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Research Article

Integral Inequality and Exponential Stability for Neutral Stochastic Partial Differential Equations with Delays

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The aim of this paper is devoted to obtain some sufficient conditions for the exponential stability in $p\,(p \geq 2)$-moment as well as almost surely exponential stability for mild solution of neutral stochastic partial differential equations with delays by establishing an integral-inequality. Some well-known results are generalized and improved. Finally, an example is given to show the effectiveness of our results.

1. Introduction

The investigation for stochastic partial differential equations with delays has attracted the considerable attention of researchers and many qualitative theories for the solutions of this kind have been derived. Many important results have been reported in [1–20]. For example, Caraballo, in [1], extended the results from Haussmann [7] to the delay equations of the same kind; Mao, in [15, 21], proved the exponential stability in mean-square sense about the strong solutions of linear stochastic differential equations with finite constant delay; by using the method in [7, 8], Caraballo and Real, in [4], considered the stability for the strong solutions of semilinear stochastic delay evolution equations; Govindan, in [5, 6], has studied the existence and stability of mild solutions for stochastic partial differential equations by the comparison theorem; Caraballo and Liu, in [2], discussed the exponential stability for mild solution of stochastic partial differential equations with delays by employing the well-known Gronwall inequality and stochastic analysis technique under the Lipschitz condition, but the requirement of the monotone decreasing behaviors of the delays should be imposed; Liu and Truman in [9] and Liu and Mao in [10] analyzed the exponential stability for mild solution of stochastic partial functional differential equations by establishing the corresponding Razuminkhin-type theorem.

In the case of delay differential equations, in the particular case when we are concerned with the mild solution of stochastic partial differential equations, Lyapunov's second method, although it is usually regarded as an important tool to study the stability and boundedness, is not suitable to consider such problem. A crucial problem is that mild solutions do not have stochastic differentials, so that the Itô formula fails to deal with this problem. Very recently, Burton has successfully utilized the fixed point theorem to investigate the stability for deterministic systems in [22]; Luo in [23] and Appleby in [22] have applied this valuable method into dealing with the stability for stochastic differential equations. Following the ideas of Burton in [22], Luo in [23], and Appleby in [24], by employing the contraction mapping principle and stochastic analysis, some sufficient conditions ensuring the trivial solution of exponential stability in p $(p \geq 2)$-moment and almost sure exponential stability for mild solution of stochastic partial differential equations with delays were obtained in [13], which did not comprise the monotone decreasing behavior of the delays.

However, comparing with stochastic partial differential equations with delays, there are only a few results about neutral stochastic partial differential equations. Precisely, Liu [11] considered a linear neutral stochastic differential equations with constant delays and some stability properties of the mild solutions in a similar way as Datko [25] in the deterministic case. Caraballo et al., in [3], have studied the almost sure exponential stability and ultimate boundedness of the solutions to a class of neutral stochastic semilinear partial delay differential equations; Mahmudov, in [14], has discussed the existence and uniqueness for mild solution of neutral stochastic differential equations by constructing a new iterative scheme under the non-Lipschitz conditions.

It should be pointed out that there exist a number of difficulties encountered in the study of the stability for mild solution to neutral stochastic partial differential equations with delays since the neutral item is present. And many methods used frequently fail to consider the exponential stability of mild solution for neutral stochastic partial differential equations with delays, for example, the comparison theorem in [5, 6], the Gronwall inequality in [2], the analytic technique in [9], and the semigroup method in [16]. The methods proposed in [1, 3, 8, 10, 18, 20] are also ineffective in dealing with this problem since mild solutions do not have stochastic differentials. So, the technique and the method dealt with such problems are in need of being developed and explored. On the other hand, to the best of our knowledge, there is no paper which investigates the exponential stability in p $(p \geq 2)$-moment and almost surely exponential stability for mild solution of such problems. Thus, we will make the first attempt to study such problem to close this gap in this paper.

The content of this paper is arranged as follows. In Section 2, some necessary definitions, notations, and lemmas used in this paper will be introduced. In Section 3, by establishing a lemma, some sufficient conditions about the exponential stability in p $(p \geq 2)$-moment and almost sure exponential stability are derived. Finally, one example is provided to illustrate the obtained results.

2. Preliminaries

Let X and Y be two real, separable Hilbert spaces and let $L(Y, X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation $\|\cdot\|$ to denote the norms in X, Y and $L(Y, X)$. Let $(\Omega, \Im, P)$ be a complete probability space equipped with some filtration $\Im_t$ $(t \geq 0)$ satisfying the usual conditions; that is, the filtration is right continuous and $\Im_0$ contains all P-null sets.

In this paper, we consider the following neutral stochastic partial differential equations with delays:

$$
\begin{aligned}
&d[x(t) - D(t, x(t-\delta(t)))] \\
&\quad = \left[Ax(t) + f(t, x(t-r(t)))\right]dt + g\big(t, x(t-\rho(t))\big)dw(t), \quad t \in [0, +\infty), \\
&x_0(\cdot) = \varphi \in C^b_{\Im_0},
\end{aligned} \tag{2.1}
$$

where φ is $\Im_0$-measurable and $\delta,\ r,\ \rho : [0, +\infty) \to [0, \tau]$ $(\tau > 0)$ are bounded and continuous functions. Let $C([-\tau, 0], X)$ be the space of all right-continuous functions with left-hand limit φ from $[-\tau, 0]$ to X with the sup-norm $\|\cdot\|_C = \sup_{-\tau \le \theta \le 0} \|\varphi(\theta)\|$ and let $C^b_{\Im_t} \equiv C^b_{\Im_t}([-\tau, 0], X)$ be the family of all almost surely bounded, $\Im_t$ $(t \ge 0)$-measurable, and $C([-\tau, 0], X)$-valued random variables. $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)$ $(t \ge 0)$ on the Hilbert space X; then it is possible under some circumstances (we refer the readers to [26] for a detailed presentations of the definition and relevant properties of $(-A)^\alpha$) to define the fractional power $(-A)^\alpha : D((-A)^\alpha) \to X$ which is a closed linear operator with its domain $D((-A)^\alpha)$, for $\alpha \in (0, 1]$. Let $f : [0, +\infty) \times X \to X$, $(-A)^\alpha D : [0, +\infty) \times X \to X$, and $g : [0, +\infty) \times X \to L^0_2(Y, X)$ be three suitable measurable mappings, where $L^0_2(Y, X)$ is introduced in detail as follows.

Let $\beta_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \Im, P)$. Set $w(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n}\beta_n(t)e_n,\ t \ge 0$, where $\lambda_n \ge 0$ $(n = 1, 2, \ldots)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in Y. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with a finite trace $\operatorname{tr}Q = \sum_{n=1}^{+\infty} \lambda_n < +\infty$. Then, the above Y-valued stochastic process $w(t)$ is called a Q-Wiener process.

Definition 2.1. Let $\sigma \in L(Y, X)$ and define

$$
\|\sigma\|^2_{L^0_2} := \operatorname{tr}(\sigma Q \sigma^*) = \left\{ \sum_{n=1}^{+\infty} \left\| \sqrt{\lambda_n}\sigma e_n \right\|^2 \right\}. \tag{2.2}
$$

If $\|\sigma\|^2_{L^0_2} < +\infty$, then σ is called a Q-Hilbert-Schmidt operator and let $L^0_2(Y, X)$ denote the space of all Q-Hilbert-Schmidt operators $\sigma : Y \to X$.

Now, for the definition of an X-valued stochastic integral of an $L^0_2(Y, X)$-valued and $\Im_t$-adapted predictable process $\Phi(t)$ with respect to the Q-Wiener process $w(t)$, the readers can refer to [26].

Definition 2.2. An X-value stochastic process $x(t)$, $t \in [-\tau, +\infty)$, is called a mild solution of the system (2.1), if

(i) $x(t)$ is a $\Im_t$ $(t \ge 0)$ adapted process;

(ii) $x(t) \in X$ has a continuous paths on $t \in [0, +\infty)$ almost surely, and f or arbitrary $t \in [0, +\infty)$,

$$
\begin{aligned}
x(t) = {} & S(t)(x(0) - D(0, x(-\delta(0)))) + D(t, x(t-\delta(t))) + \int_0^t AS(t-s) D(s, x(s-\delta(s))) ds \\
& + \int_0^t S(t-s) f(s, x(s-r(s))) ds + \int_0^t S(t-s) G(s, x(s-\rho(s))) dw(s),
\end{aligned}
\tag{2.3}
$$

where $x_0(\cdot) = \varphi \in C^b_{\mathfrak{I}_0}$.

Definition 2.3. The mild solution of system (2.1) is said to be exponentially stable in p $(p \geq 2)$-moment, if there exists a pair of positive constants $\gamma > 0$ and $M_1 > 0$, for any initial value $\varphi \in C^b_{\mathfrak{I}_0}$ such that

$$
E\|x(t)\|^p \leq M_1 E\|\varphi\|_C^p e^{-\gamma t}, \quad t \geq 0, \ p \geq 2. \tag{2.4}
$$

Definition 2.4. The mild solution of (2.1) is said to be almost surely exponential stability if there exists a positive constant $\alpha > 0$, for any initial value $\varphi \in C^b_{\mathfrak{I}_0}$, such that

$$
\lim_{t \to +\infty} \sup \frac{1}{t} \ln(\|x(t)\|) < -\alpha, \quad P - \text{a.s.} \tag{2.5}
$$

Lemma 2.5 (see [26]). *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)(t \geq 0)$, on the separable Hilbert space X. If $0 \in \rho(-A)$, then one has the following.*

(a) *There exist a constant $M \geq 1$ and a real number $\beta > 0$ such that $\|S(t)h\| \leq Me^{-\beta t}\|h\|$, $t \geq 0$, for any $h \in X$.*

(b) *The fractional power $(-A)^\alpha$ satisfies that $\|(-A)^\alpha S(t)h\| \leq M_\alpha e^{-\beta t} t^{-\alpha}\|h\|$, $t > 0$, for any $h \in X$, where $M_\alpha \geq 1$, $\alpha \in (0,1]$.*

(c) $\|(S(t) - I)h\| \leq N_\alpha t^\alpha \|(-A)^\alpha h\|$, $h \in D((-A)^\alpha)$, $N_\alpha \geq 1$, $t \geq 0$.

Lemma 2.6 (see [21, 27]). *Let $p \in [1, +\infty)$ and $\nu \in (0,1)$. For any two real positive numbers a, $b > 0$, then*

$$
(a+b)^p \leq \nu^{1-p} a^p + (1-\nu)^{1-p} b^p. \tag{2.6}
$$

Lemma 2.7 (see [28]). *For any $r \geq 1$ and for arbitrary $L_2^0(Y, X)$-valued predictable process $\Phi(\cdot)$,*

$$
\sup_{s \in [0,t]} E\left\|\int_0^s \Phi(u) dw(u)\right\|^{2r} \leq C_r \left(\int_0^t \left(E\|\Phi(s)\|^{2r}_{L_2^0} \right)^{1/r} ds \right)^r, \quad t \in [0, +\infty), \tag{2.7}
$$

where $C_r = (r(2r-1))^r$.

Lemma 2.8 (see [2]). *Let $\|S(t)\| \leq M$. For all $t \geq 0$ let $\Phi : [0, +\infty) \to L_2^0$ be a predictable, $\Im_t$-adapted process such that $\int_0^t E\|\Phi(s)\|_{L_2^0}^p ds < +\infty$ for some integer $p > 2$ and any $t \geq 0$. Then, there exists a constant $c(p) > 0$ such that for any fixed natural number $N > t_0$,*

$$E\left\{ \sup_{N \leq t \leq N+1} \left\| \int_N^t S(t-s)\Phi(s)dw(s) \right\|^p \right\} \leq c(p) \int_N^{N+1} E\|\Phi(s)\|_{L_2^0}^p \, ds. \tag{2.8}$$

Lemma 2.9 (see [2]). *Let A be the infinitesimal generator of a contraction semigroup. Let $\Phi : [0, +\infty) \to L_2^0(Y, X)$ be a predictable $\Im_t$-adapted process such that $\int_0^t E\|\Phi(s)\|_{L_2^0}^2 ds < +\infty$, for any $t \geq 0$. Then there exists a constant $K_0 > 0$, independent of N, such that for any fixed natural number $N > 0$,*

$$E\left\{ \sup_{N \leq t \leq N+1} \left\| \int_N^t S(t-s)\Phi(s)dw(s) \right\|^2 \right\} \leq K_0 \int_N^{N+1} E\|\Phi(s)\|_{L_2^0}^2 ds. \tag{2.9}$$

Furthermore, one imposes the following important assumptions.

(H_1) *The mappings $f(t, \cdot)$ and $g(t, \cdot)$ satisfy the uniformly Lipschitz condition: there exist two positive constants $C_1, C_2 > 0$, for any $x, y \in X$ and $t \geq 0$ such that*

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leq C_1\|x-y\|, \quad f(t,0) = 0, \\ \|g(t,x) - g(t,y)\|_{L_2^0} &\leq C_2\|x-y\|, \quad g(t,0) = 0. \end{aligned} \tag{2.10}$$

(H_2) *The mapping $(-A)^\alpha D(t, \cdot)$ also satisfies the uniformly Lipschitz condition: there exists one positive constant $C_3 > 0$, for any $x,\ y \in X$ such that*

$$\|(-A)^\alpha D(t,x) - (-A)^\alpha D(t,y)\| \leq C_3\|x-y\|, \quad D(t,0) = 0,\ t \geq 0, \tag{2.11}$$

for $\alpha \in (1/p, 1]$ $(p \geq 2)$ and $D(t, \cdot) \in D((-A)^\alpha)$.

(H_3) *For $\alpha \in (1/p, 1]$ $(p \geq 2)$, $\kappa = \|(-A)^{-\alpha}\|C_3 < 1$.*

Remark 2.10. Under the condition: (H_1)–(H_3), the existence and uniqueness of mild solution to the neutral stochastic partial differential equations with delays (2.1) is easily shown by using the proposed method in [14] and the proof of this problem is very similar to the proof of [14, Theorem 6]. Here, we omit it. In particular, the system (2.1) obviously has a trivial mild solution when $\varphi = 0$.

3. Main Results

In this section, in order to establish some sufficient conditions ensuring the exponential stability in p $(p \geq 2)$-moment and almost sure exponential stability for mild solution of system (2.1), we are in need of establishing the following integral-inequality to overcome the difficulty when the neutral item is present.

Lemma 3.1. *For $\gamma > 0$, there exist three positive constants: $\lambda_i > 0$ $(i = 1,2,3)$ and a function $y : [-\tau, +\infty) \to [0, +\infty)$. If $\lambda_2 + (\lambda_3/\gamma) < 1$, the following inequality*

$$y(t) \le \begin{cases} \lambda_1 e^{-\gamma t} + \lambda_2 \sup\limits_{\theta\in[-\tau,0]} y(t+\theta) + \lambda_3 \displaystyle\int_0^t e^{-\gamma(t-s)} \sup\limits_{\theta\in[-\tau,0]} y(s+\theta)ds, & t \ge 0, \\ \lambda_1 e^{-\gamma t}, & t \in [-\tau, 0] \end{cases} \tag{3.1}$$

holds. Then one has: $y(t) \le M_2 e^{-\mu t}$ $(t \ge -\tau)$, where μ is a positive root of the algebra equation: $(\lambda_2 + (\lambda_3/\gamma - \mu))e^{\mu\tau} = 1$ and $M_2 = \max\{\lambda_1(\gamma-\mu)/\lambda_3 e^{\mu\tau}, \lambda_1\} > 0$.

Proof. Letting $F(\lambda) = (\lambda_2 + (\lambda_3/\gamma - \lambda))e^{\lambda\tau} - 1$, we have $F(0)\ F(\gamma-) < 0$ holds. That is, there exists a positive constant $\mu \in (0, \gamma)$, such that $F(\mu) = 0$.

For any $\varepsilon > 0$ let

$$C_\varepsilon = \max\left\{ (\varepsilon + \lambda_1)\frac{(\gamma-\mu)}{\lambda_3 e^{\mu\tau}}, \lambda_1 + \varepsilon \right\}. \tag{3.2}$$

Now, in order to show this lemma, we only claim that (3.1) implies

$$y(t) \le C_\varepsilon e^{-\mu t}, \quad t \ge -\tau. \tag{3.3}$$

Easily, for any $t \in [-\tau, 0]$, (3.3) holds. Assume, for the sake of contradiction, that there exists a $t_1 > 0$ such that

$$y(t) < C_\varepsilon e^{-\mu t}, \quad t \in [-\tau, t_1), \qquad y(t_1) = C_\varepsilon e^{-\mu t_1}. \tag{3.4}$$

Then, (3.1) implies

$$\begin{aligned} y(t_1) &\le \lambda_1 e^{-\gamma t_1} + \lambda_2 C_\varepsilon \sup_{\theta\in[-\tau,0]} e^{-\mu(t_1+\theta)} + \lambda_3 C_\varepsilon \int_0^{t_1} e^{-\gamma(t_1-s)} \sup_{\theta\in[-\tau,0]} e^{-\mu(s+\theta)} ds \\ &\le \lambda_1 e^{-\gamma t_1} + \lambda_2 C_\varepsilon e^{-\mu t_1} e^{\mu\tau} + \lambda_3 C_\varepsilon e^{-\gamma t_1} \int_0^{t_1} e^{(\gamma-\mu)s} ds e^{\mu\tau} \\ &= \lambda_1 e^{-\gamma t_1} - \frac{\lambda_3 C_\varepsilon e^{\mu\tau}}{\gamma-\mu} e^{-\gamma t_1} + \left(\lambda_2 + \frac{\lambda_3}{\gamma-\mu}\right) C_\varepsilon e^{\mu\tau} e^{-\mu t_1}. \end{aligned} \tag{3.5}$$

From the definitions of μ and C_ε, we have

$$\begin{gathered} \left(\lambda_2 + \frac{\lambda_3}{\gamma-\mu}\right) e^{\mu\tau} = 1, \\ \lambda_1 e^{-\gamma t_1} - \frac{\lambda_3 C_\varepsilon e^{\mu\tau}}{\gamma-\mu} e^{-\gamma t_1} \le \lambda_1 e^{-\gamma t_1} - \frac{\lambda_3 e^{\mu\tau}}{\gamma-\mu} e^{-\gamma t_1} (\varepsilon + \lambda_1) \frac{(\gamma-\mu)}{\lambda_3 e^{\mu\tau}} < 0. \end{gathered} \tag{3.6}$$

Thus, (3.5) yields

$$y(t_1) < C_\varepsilon e^{-\mu t_1}, \tag{3.7}$$

which contradicts (3.4); that is, (3.3) holds.

As $\varepsilon > 0$ is arbitrarily small, in view of (3.3), it follows

$$y(t) \le M_2 e^{-\mu t}, \quad t \ge -\tau, \tag{3.8}$$

where $M_2 = \max\{\lambda_1((\gamma - \mu)/\lambda_3 e^{\mu\tau}),\ \lambda_1\} > 0$. The proof of this lemma is completed. □

Theorem 3.2. *Supposed that the conditions* (H_1)*–*(H_3) *are satisfied, then the mild solutionto system* (2.1) *is exponential stability in* $p(p \ge 2)$*-moment, if the following inequality*

$$\begin{aligned}&\frac{6^{p-1}(q\beta)^{p\alpha-(p/q)} M_{1-\alpha}^p (\Gamma(1-q(1-\alpha)))^{q/p} C_3^p}{(1-\kappa)^p}\\ &\quad + \frac{3^{p-1}M^p\left(\beta^{1-p}C_1^p + C_2^p (p(p-1)/2)^{p/2}(2\beta(p-1)/(p-2))^{1-(p/2)}\right)}{(1-\kappa)^p} < \beta\end{aligned} \tag{3.9}$$

holds, where $(1/p) + (1/q) = 1\ (p \ge 2,\ 1 < q \le 2)$.

Proof. By virtue of the inequality (3.9) and the condition (H_3), we can always find a number $\varepsilon > 0$ small enough such that

$$\begin{aligned}\kappa + \Bigg[&{}^{p-1}(1+\varepsilon)^{p-1}(q\beta)^{p\alpha-(p/q)} M_{1-\alpha}^p (\Gamma(1-q(1-\alpha)))^{q/p} C_3^p\\ &+3^{p-1}M^p\left(\beta^{1-p}C_1^p + C_2^p\left(\frac{p(p-1)}{2}\right)^{p/2}\left(\frac{2\beta(p-1)}{p-2}\right)^{1-(p/2)}\right)\Bigg] / (1-\kappa)^{p-1}\beta < 1.\end{aligned} \tag{3.10}$$

Based on an elementary inequality, for any real numbers $a,\ b,\ c,\ d$, and e, it follows

$$\begin{aligned}(a+b+c+d+e)^p &\le 3^{p-1}(a+b+c)^p + 3^{p-1}d^p + 3^{p-1}e^p\\ &\le 3^{p-1}\left[\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + (1+\varepsilon)^{p-1}(b+c)^p\right] + 3^{p-1}d^p + 3^{p-1}e^p\\ &\le 3^{p-1}\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + 6^{p-1}(1+\varepsilon)^{p-1}(b^p + c^p) + 3^{p-1}d^p + 3^{p-1}e^p.\end{aligned} \tag{3.11}$$

From (2.3), the condition (H_3), and Lemma 2.6, we have

$$
\begin{aligned}
E\|x(t)\|^p \leq{}& \frac{E\|D(t,x(t-\delta(t)))\|^p}{\kappa^{p-1}} + \frac{1}{(1-\kappa)^{p-1}} E\| S(t)(x(0)-D(0,x(-\delta(0))))\\
&+ \int_0^t AS(t-s)D(s,x(s-\delta(s)))ds + \int_0^t S(t-s)f(s,x(s-r(s)))ds\\
&+ \int_0^t S(t-s)g\big(s,x(s-\rho(s))\big)dw(s)\Big\|^p\\
\leq{}& \frac{E\|D(t,x(t-\delta(t)))\|^p}{\kappa^{p-1}} + \frac{3^{p-1}}{(1-\kappa)^{p-1}}\left(1+\frac{1}{\varepsilon}\right)^{p-1} E\|S(t)x(0)\|^p\\
&+ \frac{6^{p-1}}{(1-\kappa)^{p-1}}(1+\varepsilon)^{p-1} E\|-S(t)D(0,x(-\delta(0)))\|^p\\
&+ \frac{6^{p-1}}{(1-\kappa)^{p-1}}(1+\varepsilon)^{p-1} M_{1-\alpha}^p \left(\int_0^t e^{-\beta(t-s)}(t-s)^{-q(1-\alpha)}ds\right)^{p/q}\\
&\times \int_0^t e^{-p\beta(t-s)} E\big\|(-A)^\alpha D(s,x(s-\delta(s)))\big\|^p ds\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}} E\left\|\int_0^t S(t-s)f(s,x(s-r(s)))ds\right\|^p\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}} E\left\|\int_0^t S(t-s)g\big(s,x(s-\rho(s))\big)dw(s)\right\|^p\\
={}& \frac{E\|D(t,x(t-\delta(t)))\|^p}{\kappa^{p-1}} + \frac{3^{p-1}}{(1-\kappa)^{p-1}}\left(1+\frac{1}{\varepsilon}\right)^{p-1} \|S(t)x(0)\|^p\\
&+ \frac{6^{p-1}}{(1-\kappa)^{p-1}}(1+\varepsilon)^{p-1} E\|-S(t)D(0,x(-\delta(0)))\|^p\\
&+ \frac{6^{p-1}}{(1-\kappa)^{p-1}}(1+\varepsilon)^{p-1} M_{1-\alpha}^p (q\beta)^{p\alpha-(p/q)}\big(\Gamma(1-q(1-\alpha))\big)^{p/q}\\
&\times \int_0^t e^{-\beta(t-s)} E\big\|(-A)^\alpha D(s,x(s-\delta(s)))\big\|^p ds\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}} E\left\|\int_0^t S(t-s)f(s,x(s-r(s)))ds\right\|^p\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}} E\left\|\int_0^t S(t-s)g\big(s,x(s-\rho(s))\big)dw(s)\right\|^p.
\end{aligned}
\tag{3.12}
$$

Then, Lemma 2.7 and (3.12) imply that

$$\begin{aligned}
E\|x(t)\|^2 \leq{}& \kappa \sup_{\theta\in[-\tau,0]} E\|x(t+\theta)\|^p + \frac{3^{p-1}M^p}{(1-\kappa)^{p-1}}\left(\left(1+\frac{1}{\varepsilon}\right)^{p-1} + 2^{p-1}(1+\varepsilon)^{p-1}\kappa^p\left\|(-A)^{-\alpha}\right\|^p\right)\\
&\times \sup_{\theta\in[-\tau,0]} E\left\|\varphi(\theta)\right\|^2 e^{-\beta t} + \frac{6^{p-1}}{(1-\kappa)^{p-1}}(1+\varepsilon)^{p-1}(q\beta)^{p\alpha-(p/q)}\left(\Gamma(1-q(1-\alpha))\right)^{p/q}C_3^p\\
&\times \int_0^t e^{-\beta(t-s)} \sup_{\theta\in[-\tau,0]} E\|x(s+\theta)\|^p ds\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}}M^p C_1^p \beta^{1-p}\int_0^t e^{-\beta(t-s)} \sup_{\theta\in[-\tau,0]} E\|x(s+\theta)\|^p ds\\
&+ \frac{3^{p-1}}{(1-\kappa)^{p-1}}M^p C_2^p\left(\frac{p(p-1)}{2}\right)^{p/2}\left(\frac{2\beta(p-1)}{p-2}\right)^{1-(p/2)}\\
&\times \int_0^t e^{-\beta(t-s)} \sup_{\theta\in[-\tau,0]} E\|x(s+\theta)\|^p ds.
\end{aligned} \tag{3.13}$$

By Lemma 3.1, we can derive that $E\|x(t)\|^p \leq M_3 e^{-\mu t}(M_3 > 0, \mu \in (0,\beta))$. That is, the exponential stability in p $(p \geq 2)$-moment for mild solution to system (2.1) is obtained. The proof is completed. □

Theorem 3.3. *Suppose that all the conditions of Theorem 4.1 hold with* $p > 2$*, then the mild solution of system* (2.1) *is almost surely exponential stability, that is,*

$$\lim_{t\to+\infty} \frac{\log\|x(t)\|}{t} \leq -\frac{\mu}{2p}, \quad \text{a.s.,} \tag{3.14}$$

where μ *is defined in Theorem 3.2.*

Proof. Let N be a sufficiently large positive integer and for $t \in [N, N+1]$, then

$$\begin{aligned}
x(t) ={}& S(t-N)(x(N) - D(N, x(N-\delta(N)))) + D(t, x(t-\delta(t)))\\
&+ \int_N^t AS(t-s)D(s, x(s-\delta(s)))ds\\
&+ \int_N^t S(t-s)f(s, x(s-r(s)))ds + \int_N^t S(t-s)g(s, x(s-\rho(s)))dw(s).
\end{aligned} \tag{3.15}$$

For arbitrary fixed $\varepsilon_N > 0$, we have

$$\begin{aligned}
P\left\{\sup_{N\le t\le N+1}\|x(t)\| > \varepsilon_N\right\} \le{}& P\left\{\sup_{N\le t\le N+1}\|S(t-N)(x(N)-D(N,x(N-\delta(N))))\| > \frac{\varepsilon_N}{5}\right\} \\
&+ P\left\{\sup_{N\le t\le N+1}\|D(t,x(t-\delta(t)))\| > \frac{\varepsilon_N}{5}\right\} \\
&+ P\left\{\sup_{N\le t\le N+1}\left\|\int_N^t AS(t-s)D(s,x(s-\delta(s)))ds\right\| > \frac{\varepsilon_N}{5}\right\} \\
&+ P\left\{\sup_{N\le t\le N+1}\left\|\int_N^t S(t-s)f(s,x(s-r(s)))ds\right\| > \frac{\varepsilon_N}{5}\right\} \\
&+ P\left\{\sup_{N\le t\le N+1}\left\|\int_N^t S(t-s)g\big(s,x\big(s-\rho(s)\big)\big)dw(s)\right\| > \frac{\varepsilon_N}{5}\right\} \\
\le{}& \left(\frac{5}{\varepsilon_N}\right)^p E\left[\sup_{N\le t\le N+1}\|S(t-N)(x(N)-D(N,x(N-\delta(N))))\|^p\right] \\
&+ \left(\frac{5}{\varepsilon_N}\right)^p E\left[\sup_{N\le t\le N+1}\|D(t,x(t-\delta(t)))\|^p\right] \\
&+ \left(\frac{5}{\varepsilon_N}\right)^p E\left[\sup_{N\le t\le N+1}\left\|\int_N^t AS(t-s)D(s,x(s-\delta(s)))ds\right\|^p\right] \\
&+ \left(\frac{5}{\varepsilon_N}\right)^p E\left[\sup_{N\le t\le N+1}\left\|\int_N^t S(t-s)f(s,x(s-r(s)))ds\right\|^p\right] \\
&+ \left(\frac{5}{\varepsilon_N}\right)^p E\left[\sup_{N\le t\le N+1}\left\|\int_N^t S(t-s)g\big(s,x\big(s-\rho(s)\big)\big)dw(s)\right\|^p\right] \\
={}& \sum_{i=1}^{5} I_i,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
I_1 &\le \left(\frac{5}{\varepsilon_N}\right)^p M^p E\|x(N)-D(N,x(N-\delta(N)))\|^p \\
&\le \left(\frac{5}{\varepsilon_N}\right)^p M^p M_3\left(\frac{1}{(1-\kappa)^{p-1}} + \kappa e^{\mu\tau}\right)e^{-\mu N}, \\
I_2 &\le \left(\frac{5}{\varepsilon_N}\right)^p \kappa^p M_3 e^{\mu\tau} e^{-\mu N},
\end{aligned}$$

$$
\begin{aligned}
I_3 &\leq \left(\frac{5}{\varepsilon_N}\right)^p M_{1-\alpha}^p C_3^p (q\beta)^{p\alpha-(p/q)} (\Gamma(1-q(1-\alpha)))^{p/q} \int_N^{N+1} E\|x(s-\delta(s))\|^p ds \\
&\leq \left(\frac{5}{\varepsilon_N}\right)^p M_{1-\alpha}^p M_3 C_3^p (q\beta)^{p\alpha-(p/q)} (\Gamma(1-q(1-\alpha)))^{p/q} e^{\mu\tau} e^{-\mu N}, \\
I_4 &\leq \left(\frac{5}{\varepsilon_N}\right)^p M^p E\left[\sup_{N\leq t\leq N+1} \left(\int_N^t e^{-q\beta(t-s)} ds\right)^{p/q} \int_N^{N+1} \|f(s,x(s-r(s)))\|^p ds\right] \\
&\leq \left(\frac{5}{\varepsilon_N}\right)^p \frac{M^p M_3 C_1^p e^{\mu\tau}}{(q\beta)^{p/q}} e^{-\mu N}, \\
I_5 &\leq \left(\frac{5}{\varepsilon_N}\right)^p M^p c(p) \int_N^{N+1} E\|g(s,x(s-\rho(s)))\|_{L_2^0}^2 ds \\
&\leq \left(\frac{5}{\varepsilon_N}\right)^p M^p C_2^p c(p) e^{\mu\tau} e^{-\mu N}.
\end{aligned}
\tag{3.17}
$$

Thus, (3.16) implies

$$
P\left\{\sup_{N\leq t\leq N+1} \|x(t)\|_H > \varepsilon_N\right\} \leq K\left(\frac{5}{\varepsilon_N}\right)^p e^{-\mu N}, \tag{3.18}
$$

where

$$
\begin{aligned}
K = {} & M^p M_3 \left(\frac{1}{(1-\kappa)^{p-1}} + \kappa e^{\mu\tau}\right) + \kappa^p M_3 e^{\mu\tau} + M_{1-\alpha}^p M_3 C_3^p (q\beta)^{p\alpha-(p/q)} (\Gamma(1-q(1-\alpha)))^{p/q} e^{\mu\tau} \\
& + \frac{M^p M_3 C_1^p e^{\mu\tau}}{(q\beta)^{p/q}} + M^p C_2^p c(p) e^{\mu\tau}.
\end{aligned}
\tag{3.19}
$$

As ε_N is arbitrarily given real number, let $\varepsilon_N = e^{-(\mu N/2p)}$, such that

$$
P\left\{\sup_{N\leq t\leq N+1} \|x(t)\| > e^{-(\mu N/2p)}\right\} \leq 5^p K e^{-(\mu N/2)}. \tag{3.20}
$$

Consequently, from the Borel-Cantelli Lemma, there exists a $T(\omega) > 0$, for all $t > T(\omega)$, and we have

$$
\|x(t)\|^p \leq e^{-(\mu N/2)}, \quad \text{a.s.} \tag{3.21}
$$

That is,

$$\lim_{t\to+\infty} \frac{\log\|x(t)\|}{t} \leq -\frac{\mu}{2p}, \quad \text{a.s.} \tag{3.22}$$

The proof is completed. □

Corollary 3.4. *Suppose that the conditions (H_1)–(H_3) hold with $p = 2$, then the mild solution of system* (2.1) *is exponential stability in mean square and almost surely exponential stability, if the following inequality*

$$\frac{6M_{1-\alpha}^2 C_3^2 \beta^{1-2\alpha}\Gamma(2\alpha-1) + 3M^2\left(\beta^{-1}C_1^2 + C_2^2\right)}{(1-\kappa)^2} < \beta \tag{3.23}$$

holds, where $\alpha \in (1/2, 1]$.

Remark 3.5. When the neutral item $D(t,\cdot)$ is removed, system (2.1) is turned into the following stochastic partial differential equations with delays:

$$\begin{gathered} dx(t) = \left[Ax(t) + f(t, x(t-r(t)))\right]dt + g\left(t, x\left(t-\rho(t)\right)\right)dw(t), \quad t \in [0, +\infty), \\ x_0 = \varphi \in C_{\mathfrak{I}_0}^b. \end{gathered} \tag{3.24}$$

The mild solution of system (3.24) is the exponential stability in p $(p \geq 2)$-moment and almost surely exponential stability provided that

$$3^{p-1}M^p\left(\beta^{1-p}C_1^p + C_2^p\left(\frac{p(p-1)}{2}\right)^{p/2}\left(\frac{2\beta(p-1)}{p-2}\right)^{1-(p/2)}\right) < \beta \tag{3.25}$$

holds, which was studied by the fixed point theorem in [13]. As the neutral item $D(t,\cdot) \equiv 0$ and the delays $r(\cdot) \equiv 0$, $\rho(\cdot) \equiv 0$, system (2.1) is considered as the stochastic evolution equations:

$$\begin{gathered} dx(t) = \left[Ax(t) + f(t, x(t))\right]dt + g(t, x(t))dw(t), \quad t \in [0, +\infty), \\ x_0 \in X. \end{gathered} \tag{3.26}$$

The mild solution to system (3.26) is guaranteed to be the exponential stability in p $(p \geq 2)$-moment and almost surely exponential stability under the inequality (3.25) in [17]. Thus, we can generalize the results in [13, 17] which are regarded as two special cases in this paper.

Remark 3.6. Caraballo and Liu, in [2], have considered the exponential stability in p $(p \geq 2)$-moment and almost surely exponential stability for mild solution to system (3.24) by utilizing the Gronwall inequality. However, the monotone decreasing behaviors of the delays are imposed in [2], that is, $r'(t) \leq 0$, $\rho'(t) \leq 0$, for for all $t \geq 0$. In particular, when $r(t) \equiv \tau$

and $\rho(t) \equiv \tau$, the condition for the exponential stability in p $(p \geq 2)$-moment and almost surely exponential stability for mild solution to system (3.24) in [2] is

$$3^{p-1} M^p \left(C_1^p \beta^{1-p} + C_2^p \left(\frac{p(p-1)}{2} \right)^{p/2} \left(\frac{2\beta(p-1)}{p-2} \right)^{1-(p/2)} \right) e^{\beta\tau} < \beta, \quad p \geq 2. \tag{3.27}$$

In this sense, this paper can generalize and improve the results in [2].

4. Example

In this section, we provide an example to illustrate the obtained results above.

We consider the following neutral stochastic partial differential equations with delays:

$$\begin{gathered} d\left[x(t,\xi) + \frac{\alpha_3}{M_{1-\alpha}\left\| (-A)^\alpha \right\|} x(t-\delta(t),\xi) \right] = \left[\frac{\partial^2}{\partial x^2} z(t,x) + \alpha_1 x(t-r(t),\xi) \right] dt + \alpha_2 x\big(t-\rho(t),\xi\big) d\beta(t), \\ x(t,0) = x(t,\pi) = 0, \quad \alpha_i > 0, \quad i = 1,2,3, \quad 0 < \delta(t), \quad r(t), \quad \rho(t) < \tau, \\ x(s,\xi) = \varphi(s,\xi), \quad \varphi(\cdot,\xi) \in C, \quad \varphi(s,\cdot) \in L^2[0,\pi], \quad -\tau \leq s \leq 0, \quad 0 \leq \xi \leq \pi,\ \tau \geq 0,\ t \geq 0, \end{gathered} \tag{4.1}$$

where $\beta(t)$ is a standard one-dimensional Wiener process and $\|\varphi\|_C < +\infty$ a.s., and $M_{1-\alpha} \geq 1 (\alpha \in (1/2, 1])$. Take $X = L^2[0,\pi]$, $Y = R^1$. Define $A : X \to X$ by $-A = \partial^2/\partial\xi^2$ with domain $D(-A) = \{\omega \in X : \omega,\ \partial\omega/\partial\xi$ are absolutely continuous, $\partial^2/\partial\xi^2 \in X,\ \omega(0) = \omega(\pi) = 0\}$. Then

$$(-A)\omega = \sum_{n=1}^{+\infty} n^2 (\omega, \omega_n)\omega_n, \quad \omega \in D(-A), \tag{4.2}$$

where $\omega_n(\xi) = \sqrt{2/\pi} \sin n\xi, n = 1,2,3,\ldots$, is orthonormal set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t) (t \geq 0)$ in X and is given (see pazy [26, page 70]) by

$$S(t)\omega = \sum_{n=1}^{+\infty} \exp\left(-n^2 t\right)(\omega, \omega_n)\omega_n, \quad \omega \in X, \tag{4.3}$$

that satisfies $\|S(t)\| \leq \exp(-\pi^2 t),\ t \geq 0$, and hence is a contraction semigroup.

Define

$$\begin{gathered} D(t, x(t-\delta(t))) = \frac{\alpha_3}{M_{1-\alpha}\left\| (-A)^\alpha \right\|} x(t-\delta(t),\xi), \qquad f(t, x(t-r(t))) = \alpha_1 x(t-r(t),\xi), \\ g\big(t, x\big(t-\rho(t)\big)\big) = \alpha_2 x\big(t-\rho(t),\xi\big). \end{gathered} \tag{4.4}$$

It is easily seen that

$$\begin{aligned} &\|f(t,x(t-r(t))) - f(t,y(t-r(t)))\| \\ &\quad \le \alpha_1 \|x(t-r(t)) - y(t-r(t))\|, \quad f(t,0)=0, \\ &\|g(t,y(t-\rho(t))) - g(t,y(t-\rho(t)))\|_{L_2^0} \\ &\quad \le \alpha_2 \|x(t-\rho(t)) - y(t-\rho(t))\|, \quad g(t,0)=0, \\ &\|(-A)^{\alpha} D(t,x(t-\delta(t))) - (-A)^{\alpha} D(t,y(t-\delta(t)))\| \\ &\quad \le \frac{\alpha_3}{M_{1-\alpha}} \|x(t-\delta(t)) - y(t-\delta(t))\|, \quad (-A)^{\alpha} D(t,0)=0, \end{aligned} \tag{4.5}$$

from the definition of $(-A)^{-\alpha}$ by

$$\|(-A)^{-\alpha}\| \le \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \|S(t)\| dt \le \frac{1}{\pi^{2\alpha}}. \tag{4.6}$$

Thus, when $\alpha_3 < M_{1-\alpha}\pi^{2\alpha}(\alpha \in (1/2, 1])$, by virtue of Corollary 3.4, the mild solution of system (4.1) is exponential stability in mean square and almost sure exponential stability provided that the following inequality

$$6\alpha_3^2 \pi^{2-4\alpha}\Gamma(2\alpha-1) + 3\left(\pi^{-2}\alpha_1^2 + \alpha_2^2\right) < \left(\pi - \frac{\alpha_3}{M_{1-\alpha}\pi^{2\alpha-1}}\right)^2, \quad \alpha \in \left(\frac{1}{2}, 1\right] \tag{4.7}$$

holds.

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