Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 369185, 11 pages
doi:10.1155/2009/369185

Research Article

Integrodifferential Inequality for Stability of Singularly Perturbed Impulsive Delay Integrodifferential Equations

Danhua He[1] and Liguang Xu[2]

[1] *Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, China*
[2] *Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China*

Correspondence should be addressed to Liguang Xu, xlg132@126.com

Received 24 February 2009; Accepted 11 June 2009

Recommended by Ondrej Dosly

The exponential stability of singularly perturbed impulsive delay integrodifferential equations (SPIDIDEs) is concerned. By establishing an impulsive delay integrodifferential inequality (IDIDI), some sufficient conditions ensuring the exponentially stable of any solution of SPIDIDEs for sufficiently small $\varepsilon > 0$ are obtained. A numerical example shows the effectiveness of our theoretical results.

1. Introduction

Integrodifferential equations (IDEs) arise from many areas of science (from physics, biology, medicine, etc.), which have extensive scientific backgrounds and realistic mathematical models, and hence have been emerging as an important area of investigation in recent years, see [1–6]. Correspondingly, the stability of impulsive delay integrodifferential equations has been studied quite well, for example, [7–9]. However, besides delay and impulsive effects, singular perturbation likewise exists in a wide models for physiological processes or diseases [10]. And many good results on the stability of singularly perturbed delay differential equations have been reported, see, for example, [11–14]. Therefore, it is necessary to consider delay, impulse and singular perturbation on the stability of integrodifferential equations. However, to the best of our knowledge, there are no results on the problems of the exponential stability of solutions for SPIDIDEs due to some theoretical and technical difficulties. Based on this, this article is devoted to the discussion of this problem.

Applying differential inequalities, in [14–17], authors investigated the stability of impulsive differential equations. In [14], Zhu et al. established a delay differential inequality with impulsive initial conditions and derived some sufficient conditions ensuring the exponential stability of solutions for the singular perturbed impulsive delay differential equations (SPIDDEs). In this paper, we will improve the inequality established in [14] such that it is effective for SPIDIDEs. By establishing an IDIDI, some sufficient conditions ensuring the exponential stability of any solution of SPIDIDEs for sufficiently small $\varepsilon > 0$ are obtained. The results extend and improve the earlier publications, and which will be shown by the Remarks 3.2 and 3.5 provided later. An example is given to illustrate the theory.

2. Preliminaries

Throughout this letter, unless otherwise specified, let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ be the set of $m \times n$ real matrices. $\mathcal{N} \triangleq \{1,2,\ldots,n\}$. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B (A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq (\leq, >, <)$". Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. In particular, let $C \triangleq C[(-\infty, 0], R^n]$ denote the family of all bounded continuous R^n-valued functions ϕ defined on $(-\infty, 0]$ with the norm $\|\phi\| = \sup_{-\infty<\theta\leq 0}|\phi(\theta)|$, where $|\cdot|$ is Euclidean norm of R^n.

$PC[I, R^n] \triangleq \{\varphi : I \to R^n \mid \varphi(t^+) = \varphi(t)$ for $t \in I, \varphi(t^-)$ exist for $t \in I, \varphi(t^-) = \varphi(t)$ for all but points $t_k \in I\}$, where $I \subset R$ is an interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the left limit and right limit of scalar function $\varphi(t)$, respectively. Especially, let $PC \triangleq PC[(-\infty, 0], R^n]$.

For $x \in R^n$ and $\varphi \in C$ or $\varphi \in PC$, we define

$$
\begin{gathered}
[x]^+ = (|x_1|, \ldots, |x_n|)^T, \qquad [A]^+ = \left(|a_{ij}|\right)_{n\times n}, \qquad [\varphi(t)]_\tau = \left([\varphi_1(t)]_\tau, \ldots, [\varphi_n(t)]_\tau\right)^T, \\
[\varphi(t)]_\tau^+ = \left[[\varphi(t)]^+\right]_\tau, \qquad [\varphi_i(t)]_\tau = \sup_{-\tau\leq s\leq 0}\{\varphi_i(t+s)\}, \quad i \in \mathcal{N}, \\
D^+\varphi(t) = \lim_{s\to 0^+}\sup \frac{\varphi(t+s)-\varphi(t)}{s}.
\end{gathered} \tag{2.1}
$$

In this paper, we consider a class of SPIDIDEs described by

$$
\begin{gathered}
\varepsilon\dot{x}(t) = A(t)x(t) + f(t, x(t-\tau(t))) + \int_{-\infty}^{t} R(t-s)G(x(s))ds, \quad t \geq t_0, \ t \neq t_k, \\
x(t_k) = J_k\left(t_k, x(t_k^-)\right), \quad k = 1, 2, \ldots,
\end{gathered} \tag{2.2}
$$

with the initial conditions

$$
x(t_0 + \theta) = \phi(\theta) \in PC, \quad \theta \in (-\infty, 0], \tag{2.3}
$$

where $0 \le \tau(t) \le \tau$, $x(t) = (x_1(t), \dots, x_n(t))^T \in PC[R, R^n]$, $A(t) = (a_{ij}(t))_{n\times n} \in PC[R, R^{n\times n}]$, $J_k \in C[R \times R^n, R^n]$, $R(t) = (r_{ij}(t))_{n\times n} \in PC[R^+, R^{n\times n}]$, $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$.

Definition 2.1. The solution of (2.2) is said to be exponentially stable for sufficiently small ε if there exist finite constant vectors $K > 0$ and $\sigma > 0$, which are independent of $\varepsilon \in (0, \varepsilon_0]$ for some ε_0, and a constant $\lambda > 0$ such that $[x(t) - y(t)]^+ \le Ke^{-\lambda(t-t_0)}$ for $t \ge t_0$ and for any initial perturbation satisfying $\sup_{s\in(-\infty,0]}[\phi(s) - \varphi(s)]^+ < \sigma$. Here $y(t)$ is the solution of (2.2) corresponding to the initial condition φ.

3. Main Results

In order to prove the main result in this paper, we first need the following technique lemma.

Lemma 3.1. *Assume that* $0 \le u(t) = (u_1(t), \dots, u_n(t))^T \in R^n$, $t \ge t_0$ *satisfy*

$$\begin{gathered} D^+u(t) \le P(t)u(t) + Q(t)[u(t)]_\tau + \int_0^\infty W(s)u(t-s)ds, \quad t \ge t_0, \\ u(t_0+\theta) = \varphi(\theta) \in PC, \quad \theta \in (-\infty, 0], \end{gathered} \tag{3.1}$$

where $P(t) = (p_{ij}(t))_{n\times n} \ge 0$ *for* $t \ge t_0$ *and* $i \ne j$, $Q(t) = (q_{ij}(t))_{n\times n} \ge 0$ *for* $t \ge t_0$, $W(s) = (w_{ij}(s))_{n\times n} \ge 0$.

If there exist a positive constant ξ *and a positive vector* $z = (z_1, \dots, z_n)^T \in R^n$ *and two positive diagonal matrices* $L = \operatorname{diag}\{L_1, \dots, L_n\}$, $H = \operatorname{diag}\{h_1, \dots, h_n\}$ *with* $0 < h_i < 1$ *such that*

$$\left(Q(t) + HP(t) + L + \int_0^\infty W(s)e^{\xi s}ds\right)z < 0, \quad t \ge t_0. \tag{3.2}$$

Then one has

$$u(t) \le ze^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{3.3}$$

where the positive constant λ *is defined as*

$$0 < \lambda < \lambda_0 = \min_{1\le i\le n}\left\{\inf_{t\ge t_0} \lambda_i(t) : \lambda_i(t)z_i + \sum_{j=1}^n \left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^\infty w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j = 0\right\}, \tag{3.4}$$

for the given z.

Proof. Note that the result is trivial if $\tau = 0$. In the following, we assume that $\tau > 0$. Denote

$$F(\lambda_i(t)) = \lambda_i(t)z_i + \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j, \quad t \geq t_0,\ i \in \mathcal{N}, \tag{3.5}$$

then for any given $t \geq t_0$, we have

$$\begin{aligned} F(0) &= \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t) + \int_0^{\infty} w_{ij}(s)ds\right)z_j \\ &\leq \sum_{j=1}^{n} p_{ij}(t)z_j - h_i\sum_{j=1}^{n} p_{ij}(t)z_j \\ &= (1-h_i)\sum_{j=1}^{n} p_{ij}(t)z_j \\ &\leq -(1-h_i)\frac{L_i}{h_i}z_i \\ &< 0, \end{aligned} \tag{3.6}$$

the first inequality and the second inequality are from (3.2), the last inequality is because $0 < h_i < 1$, $L_i > 0$, $z_i > 0$, $i \in \mathcal{N}$.

We also have

$$\lim_{\lambda_i(t)\to\infty} F(\lambda_i(t)) = \infty, \qquad F'(\lambda_i(t)) = z_i + \sum_{j=1}^{n}\left(q_{ij}(t)\tau e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)se^{\lambda_i(t)s}ds\right)z_j > 0. \tag{3.7}$$

So by (3.6) and (3.7), for any $t \geq t_0$, there is a unique positive $\lambda_i(t)$ such that

$$\lambda_i(t)z_i + \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j = 0, \quad i \in \mathcal{N}. \tag{3.8}$$

Therefore, from the definition of λ_0, one can know that $\lambda_0 \geq 0$.

Next, we will show that $\lambda_0 \neq 0$.

If this is not true, fix v_i satisfying $0 < h_i < v_i < 1$ and $1 - h_i/v_i - h_i > 0$, $i \in \mathcal{N}$, there exist a $t^* \geq t_0$ and some integer l such that $\overline{\lambda}_l(t^*) < \delta$, where $0 < \delta < \min\{(1 - h_l/v_l - h_l)(L_l/h_l),\ (l/\tau)\ln(1/v_l), \xi\}$, such that

$$\overline{\lambda}_l(t^*)z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\overline{\lambda}_l(t^*)\tau} + \int_0^{\infty} w_{ij}(s)e^{\overline{\lambda}_l(t^*)s}ds\right)z_j = 0. \tag{3.9}$$

Then, we have

$$
\begin{aligned}
0 &= \overline{\lambda}_l(t^*)z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\overline{\lambda}_l(t^*)\tau} + \int_0^{\infty} w_{lj}(s)e^{\overline{\lambda}_l(t^*)s}ds\right)z_j \\
&< \delta z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\delta\tau}\right)z_j + \sum_{j=1}^{n}\int_0^{\infty} w_{lj}(s)z_j e^{\delta s}ds \\
&< \delta z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + \frac{1}{v_l}q_{lj}(t)\right)z_j + \sum_{j=1}^{n}\int_0^{\infty} w_{lj}(s)z_j e^{\xi s}ds \\
&\le \delta z_l + \sum_{j=1}^{n}p_{lj}(t)z_j - \frac{h_l}{v_l}\sum_{j=1}^{n}p_{lj}(t)z_j - h_l\sum_{j=1}^{n}p_{lj}(t)z_j \\
&= \delta z_l + \left(1 - \frac{h_l}{v_l} - h_l\right)\sum_{j=1}^{n}p_{lj}(t)z_j \\
&\le \delta z_l - \left(1 - \frac{h_l}{v_l} - h_l\right)\frac{L_l}{h_l}z_l \\
&< 0,
\end{aligned}
\tag{3.10}
$$

this contradiction shows that $\lambda_0 > 0$, so there at least exists a positive constant λ_0 such that $0 < \lambda < \lambda_0$, that is, the definition of λ for (3.3) is reasonable.

Since $\varphi(t) \in PC$ is bounded, we always can choose a sufficiently large $z > 0$ such that

$$
u(t) \le z e^{-\lambda(t-t_0)}, \quad -\infty < t \le t_0. \tag{3.11}
$$

In order to prove (3.3), we first prove for any given $k > 1$,

$$
u_i(t) < k z_i e^{-\lambda(t-t_0)} \equiv v_i(t), \quad t \ge t_0,\ i \in \mathcal{N}. \tag{3.12}
$$

If (3.12) is not true, then by continuity of $u(t)$, there must exist some integer m and $\hat{t} > t_0$ such that

$$
u_m\left(\hat{t}\right) = v_m\left(\hat{t}\right), \qquad D^+u_m\left(\hat{t}\right) \ge v_m'\left(\hat{t}\right), \tag{3.13}
$$

$$
u_i(t) \le v_i(t), \quad -\infty < t \le \hat{t},\ i \in \mathcal{N}. \tag{3.14}
$$

So, by (3.1), the equality of (3.13), (3.14) and $p_{ij}(t) \geq 0$ and $i \neq j$, $q_{ij}(t) \geq 0$, for $t \geq t_0$, and the definition of λ, we derive that

$$\begin{aligned}
D^+u_m\left(\hat{t}\right) &\leq \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right)u_j\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)u_j\left(\hat{t}-\tau\right)\right) + \sum_{j=1}^{n}\int_0^{\infty} w_{mj}(s)u_j\left(\hat{t}-s\right)ds \\
&\leq \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right)kz_je^{-\lambda(\hat{t}-t_0)} + q_{mj}\left(\hat{t}\right)kz_je^{-\lambda(\hat{t}-\tau-t_0)}\right) + \sum_{j=1}^{n}\int_0^{\infty} w_{mj}(s)kz_je^{-\lambda(\hat{t}-s-t_0)}ds \\
&= \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)e^{\lambda\tau} + \int_0^{\infty} w_{mj}(s)e^{\lambda s}ds\right)kz_je^{-\lambda(\hat{t}-t_0)} \\
&< \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)e^{\lambda_m(\hat{t})\tau} + \int_0^{\infty} w_{mj}(s)e^{\lambda_m(\hat{t})\tau}ds\right)kz_je^{-\lambda(\hat{t}-t_0)} \\
&= -\lambda_m\left(\hat{t}\right)z_mke^{-\lambda(\hat{t}-t_0)} \\
&< -\lambda z_mke^{-\lambda(\hat{t}-t_0)} \\
&= v_m'\left(\hat{t}\right),
\end{aligned} \tag{3.15}$$

which contradicts the inequality in (3.13), and so (3.12) holds for all $t \geq t_0$. Letting $k \to 1$, then (3.3) holds, and the proof is completed. □

Remark 3.2. If $W(s) = (w_{ij}(s))_{n\times n} = 0$ in Lemma 3.1, then we get [14, Lemma 1].

Theorem 3.3. *Assume that $A(t) = (a_{ij}(t))_{n\times n} \geq 0$ for $t \geq t_0$ and $i \neq j$, further suppose the following*

(H_1) *For any $x, y \in R^n$, there exist nonnegative matrices $U(t) = (u_{ij}(t))_{n\times n}$ and $B = (b_{ij})_{n\times n}$, $t \geq t_0$, such that*

$$\begin{gathered}
[f(t,x) - f(t,y)]^+ \leq U(t)[x-y]^+, \quad t \geq t_0, \\
[G(x) - G(y)]^+ \leq B[x-y]^+, \quad t \geq t_0.
\end{gathered} \tag{3.16}$$

(H_2) *For any $x, y \in R^n$,there exist nonnegative constant matrices M_k such that*

$$[J_k(t,x) - J_k(t,y)]^+ \leq M_k[x-y]^+, \quad t \geq t_0. \tag{3.17}$$

(H_3) *There exist a positive constant $\bar{\xi}$ and a positive vector $z = (z_1, z_2, \ldots, z_n)^T \in R^n$ and two positive diagonal matrices $V = \operatorname{diag}\{v_1, \ldots, v_n\}$, $S = \operatorname{diag}\{s_1, s_2, \ldots, s_n\}$, with $0 < s_i < 1$, $i \in \mathcal{N}$ such that*

$$\left(U(t) + SA(t) + V + \int_0^{\infty} N(s)e^{\bar{\xi}s}ds\right)z < 0, \quad t \geq t_0, \tag{3.18}$$

where $N(s) = (n_{ij}(s))_{n\times n} \geq 0$, $n_{ij}(s) = |r_{ij}(s)|b_{ij}$.

(H_4) *There exists a positive constant η satisfying*

$$\frac{\ln \eta_k}{t_k - t_{k-1}} \le \eta < \lambda(\varepsilon), \quad k = 1, 2, \ldots, \tag{3.19}$$

where η_k satisfy

$$\eta_k \ge 1, \quad \eta_k z \ge M_k z, \tag{3.20}$$

and $\lambda(\varepsilon)$ is defined as

$$0 < \lambda(\varepsilon) < \lambda_0(\varepsilon) = \min_{1\le i\le n}\left\{ \inf_{t\ge t_0} \lambda_i(t,\varepsilon) : \lambda_i z_i + \sum_{j=1}^{n} \left(\frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} + \int_0^\infty \frac{n_{ij}(s)}{\varepsilon} e^{\lambda_i s} ds \right) z_j = 0 \right\}. \tag{3.21}$$

for the given z.

Then there exists a small $\varepsilon_0 > 0$ such that the solution of (2.2) *is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$.*

Proof. By a similar argument with (3.4), one can know that the $\lambda(\varepsilon)$ defined by (3.21) is reasonable. For any $\phi, \varphi \in PC$, let $x(t)$, $y(t)$ be two solutions of (2.2) through (t_0, ϕ), (t_0, φ), respectively. Since $\phi, \varphi \in PC$ are bounded, we can always choose a positive vector z such that

$$[x(t) - y(t)]^+ \le z e^{-\lambda(t-t_0)}, \quad t \in (-\infty, t_0]. \tag{3.22}$$

Calculating the upper right derivative $D^+[x(t)-y(t)]^+$ along the solution of (2.2), by condition (H_1), we have

$$\begin{aligned}
D^+[x(t) - y(t)]^+ & \\
&= \operatorname{Sgn}(x(t) - y(t))(x(t) - y(t))' \\
&\le \operatorname{Sgn}\big(x(t) - y(t)\big) \frac{A(t)}{\varepsilon}\big(x(t) - y(t)\big) + \frac{1}{\varepsilon}\big[f\big(t, x(t-\tau(t))\big) - f\big(t, y(t-\tau(t))\big)\big]^+ \\
&\quad + \frac{1}{\varepsilon}\int_{-\infty}^{t} R(t-s)\big[G(x(s)) - G\big(y(s)\big)\big]^+ ds \\
&\le \frac{A(t)}{\varepsilon}[x(t) - y(t)]^+ + \frac{U(t)}{\varepsilon}[x(t) - y(t)]_\tau^+ + \frac{1}{\varepsilon}\int_0^\infty [R(s)]^+ B[x(t-s) - y(t-s)]^+ ds.
\end{aligned} \tag{3.23}$$

From condition (H_3), we have

$$\left(\frac{U(t)}{\varepsilon} + S\frac{A(t)}{\varepsilon} + \frac{V}{\varepsilon} + \int_0^\infty \frac{N(s)}{\varepsilon} e^{\bar{\xi} s} ds \right) z < 0, \quad t \ge t_0. \tag{3.24}$$

Therefore, (3.23) and (3.24) imply that all the assumptions of Lemma 3.1 are true. So we have

$$[x(t) - y(t)]^+ \leq z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_0, t_1), \tag{3.25}$$

where $\lambda(\varepsilon)$ is determined by (3.21) and the positive constant vector z is determined by (3.18).

Using the discrete part of (2.2), condition (H_2), (3.20) and (3.25), we can obtain that

$$\begin{aligned}
[x(t_1) - y(t_1)]^+ &= \left[J_1\left(t_1, x\left(t_1^-\right)\right) - J_1\left(t_1, y\left(t_1^-\right)\right)\right]^+ \\
&\leq M_1\left[x\left(t_1^-\right) - y\left(t_1^-\right)\right]^+ \\
&\leq M_1 z e^{-\lambda(\varepsilon)(t_1-t_0)} \\
&\leq \eta_1 z e^{-\lambda(\varepsilon)(t_1-t_0)},
\end{aligned} \tag{3.26}$$

and so, we have

$$[x(t) - y(t)]^+ \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in (-\infty, t_1]. \tag{3.27}$$

By a similar argument with (3.25), we can use (3.27) derive that

$$[x(t) - y(t)]^+ \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_1, t_2). \tag{3.28}$$

Therefore, by simple induction, we have

$$[x(t) - y(t)]^+ \leq \eta_1 \cdots \eta_{k-1} z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots. \tag{3.29}$$

From (3.19) and (3.29), we obtain

$$[x(t) - y(t)]^+ \leq z e^{-(\lambda(\varepsilon)-\eta)(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots,\ \forall \varepsilon > 0. \tag{3.30}$$

For any $t \geq t_0$, let $\lambda_i(t, \varepsilon)$ be defined as the unique positive zero of

$$\lambda_i z_i + \sum_{j=1}^{n} \left(\frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} + \int_0^\infty \frac{n_{ij}(s)}{\varepsilon} e^{\lambda_i s} ds \right) z_j = 0. \tag{3.31}$$

Differentiate both sides of (3.31) with respect to the variable ε, we have

$$\frac{d}{d\varepsilon}\lambda_i(t,\varepsilon) = \frac{-\lambda_i z_i}{\varepsilon z_i + \sum_{j=1}^{n} u_{ij}(t)\tau e^{\lambda_i \tau} z_j + \sum_{j=1}^{n}\int_0^\infty n_{ij}(s) s z_j e^{\lambda_i s} ds} < 0, \tag{3.32}$$

so $\lambda_i(t,\varepsilon)$ is monotonically decreasing with respect to the variable ε, which implies that $\lambda_0(\varepsilon)$ is also monotonically decreasing with respect to the variable ε. So we can choose the $\lambda(\varepsilon)$ in (3.21) satisfying the same monotonicity with $\lambda_0(\varepsilon)$, for example, $\lambda(\varepsilon) = \lambda_0(\varepsilon) - \delta$, where $0 < \delta < \lambda_0(\varepsilon) - \lambda(\varepsilon)$. Hence we can deduce that there exists a small $\varepsilon_0 > 0$ such that the solution of (2.2) is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$. The proof is completed. □

Remark 3.4. Suppose that $N(s) = (n_{ij}(s))_{n\times n} = 0$ in Theorem 3.3, then we can easily get [14, Theorem 1]. In fact, "$\eta_k \overset{\Delta}{=} \max\{\|M_k\|, 1\}$" of condition (H_4) in [14, Theorem 1] ensure that the above (3.20) holds.

Remark 3.5. If $J_k(t,x) = x$, $t \geq t_0$, that is there have no impulses in (2.2), then by Theorem 3.3, we can obtain the following result.

Corollary 3.6. *Assume that $A(t) = (a_{ij}(t))_{n\times n} \geq 0$ for $t \geq t_0$ and $i \neq j$, $N(s) = (n_{ij}(s))_{n\times n} \geq 0$, further suppose that (H_1) and (H_3) hold. Then there exists a small $\varepsilon_0 > 0$ such that the solution of* (2.2) *is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$.*

Remark 3.7. From Lemma 3.1 and the proof of Theorem 3.3, it is obvious that the results obtained in this paper still hold for $\varepsilon = 1$. So this type of exponential stability can obviously be applied to general impulsive delay integrodifferential equations.

Remark 3.8. When $\varepsilon = 1$ and $r_{ij}(s) = 0$, the global exponential stability criteria for (2.2) have been established in [18] by utilizing the Lyapunov functional method. However, the additional assumption that f_j is bounded is required in [18].

4. An Illustrative Example

In this section, we will give an example to illustrate the exponential stability of (2.2).

Example 4.1. Consider the following SPIDIDEs:

$$\begin{aligned}
\varepsilon\dot{x}_1(t) &= (-10 - \sin t)x_1(t) + (2 + \sin t)\arctan x_1(t - \tau(t)) \\
&\qquad + (1 + \cos t)\arctan x_2(t - \tau(t)) + \int_{-\infty}^{t} e^{-(t-s)} x_1(s) ds, \quad t \neq t_k, \\
\varepsilon\dot{x}_2(t) &= (-8 + 2\cos t)x_1(t) + \sin^2 t \arctan x_1(t - \tau(t)) \\
&\qquad + (1 - \cos t)\arctan x_2(t - \tau(t)) + \int_{-\infty}^{t} e^{-2(t-s)} x_2(s) ds, \quad t \neq t_k, \\
x_1(t_k) &= \alpha_{1k} x_1\left(t_k^-\right) - \beta_{1k} x_2\left(t_k^-\right), \\
x_2(t_k) &= \beta_{2k} x_1\left(t_k^-\right) + \alpha_{2k} x_2\left(t_k^-\right),
\end{aligned} \tag{4.1}$$

where $\alpha_{ik}, \beta_{ik} \geq 0$ are constants, $\tau(t) = e^{-t} \leq 1 \overset{\Delta}{=} \tau$, $t \geq t_0$, $t_k = t_{k-1} + 3k$, $k = 1, 2, \ldots$.

We can easily find that conditions (H_1) and (H_2) are satisfied with

$$A(t) = \begin{pmatrix} -10 - \sin t & 0 \\ 0 & -8 + 2\cos t \end{pmatrix}, \qquad U(t) = \begin{pmatrix} 2 + \sin t & 1 + \cos t \\ \sin^2 t & 1 - \cos t \end{pmatrix},$$
$$N(s) = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}, \qquad M_k = \begin{pmatrix} \alpha_{1k} & \beta_{1k} \\ \beta_{2k} & \alpha_{2k} \end{pmatrix}. \tag{4.2}$$

So there exist $\bar{\xi} = 0.5$, $z = (1,1)^T$, $V = \operatorname{diag}\{1, 1/3\}$ and $S = \operatorname{diag}\{0.8, 0.5\}$ such that

$$\left(U(t) + SA(t) + V + \int_0^\infty N(s) e^{\bar{\xi} s} ds\right) z = \left(-2 + 0.2\sin t + \cos t, -2 + \sin^2 t\right) < 0, \quad t \ge t_0. \tag{4.3}$$

Let $\eta_k = \max\{\alpha_{1k} + \beta_{1k}, \alpha_{2k} + \beta_{2k}\}$, we can obtain η_k satisfy $\eta_k z \ge M_k z$.

Case 1. Let $\alpha_{1k} = 0.2e^{0.3k}$, $\alpha_{2k} = 0.7e^{0.3k}$, $\beta_{1k} = 0.5e^{0.3k}$, $\beta_{2k} = 0.3e^{0.3k}$, then we obtain that there exists an $\eta = 0.1 > 0$ such that

$$\eta_k = e^{0.3k} \ge 1, \qquad \frac{\ln \eta_k}{t_k - t_{k-1}} = \frac{0.3k}{3k} = 0.1 = \eta, \tag{4.4}$$

and for $\varepsilon > 0$, the positive constant $\lambda(\varepsilon)$ is determined by the following equations:

$$\lambda_1(t) + \frac{1}{\varepsilon}\left(-10 - \sin t + (3 + \sin t + \cos t) e^{\lambda_1(t)} + \int_0^\infty e^{-s} e^{\lambda_1(s)} ds\right) = 0,$$
$$\lambda_2(t) + \frac{1}{\varepsilon}\left(-8 + 2\cos t + \left(1 + \sin^2 t - \cos t\right) e^{\lambda_2(t)} + \int_0^\infty e^{-2s} e^{\lambda_2(s)} ds\right) = 0. \tag{4.5}$$

So for a given ε, we can obtain the corresponding λ by (4.5). By the proof of Theorem 3.3, we know that λ is monotonically decreasing with respect to the variable ε, then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have $\lambda > \eta$. Therefore, all the conditions of Theorem 3.3 are satisfied, we conclude that the solution of (4.1) is exponentially stable for sufficiently small $\varepsilon > 0$.

Case 2. Let $\alpha_{1k} = \alpha_{2k} = 1$ and $\beta_{1k} = \beta_{2k} = 0$, then (4.1) becomes the singularly perturbed delay integrodifferential equations without impulses. So by Corollary 3.6, the solution of (4.1) is exponentially stable for sufficiently small $\varepsilon > 0$.

Remark 4.2. Obviously, the delay differential inequality which established in [14] is ineffective for studing the stability of SPIDIDEs (4.1).

Acknowledgments

The authors would like to thank the editor and the referees for their very helpful suggestions. The work is supported by National Natural Science Foundation of China under Grant 10671133.

References

[1] J. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences, Springer, New York, NY, USA, 2nd edition, 1977.

[2] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, vol. 74 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[3] V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[4] S. Gan, "Dissipativity of θ-methods for nonlinear Volterra delay-integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 898–907, 2007.

[5] A. Alsaedi and B. Ahmad, "Existence and analytic approximation of solutions of duffing type nonlinear integro-differential equation with integral boundary conditions," *Journal of Inequalities and Applications*, vol. 2009, Article ID 193169, 19 pages, 2009.

[6] B. M. Brown and V. G. Kirby, "A computational investigation of an integro-differential inequality with periodic potential," *Journal of Inequalities and Applications*, vol. 5, no. 4, pp. 321–342, 2000.

[7] L. Xu and D. Xu, "Exponential stability of nonlinear impulsive neutral integro-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2910–2923, 2008.

[8] M. U. Akhmetov, A. Zafer, and R. D. Sejilova, "The control of boundary value problems for quasilinear impulsive integro-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 48, no. 2, pp. 271–286, 2002.

[9] J. J. Nieto and R. Rodríguez-López, "New comparison results for impulsive integro-differential equations and applications," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1343–1368, 2007.

[10] M. C. Mackey and L. Glass, "Oscillation and chaos in physiological control systems," *Science*, vol. 197, no. 4300, pp. 287–289, 1977.

[11] H. Tian, "The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 1, pp. 143–149, 2002.

[12] J. H. da Cruz and P. Z. Táboas, "Periodic solutions and stability for a singularly perturbed linear delay differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 6, pp. 1657–1667, 2007.

[13] X. Liu, X. Shen, and Y. Zhang, "Exponential stability of singularly perturbed systems with time delay," *Applicable Analysis*, vol. 82, no. 2, pp. 117–130, 2003.

[14] W. Zhu, D. Xu, and C. Yang, "Exponential stability of singularly perturbed impulsive delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1161–1172, 2007.

[15] D. Xu, Z. Yang, and Z. Yang, "Exponential stability of nonlinear impulsive neutral differential equations with delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 5, pp. 1426–1439, 2007.

[16] Z.-H. Guan, "Impulsive inequalities and stability of large scale singular measure differential systems with unbounded delays," *Dynamics of Continuous, Discrete and Impulsive Systems for Theory and Applications*, vol. 3, no. 1, pp. 113–130, 1997.

[17] D. Xu and Z. Yang, "Impulsive delay differential inequality and stability of neural networks," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 107–120, 2005.

[18] Z.-T. Huang, Q.-G. Yang, and X.-S. Luo, "Exponential stability of impulsive neural networks with time-varying delays," *Chaos, Solitons & Fractals*, vol. 35, no. 4, pp. 770–780, 2008.