, d\mu = 0$. So assume $x \in \mathrm{X}$ such that $\int_{S(x)} g \, d\mu > 0$.

First, we show that the function z is μ-almost measurable on $S(x)$. It is an easy consequence of (A_2^1) and Lemma 2.4 that the functions

$$s \longrightarrow \mu(S(s)) = \int_{S(s)} 1 \, d\mu, \quad s \in \mathrm{X}, \tag{3.6}$$

$$s \longrightarrow \int_{S(s)} f \, d\mu, \qquad s \longrightarrow \int_{S(s)} g \, d\mu, \quad s \in \mathrm{X} \tag{3.7}$$

are μ-almost measurable on $S(x)$. This means that there exists a measurable subset C of $S(x)$ such that $\mu(S(x) \setminus C) = 0$ and (3.6), (3.7) are measurable on C. Further, since g is μ-almost measurable on $S(x)$, it can be supposed that g is measurable on C. Thus we need to show that the function

$$s \longrightarrow q(t(s)), \quad s \in X \tag{3.8}$$

is measurable on C. To prove this let

$$D := \{s \in C \mid \mu(S(s)) > 0\}. \tag{3.9}$$

The measurability of (3.6) on C implies that $D \in \mathcal{A}$. Since

$$q(t(s)) = q(0), \quad s \in C \setminus D, \tag{3.10}$$

it is enough to show that (3.8) is measurable on D. It follows from the definitions of C and D that the function

$$s \longrightarrow \left(\frac{1}{\mu(S(s))}\int_{S(s)} f\,d\mu, \int_{S(s)} g\,d\mu\right), \quad s \in D \tag{3.11}$$

is measurable. Hence, by Lemma 2.3 (d), (3.8) is measurable on D.

It is now clear that z is μ-almost measurable on $S(x)$.

Next, we prove that z is μ-integrable over $S(x)$.

To prove this, it is enough to show that the function

$$s \longrightarrow \mu(S(s))q(t(s)), \quad s \in X \tag{3.12}$$

is bounded on C. Since

$$\mu(S(s))q(t(s)) = 0, \quad s \in C \setminus D, \tag{3.13}$$

we need to verify that (3.12) is bounded on D. By (A_3^2), we can find a $t_0 > 0$ such that

$$q(t) < \left(2\int_{S(x)} g\,d\mu\right)^{-1} \cdot t \quad \forall\, t > t_0. \tag{3.14}$$

It therefore follows from

$$t(s) = a_s + b_s q(t(s)), \quad s \in D \tag{3.15}$$

that

$$\mu(S(s))t(s) \le \int_{S(s)} f\,d\mu + \mu(S(s))\int_{S(s)} g\,d\mu\left(2\int_{S(x)} g\,d\mu\right)^{-1} t(s) \tag{3.16}$$

for all $s \in E := \{s \in D \mid t(s) > t_0\}$, and hence

$$\frac{1}{2}\mu(S(s))t(s) \le \mu(S(s))t(s)\left(1 - \int_{S(s)} g\,d\mu\left(2\int_{S(x)} g\,d\mu\right)^{-1}\right) \le \int_{S(s)} f\,d\mu, \quad s \in E. \tag{3.17}$$

Thus

$$\mu(S(s))t(s) \le 2\int_{S(s)} f\,d\mu, \quad s \in E, \tag{3.18}$$

and therefore another application of (3.14) gives

$$\begin{aligned}
&\mu(S(s))q(t(s)) \\
&\quad\le \begin{cases} \mu(S(s))\left(2\int_{S(x)} g\,d\mu\right)^{-1} t(s) \le \int_{S(s)} f\,d\mu\left(\int_{S(x)} g\,d\mu\right)^{-1}, & \text{if } s \in E \\ \mu(S(x))q(t_0), & \text{if } s \in D \setminus E \end{cases} \\
&\quad\le \begin{cases} \int_{S(x)} f\,d\mu\left(\int_{S(x)} g\,d\mu\right)^{-1}, & \text{if } s \in E \\ \mu(S(x))q(t_0), & \text{if } s \in D \setminus E, \end{cases}
\end{aligned} \tag{3.19}$$

and from this the claim follows. Consequently, z is μ-integrable over $S(x)$, as required.

The result is completely proved. □

Now we are in a position to prove Theorem 1.5.

Proof. We begin with the proof of (c) and (d).

(c) To prove that $y_n \in \mathcal{L}_{\mathrm{loc}}(D_y)$ we use induction on n. Clearly y_0 belongs to $\mathcal{L}_{\mathrm{loc}}(D_y)$. Let $n \in \mathbb{N}$ such that the assertion holds. Then Lemma 2.6 yields that $y_{n+1} \in \mathcal{L}_{\mathrm{loc}}(D_y)$. We show now that the sequence (y_n) is increasing. By our hypotheses on y_0, it follows that $y_0 \le y_1$, and we again complete the proof by induction. Suppose $n \in \mathbb{N}$ such that $y_n \le y_{n+1}$. Then, by Lemma 2.1 and the induction hypothesis

$$\begin{aligned}
y_{n+2}(x) &:= f(x) + g(x)\int_{S(x)} q \circ y_{n+1}\,d\mu \\
&\ge f(x) + g(x)\int_{S(x)} q \circ y_n\,d\mu = y_{n+1}(x), \quad x \in D_y.
\end{aligned} \tag{3.20}$$

Since

$$y_n(x) \le y_{n+1}(x) := f(x) + g(x)\int_{S(x)} q \circ y_n \, d\mu, \quad x \in D_y, \ n \in \mathbb{N}, \tag{3.21}$$

Theorem 1.4(a) implies that

$$y_n(x) \le f(x) + g(x)\mu(S(x))q(z(x)), \quad x \in D_y, \ n \in \mathbb{N}. \tag{3.22}$$

Because of (3.22) and the fact that (y_n) is increasing, there exists a function $\widehat{y} : D_y \to [0,\infty[$ such that (y_n) converges pointwise on D_y to $\widehat{y}$. Then $\widehat{y} \in \mathcal{L}_{\mathrm{loc}}(D_y)$ is a consequence of $y_n \in \mathcal{L}_{\mathrm{loc}}(D_y)$ $(n \in \mathbb{N})$ together with (3.22), Lemma 2.1(a) and, Theorem 1.4(b). If $x \in D_y$ with $\widehat{y}(x) > 0$, then according to the continuity of q on $]0,\infty[$, $q(y_n(x))$ converges to $\widehat{y}(x)$. Let $x \in D_y$ with $\widehat{y}(x) = 0$. As we have seen $(y_n(x))$ is increasing, and therefore $y_n(x) = 0$ for all $n \in \mathbb{N}$. Consequently

$$q\big(y_n(x)\big) = q(0) = q\big(\widehat{y}(x)\big). \tag{3.23}$$

Now (1.11) and the monotone convergence theorem give that $\widehat{y}$ is a solution of (1.2).

(d) We can see exactly as in the proof of (c) that $y_n \in \mathcal{L}_{\mathrm{loc}}(D_y)$, $n \in \mathbb{N}$, and the sequence (y_n) is decreasing. It follows from an easy induction argument that y_n is nonnegative for all $n \in \mathbb{N}$. Linking up with the foregoing, there exists a function $\widehat{y} : D_y \to [0,\infty[$ such that (y_n) converges pointwise on D_y to $\widehat{y}$. $\widehat{y} \in \mathcal{L}_{\mathrm{loc}}(D_y)$ can be shown as in the proof of (c).

The continuity of q on $[0,\infty[$ implies that $q(y_n(x))$ converges to $q(\widehat{y}(x))$ for every $x \in D_y$.

Assume now that $q(0) > 0$. If $x \in D_y$ such that $y_n(x) = 0$ for every large enough $n \in \mathbb{N}$, then

$$q\big(y_n(x)\big) = q(0) = q\big(\widehat{y}(x)\big). \tag{3.24}$$

If $x \in D_y$ such that $y_n(x) > 0$ for every $n \in \mathbb{N}$, then

$$y_{n+1}(x) = f(x) + g(x)\int_{S(x)} q \circ y_n \, d\mu \ge f(x) + g(x)q(0)\mu(S(x)) > 0, \quad n \in \mathbb{N}, \tag{3.25}$$

which leads to $\widehat{y}(x) > 0$. In both cases $q(y_n(x))$ converges to $q(\widehat{y}(x))$ for every $x \in D_y$.

According to (1.11) and the monotone convergence theorem $\widehat{y}$ is a solution of (1.2).

(a_1) For convergence of the successive approximations

$$y_0 := f, \qquad y_{n+1}(x) := f(x) + g(x)\int_{S(x)} q \circ y_n \, d\mu, \quad x \in X, \ n \in \mathbb{N} \tag{3.26}$$

to a solution $y_{\min} : X \to \mathbb{R}$ of (1.2) it suffices, in view of (c), to show that f is a solution of (1.1), which is evident. It remains to prove that if $y : D_y \to \mathbb{R}$ is a solution of (1.6), then $y_{\min}(x) \le y(x)$ for every $x \in D_y$. To this end, it is enough to show that

$$y_n(x) \le y(x), \quad x \in D_y,\ n \in \mathbb{N}. \tag{3.27}$$

This is true for $n = 0$, since the functions g and q are nonnegative. Let $n \in \mathbb{N}$ for which the result holds. Then, because of the nonnegativity of g and the fact that q is increasing,

$$\begin{aligned} y_{n+1}(x) &:= f(x) + g(x)\int_{S(x)} q \circ y_n \, d\mu \\ &\le f(x) + g(x)\int_{S(x)} q \circ y \, d\mu \le y(x), \quad x \in X,\ n \in \mathbb{N}, \end{aligned} \tag{3.28}$$

and the proof of the induction step is complete.

(a$_2$) Let

$$X_p := \{x \in X \mid \mu(S(x)) > 0\}. \tag{3.29}$$

Choose $x_0 \in X_p$. The set of the upper bounds from $\mathcal{L}(S(x_0))$ for the solutions of (1.1) on $S(x_0)$ is denoted by U. By Theorem 1.4, U is not empty. Let

$$\overline{U} := \{\overline{u} \mid u \in U\}. \tag{3.30}$$

Then $\overline{U}$ is a minorized subset of $(L(S(x_0)), \|\cdot\|, \preceq)$ (the elements of U are nonnegative). Since this space is order complete (see [12]),

$$\overline{v} := \inf \overline{U} \tag{3.31}$$

exists. Let $v \in \overline{v}$, and let $y : D_y \to \mathbb{R}$ be a solution of (1.1) such that $S(x_0) \subset D_y$. $y(x) \le u(x)$ $(x \in S(x_0), u \in U)$ gives that

$$y(x) \le v(x) \quad \mu \text{ a.e. on } S(x_0). \tag{3.32}$$

It follows that

$$y(x) \le f(x) + g(x)\int_{S(x)} q \circ v \, d\mu, \quad x \in S(x_0), \tag{3.33}$$

because q is increasing. Since $v \in \mathcal{L}(S(x_0))$ the proof of Lemma 2.5 shows that $q \circ v \in \mathcal{L}(S(x_0))$, and hence by (A$_4$), the function $v_1 : S(x_0) \to \mathbb{R}$ defined by

$$v_1(x) := f(x) + g(x)\int_{S(x)} q \circ v \, d\mu \tag{3.34}$$

belongs to $\mathcal{L}(S(x_0))$, and therefore $v_1 \in U$. It now comes from (3.33) and the definition of v that

$$v(x) \leq v_1(x) \quad \mu \text{ a.e. on } S(x_0). \tag{3.35}$$

Since q is increasing,

$$v_1(x) \leq f(x) + g(x) \int_{S(x)} q \circ v_1 \, d\mu, \quad x \in S(x_0), \tag{3.36}$$

and thus

$$v_1(x) \leq v(x) \quad \mu \text{ a.e. on } S(x_0). \tag{3.37}$$

We can see that

$$v(x) = f(x) + g(x) \int_{S(x)} q \circ v \, d\mu \quad \mu \text{ a.e. on } S(x_0). \tag{3.38}$$

If we set

$$v_{x_0} : S(x_0) \longrightarrow \mathbb{R}, \qquad v_{x_0}(x) := f(x) + g(x) \int_{S(x)} q \circ v \, d\mu, \tag{3.39}$$

then (3.38) shows that v_{x_0} is a solution of (1.2). (3.33) gives that

$$y(x) \leq v_{x_0}(x), \quad x \in S(x_0) \tag{3.40}$$

for every solutions $y : D_y \to \mathbb{R}$ of (1.1) for which $S(x_0) \subset D_y$.

Repeat the preceding construction for every $x \in X_p$. We thus obtain a set of functions $v_x \in \mathcal{L}(S(x))$, each a solution of (1.2) on its domain. Moreover, if $y : D_y \to \mathbb{R}$ is a solution of (1.1), then for every $x \in D_y \cap X_p$ we have

$$y(s) \leq v_x(s), \quad s \in S(x). \tag{3.41}$$

Introduce the next functions: for every $x \in X \setminus X_p$ with $S(x) \neq \emptyset$ let the function v_x be defined on $S(x)$ by

$$v_x(s) := f(s). \tag{3.42}$$

Obviously, these functions are also solutions of (1.2) on their domains.

Now let $x_1, x_2 \in X$ such that $S(x_1) \neq \emptyset$ and $x_1 \in S(x_2)$. Using (3.41), it is easy to verify that

$$v_{x_1}(s) = v_{x_2}(s), \quad s \in S(x_1), \tag{3.43}$$

and therefore Lemma 2.10 is applicable to the solutions v_x. This gives a unique solution $y_{\max}$: $X \to \mathbb{R}$ of (1.2) for which

$$y_{\max} \mid S(x) = v_x, \quad x \in X \text{ with } S(x) \neq \emptyset. \tag{3.44}$$

It remains to prove that $y_{\max}$ is maximal. Let $y : D_y \to \mathbb{R}$ be a solution of (1.1), and let $x \in D_y$. If $\mu(S(x)) = 0$, then

$$y(x) \leq f(x) = y_{\max}(x), \tag{3.45}$$

while if $\mu(S(x)) > 0$, then by (3.41)

$$y(s) \leq y_{\max}(s), \quad s \in S(x), \tag{3.46}$$

so that Lemma 2.1 implies that

$$y(x) \leq y_{\max}(x). \tag{3.47}$$

(b) We first show that every solution $y : D_y \to \mathbb{R}$ of (1.2) with $D_y \neq X$ has an extension $y_1 : D_{y_1} \to \mathbb{R}$ that is a solution of (1.2) such that D_y is a proper subset of D_{y_1}. This follows from (c) as soon as it is realized that every solution $y : D_y \to \mathbb{R}$ of (1.2) with $D_y \neq X$ has an extension $y_2 : D_{y_2} \to \mathbb{R}$ that is a solution of (1.1) such that D_y is a proper subset of D_{y_2}. Really, in this case the successive approximations determined by y_2 converge to a solution $y_1 : D_{y_2} \to \mathbb{R}$ of (1.2), which is obviously an extension of y.

That realization can be reached in finitely many steps.

Let $y : D_y \to \mathbb{R}$ be a solution of (1.2) with $D_y \neq X$, and let $x_0 \in X \setminus D_y$.

(i) Suppose $S(x_0) \subset X \setminus D_y$.

If $S(x_0) = \emptyset$, then

$$y_1 : D_y \cup \{x_0\} \longrightarrow \mathbb{R}, \quad y_1 := \begin{cases} y(x), & \text{if } x \in D_y, \\ f(x), & \text{if } x = x_0 \end{cases} \tag{3.48}$$

is an appropriate solution.

If $S(x_0) \neq \emptyset$, then

$$y_2 : D_y \cup S(x_0) \longrightarrow \mathbb{R}, \quad y_2 := \begin{cases} y(x), & \text{if } x \in D_y, \\ f(x), & \text{if } x \in S(x_0) \end{cases} \tag{3.49}$$

is a solution of (1.1) that agrees with y on D_y.

(ii) Suppose $S(x_0) \cap D_y \neq \emptyset$ and $S(x_0) \setminus D_y \neq \emptyset$. Let

$$E := \{x \in S(x_0) \cap D_y \mid \mu(S(x)) > 0\}. \tag{3.50}$$

If $E = \emptyset$, then the function

$$y_2 : D_y \cup S(x_0) \longrightarrow \mathbb{R}, \quad y_2 := \begin{cases} y(x), & \text{if } x \in D_y, \\ f(x), & \text{if } x \in S(x_0) \setminus D_y \end{cases} \tag{3.51}$$

is a solution of (1.1) ($y(x) = f(x)$, $x \in S(x_0) \cap D_y$, thus $y_2 \in \mathcal{L}_{\text{loc}}(D_y \cup S(x_0)))$ that agrees with y on D_y.

If $E \neq \emptyset$, then we introduce the functions $u_x : S(x_0) \to \mathbb{R}$, $x \in E$,

$$u_x(s) := \begin{cases} y(s), & \text{if } s \in S(x), \\ f(s), & \text{if } s \in S(x_0) \setminus S(x), \end{cases} \tag{3.52}$$

which all lie in $\mathcal{L}(S(x_0))$, and they are all solutions of (1.1). By Theorem 1.4 (b), $\{\overline{u_x} \mid x \in E\}$ is a majorized subset of $(L(S(x_0)), \|\cdot\|, \preceq)$. Since this space is order complete (see [12]),

$$\overline{u} := \sup_{x \in E} \overline{u_x} \tag{3.53}$$

exists. Choose $u \in \overline{u}$. Then $u \geq u_x$ μ-almost everywhere on $S(x_0)$ for every $x \in E$, and therefore Lemma 2.1 yields that

$$u_x(s) \leq f(s) + g(s) \int_{S(s)} q \circ u \, d\mu, \quad s \in S(x_0),\ x \in E. \tag{3.54}$$

Since $u \in \mathcal{L}(S(x_0))$ the proof of Lemma 2.5 shows that $q \circ u \in \mathcal{L}(S(x_0))$, and hence by ($A_4$), the function

$$s \longrightarrow f(s) + g(s) \int_{S(s)} q \circ u \, d\mu, \quad s \in S(x_0) \tag{3.55}$$

belongs to $\mathcal{L}(S(x_0))$ too. It now follows from $u \in \overline{u}$ and (3.54) that

$$u(s) \leq f(s) + g(s) \int_{S(s)} q \circ u \, d\mu, \quad \mu \text{ a.e. on } S(x_0). \tag{3.56}$$

We set

$$\widehat{u} : S(x_0) \longrightarrow \mathbb{R}, \qquad \widehat{u}(s) := f(s) + g(s) \int_{S(s)} q \circ u \, d\mu. \tag{3.57}$$

By (3.56)

$$\widehat{u}(s) \leq f(s) + g(s) \int_{S(s)} q \circ \widehat{u} \, d\mu, \quad s \in S(x_0). \tag{3.58}$$

Fix $\hat{x} \in E$. From Lemma 2.9 (with F and G there being $\{\overline{u_x} \mid x \in E\}$ and $\{\overline{u_{\hat{x}}} \mid x \in E\}$ resp.) we get that $u(s) = y(s)$ μ-almost everywhere on $S(\hat{x})$, and hence

$$\hat{u}(s) = y(s), \quad s \in S(x),\ x \in E. \tag{3.59}$$

Consequently, $\hat{u}(x) = y(x)$, $x \in E$. If $x \in S(x_0) \cap D_y$ with $\mu(S(x)) = 0$, then $\hat{u}(x) = f(x) = y(x)$. We can see that

$$\hat{u}(x) = y(x), \quad x \in S(x_0) \cap D_y. \tag{3.60}$$

By what we have already proved that

$$y_2 : D_y \cup S(x_0) \longrightarrow \mathbb{R}, \quad y_2 := \begin{cases} y(x), & \text{if } x \in D_y, \\ \hat{u}(x), & \text{if } x \in S(x_0) \setminus D_y \end{cases} \tag{3.61}$$

is a solution of (1.1) that agrees with y on D_y.

(iii) Suppose $S(x_0) \neq \emptyset$ and $S(x_0) \subset D_y$. By an argument entirely similar to that for the case (ii), we can get a solution $y_2 : D_y \cup \{x_0\} \to \mathbb{R}$ of (1.1) that agrees with y on D_y.

After these preparations, the proof can be concluded quickly. Let $y : D_y \to \mathbb{R}$ be a solution of (1.2), and let P be the set of all solutions of (1.2) which agree with y on D_y. Since $y \in P$, P is not empty. Partially order P by declaring $u_1 \preceq u_2$ to mean that the restriction of u_2 to the domain of u_1 agrees with u_1. By Hausdorff's maximality theorem, there exists a maximal totally ordered subcollection Q of P. Let D be the union of the domains of all members of Q, and define $\hat{y} : D \to \mathbb{R}$ by $\hat{y}(x) := u(x)$, where u occurs in Q. It is easy to check that $\hat{y}$ is well defined, and it is a solution of (1.2). If D were a proper subset of X, then the first part of the proof would give a further extension of $\hat{y}$, and this would contradict the maximality of Q.

(e) Let $y : D_y \to \mathbb{R}$ be a solution of (1.2), and let $x \in D_y$.

If $\mu(S(x)) = 0$, then $y \mid S(x) = f \mid S(x)$, so that y is bounded on $S(x)$.

Suppose $\mu(S(x)) > 0$. Since f and g are bounded on $S(x)$ and $\mu(S(x))$ is finite, Theorem 1.4 (a) implies that it is enough to prove the boundedness of the function

$$s \longrightarrow q(t(s)), \quad s \in X, \tag{3.62}$$

on $S(x)$. Moreover, we have only to observe that the function

$$s \longrightarrow t(s), \quad s \in X, \tag{3.63}$$

is bounded on $S(x)$. To prove this, let

$$A := \{s \in S(x) \mid \mu(S(s)) > 0\}. \tag{3.64}$$

If $f(s) \leq c$ and $g(s) \leq c$, $s \in S(x)$, then $a_s \leq c$ and $b_s \leq c$ for every $s \in A$, and therefore by Lemma 2.3(d), the function (3.63) is bounded on A. If $s \in S(x) \setminus A$, then the definition of the function t gives that $t(s) = 0$. The claim about t is therewith confirmed.

The proof of the theorem is now complete. □

Acknowledgment

This work was supported by Hungarian Foundation for Scientific Research Grant no. K73274.

References

[1] L. Horváth, "On the associativity of the product of measure spaces," *Acta Mathematica Hungarica*, vol. 98, no. 4, pp. 301–310, 2003.

[2] J. LaSalle, "Uniqueness theorems and successive approximations," *Annals of Mathematics*, vol. 50, pp. 722–730, 1949.

[3] I. Bihari, "A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 7, pp. 81–94, 1956.

[4] D. Baĭnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[5] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, vol. 53 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[6] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, vol. 205 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[7] L. Horváth, "Integral inequalities in measure spaces," *Journal of Mathematical Analysis and Applications*, vol. 231, no. 1, pp. 278–300, 1999.

[8] L. Horváth, "Generalizations of special Bihari type integral inequalities," *Mathematical Inequalities & Applications*, vol. 8, no. 3, pp. 441–449, 2005.

[9] L. Horváth, "Generalization of a Bihari type integral inequality for abstract Lebesgue integral," *Journal of Mathematical Inequalities*, vol. 2, no. 1, pp. 115–128, 2008.

[10] L. Horváth, "Integral equations in measure spaces," *Integral Equations and Operator Theory*, vol. 45, no. 2, pp. 155–176, 2003.

[11] L. Horváth, "Nonlinear integral equations with increasing operators in measure spaces," *Journal of Integral Equations and Applications*, vol. 17, no. 4, pp. 413–437, 2005.

[12] H. H. Schaefer, *Banach Lattices and Positive Operators*, Die Grundlehren der Mathematischen Wissenschaften, Band 21, Springer, New York, NY, USA, 1974.

[13] E. Hewitt and K. R. Stromberg, *Real and Abstract Analysis*, vol. 25 of *Graduate Texts in Mathematics*, Springer, Berlin, Germany, 1965.