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Review Article

Refinements, Generalizations, and Applications of Jordan's Inequality and Related Problems

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This is a survey and expository article. Some new developments on refinements, generalizations, and applications of Jordan's inequality and related problems, including some results about Wilker-Anglesio's inequality, some estimates for three kinds of complete elliptic integrals, and several inequalities for the remainder of power series expansion of the exponential function, are summarized.

1. Jordan's Inequality and Related Inequalities

1.1. Jordan's Inequality

The well-known Jordan's inequality (see [1, page 143], [2], [3, page 269], and [4, page 33]) reads that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1.1}$$

for $0 < |x| \le \pi/2$. The equality in (1.1) is valid if and only if $x = \pi/2$.

Remark 1.1. The inequality (1.1) is an immediate consequence of the concavity of the function $x \mapsto \sin x$ on the interval $[0, \pi/2]$. The straight line $y = (2/\pi)x$ is a chord of $y = \sin x$, which joints the points $(0,0)$ and $(\pi/2,1)$. The straight line $y = x$ is a tangent to $y = \sin x$ at the origin. Hence, the graph of $y = \sin x$ for $x \in [0,\pi/2]$ lies between these straight lines. See [4, page 33, Remark 1].

Remark 1.2. The very origin of Jordan's inequality (1.1) is not found in the references listed in this paper; therefore, it is unknown that why the inequality (1.1) is named after Jordan and to which Jordan, to the best of our knowledge. Although the Name Index on [4, page 391] hints us that the inequality (1.1) is due to C. Jordan (1838–1922), but no references related to C. Jordan were listed. Someone says that may be Jordan's inequality is coming from Jordan's lemma in Complex Analysis.

1.2. Kober's Inequality

The following inequality is due to Kober [5, page 22]:

$$1-\frac{2}{\pi}x \le \cos x \le 1-\frac{x^2}{\pi}, \quad x \in \left[0,\frac{\pi}{2}\right]. \tag{1.2}$$

See also [3, pages 274–275].

In [6] and [7, page 313], it was listed that

$$\cos x \le 1-\frac{2}{\pi^2}x^2, \quad x \in [0,\pi]. \tag{1.3}$$

Remark 1.3. The left-hand side inequalities in (1.1) and (1.2) are equivalent to each other, since they can be deduced from each other via the transformation $x \to \pi/2 - x$, as said in [8]. Applying this transformation to the right-hand side of inequality (1.2) acquires

$$\sin x \le 1-\frac{(\pi-2x)^2}{4\pi}, \quad x \in \left[0,\frac{\pi}{2}\right], \tag{1.4}$$

which cannot be compared with the right-hand side of (1.1) on $[0,\pi/2]$.

1.3. Redheffer-Williams's Inequality and Li-Li's Refinement

1.3.1. Redheffer-Williams's Inequality

In [9, 10], it was proposed that

$$\frac{\sin x}{x} \ge \frac{\pi^2-x^2}{\pi^2+x^2}, \quad x \ne 0. \tag{1.5}$$

In [11], the inequality (1.5) was verified as follows: for $x \geq 1$,

$$\begin{aligned} \frac{1-x^2}{1+x^2} - \frac{\sin(\pi x)}{\pi x} &= \frac{1-x^2}{1+x^2} + \frac{\sin[\pi(x-1)]}{\pi(x-1)} \cdot \frac{x-1}{x} \\ &\leq \frac{1-x^2}{1+x^2} + \frac{x-1}{x} \\ &= -\frac{(1-x)^2}{x(1+x^2)} \leq 0. \end{aligned} \tag{1.6}$$

For $0 < x < 1$, since

$$\frac{\sin(\pi x)}{\pi x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right), \tag{1.7}$$

it is enough to prove that $(1+x^2)P_n \geq 1$ for $n \geq 2$, where

$$P_n = \prod_{k=2}^{n}\left(1 - \frac{x^2}{k^2}\right). \tag{1.8}$$

Actually, by a simple induction argument based on the relation

$$P_{n+1} = \left[1 - \frac{x^2}{(n+1)^2}\right]P_n, \tag{1.9}$$

it is deduced that

$$\left(1+x^2\right)P_n \geq 1 + \frac{x^2}{n}, \quad 0 < x < 1. \tag{1.10}$$

The inequality (1.5) follows readily.

1.3.2. Li-Li's Refinement

In [12, Theorem 4.1], the inequality (1.5) was refined as

$$\frac{(1-x^2)(4-x^2)(9-x^2)}{x^6 - 2x^4 + 13x^2 + 36} \leq \frac{\sin(\pi x)}{\pi x} \leq \frac{1-x^2}{\sqrt{1+3x^4}}, \quad 0 < x < 1. \tag{1.11}$$

1.4. Mercer-Caccia's Inequality

In [13], it was proposed that

$$\sin\theta \geq \frac{2}{\pi}\theta + \frac{1}{12\pi}\theta\left(\pi^2 - 4\theta^2\right) \tag{1.12}$$

for $\theta \in [0, \pi/2]$. By finding the minimum of the function

$$\begin{cases} 1, & x = 0, \\ \dfrac{\sin x}{x} + \dfrac{x^2}{3\pi}, & x \in \left(0, \dfrac{\pi}{2}\right], \end{cases} \tag{1.13}$$

the inequality (1.12) was not only proved but also improved in [14] as

$$\sin\theta \geq \frac{2}{\pi}\theta + \frac{1}{\pi^3}\theta\left(\pi^2 - 4\theta^2\right) \tag{1.14}$$

for $\theta \in [0, \pi/2]$. The inequality (1.14) is sharp in the sense that $1/\pi^3$ cannot be replaced by a larger constant.

1.5. Prestin's Inequality

In [15], the following inequality was given: for $0 < |x| \leq \pi/2$,

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| \leq 1 - \frac{2}{\pi}. \tag{1.15}$$

See also [3, page 270].

Remark 1.4. For $0 < x \leq \pi/2$, the inequality (1.15) can be rewritten as

$$\frac{x}{\sin x} \leq 1 + \left(1 - \frac{2}{\pi}\right)x \quad \text{or} \quad \sin x \geq \frac{x}{1 + (1 - 2/\pi)x}. \tag{1.16}$$

This inequality and the inequality (1.14) are not included each other on $(0, \pi/2]$.

1.6. Some Inequalities Obtained from Taylor's Formula

In [16, pages 101-102], [7, page 313], and [3, page 269], the following inequalities are listed: for $x \in [0, \pi/2]$,

$$x - \frac{1}{6}x^3 \leq \sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5, \tag{1.17}$$

$$1 - \frac{1}{2}x^2 \leq \cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4, \tag{1.18}$$

$$(-1)^n\left[\sin x - \sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}\right] \leq \frac{x^{2n+1}}{(2n+1)!}, \tag{1.19}$$

$$(-1)^{n+1}\left[\cos x - \sum_{k=0}^{n}(-1)^{k}\frac{x^{2k}}{(2k)!}\right] \leq \frac{x^{2n+2}}{(2n+2)!}. \tag{1.20}$$

Remark 1.5. It is obvious that these inequalities are established based on Taylor's formula.

Remark 1.6. In [17, 18], the inequality (1.17) was applied to obtain the lower and upper estimations of $\zeta(3)$ in virtue of

$$\sum_{i=0}^{\infty} \frac{1}{(2i+1)^3} = \frac{1}{4}\int_0^{\pi/2} \frac{x(\pi - x)}{\sin x} \mathrm{d}x = \frac{7}{8}\zeta(3). \tag{1.21}$$

In [19, Theorem 1.7], as a by-product, the very closer lower and upper bounds for $\zeta(3)$ are deduced by a different approach from [17, 18].

1.7. Cusa-Huygens' and Related Inequalities

Nicolaus da Cusa (1401–1464) found by a geometrical method that

$$\frac{\sin x}{x} \le \frac{2 + \cos x}{3}, \tag{1.22}$$

for $0 < x \le \pi/2$. Christian Huygens (1629–1695) proved (1.22) explicitly when he approximated π. See [20, 21] and related references therein.

In [22], by using Techebysheff's integral inequality, it was constructed that

$$\frac{\sin x}{x} \ge \frac{1 + \cos x}{2}. \tag{1.23}$$

In [4, page 238, 3.4.15], the following double inequality

$$\frac{2(1 + a\cos x)}{\pi} \le \frac{\sin x}{x} \le \frac{1 + a\cos x}{a+1} \tag{1.24}$$

was given for $a \in (0, 1/2]$ and $x \in [0, \pi/2]$.

Recently, inequalities (1.22) and (1.23) were refined in [23].

This topic is related or similar to the so-called Carlson's, Oppenheim's, Shafer's, and Shafer-Fink's double inequalities for the arc sine, arc cosine, and arc tangent functions. For detailed information, please refer to [24–36] and closely related references therein.

1.8. Some Inequalities Related to Trigonometric Functions

In [37–39], the following inequalities were presented: for $0 < x < 1$,

$$\frac{2}{\pi} \cdot \frac{x}{1 - x^2} < \frac{1}{\pi x} - \cot(\pi x) < \frac{\pi}{3} \cdot \frac{x}{1 - x^2}, \tag{1.25}$$

$$\frac{\pi^2}{8} \cdot \frac{x}{1 - x^2} < \sec\frac{\pi x}{2} - 1 < \frac{4}{\pi} \cdot \frac{x}{1 - x^2}, \tag{1.26}$$

$$\frac{\pi}{6} \cdot \frac{x}{1 - x^2} < \csc(\pi x) - \frac{1}{\pi x} < \frac{2}{\pi} \cdot \frac{x}{1 - x^2}. \tag{1.27}$$

For $0 < |x| < 1$,

$$\begin{aligned} \ln\left(\frac{\pi x}{\sin(\pi x)}\right) &< \frac{\pi^2}{6} \cdot \frac{x^2}{1-x^2}, \\ \ln\left(\sec\frac{\pi x}{2}\right) &< \frac{\pi^2}{8} \cdot \frac{x^2}{1-x^2}, \\ \ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) &< \frac{\pi^2}{12} \cdot \frac{x^2}{1-x^2}. \end{aligned} \tag{1.28}$$

The constants $2/\pi$ and $\pi/3$ in (1.25), $\pi^2/8$ and $4/\pi$ in (1.26), $\pi/6$ and $2/\pi$ in (1.27) are the best possible. So are the constants $\pi^2/6$, $\pi^2/8$, and $\pi^2/12$ in (1.28).

For $x \in (0, \pi/2)$ and $n \in \mathbb{N}$, it was proved in [40, 41] that

$$\frac{2^{2(n+1)}\left[2^{2(n+1)}-1\right]B_{n+1}}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x, \tag{1.29}$$

where

$$S_n(x) = \sum_{i=1}^{n} \frac{2^{2i}\left(2^{2i}-1\right)B_i}{(2i)!}x^{2i-1} \tag{1.30}$$

and B_i for $i \in \mathbb{N}$ are the well-known Bernoulli's numbers defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j}\frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi \tag{1.31}$$

and the first several Bernoulli's numbers are

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = -\frac{1}{30}. \tag{1.32}$$

Let

$$\begin{aligned} p(\theta) &= \begin{cases} \left(\dfrac{\pi^2}{8} - \dfrac{1}{2}\theta\right)\sec^2\theta - \theta\tan\theta - \dfrac{1}{2}, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ 0, & \theta = \pm\dfrac{\pi}{2}, \end{cases} \\ q(\theta) &= \begin{cases} \dfrac{2}{\cos^2\theta}\displaystyle\int_{\theta}^{\pi/2} t\cos^2 t \,\mathrm{d}t, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ 0, & \theta = \pm\dfrac{\pi}{2}, \end{cases} \\ \phi(\theta) &= \begin{cases} \dfrac{\pi}{4}\left(\theta\sec^2\theta + \tan\theta\right) - 2\tan\theta\sec\theta, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ \pm 1, & \theta = \pm\dfrac{\pi}{2}. \end{cases} \end{aligned} \tag{1.33}$$

These functions originate from estimates of the eigenvalues of Laplace operator on compact Riemannian manifolds. Their monotonicity and estimates have been investigated by several mathematicians. For more detailed information, please refer to [42–44] and related references therein.

2. Refinements and Generalizations of Jordan's and Related Inequalities

2.1. Qi-Guo's Refinements of Kober's and Jordan's Inequality

2.1.1. Refinements of Kober's Inequality

In [45], by the help of two auxiliary functions

$$\cos x - 1 + \frac{2}{\pi}x - \alpha x\left(\pi^2 - x^2\right), \qquad \cos x - 1 + \frac{2}{\pi}x - \beta x(\pi - 2x) \tag{2.1}$$

with undetermined positive constants α and β for $x \in [0, \pi/2]$, Kober's inequality (1.2) was refined as

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{\pi^2}x(\pi - 2x) \leq \cos x \leq 1 - \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi - 2x), \tag{2.2}$$

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{2\pi^3}x\left(\pi^2 - 4x^2\right) \leq \cos x \leq 1 - \frac{2}{\pi}x + \frac{2}{\pi^3}x\left(\pi^2 - 4x^2\right). \tag{2.3}$$

These two double inequalities are sharp in the sense that the constants $(\pi - 2)/\pi^2$, $2/\pi^2$, $(\pi - 2)/2\pi^3$, and $2/\pi^3$ cannot be replaced by larger or smaller ones, respectively.

Remark 2.1. The inequality (2.2) is better than (2.3) and may be rewritten as

$$1 - \frac{4 - \pi}{\pi}x - \frac{2(\pi - 2)}{\pi^2}x^2 \leq \cos x \leq 1 - \frac{4}{\pi^2}x^2. \tag{2.4}$$

The double inequality (2.4) is stronger than (1.2) on $[0, \pi/2]$.

Remark 2.2. Replacing x by $\pi/2 - x$ in (2.4) gives

$$x - \frac{2(\pi - 2)}{\pi^2}x^2 \leq \sin x \leq \frac{4}{\pi}x - \frac{4}{\pi^2}x^2, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2.5}$$

The lower bound in (2.5) is better than the corresponding one in (1.16) and it is not included or includes the inequality (1.14).

2.1.2. Refinements of Jordan's Inequality

In [46], by considering auxiliary functions

$$
\begin{gathered}
\sin x-\frac{2}{\pi}x-\alpha x\left(\pi^2-4x^2\right),\\
\sin x-\frac{2}{\pi}x-\beta x^2(\pi-2x),\\
\sin x-\frac{2}{\pi}x-\theta x(\pi-2x)
\end{gathered}
\tag{2.6}
$$

on $[0,\pi/2]$, the inequality (1.14) was recovered, and the following inequalities were also obtained:

$$\sin x\le\frac{2}{\pi}x+\frac{\pi-2}{\pi^3}x\left(\pi^2-4x^2\right),\tag{2.7}$$

$$\sin x\ge\frac{2}{\pi}x+\frac{4}{\pi^3}x^2(\pi-2x),\tag{2.8}$$

$$\frac{2}{\pi}x+\frac{\pi-2}{\pi^2}x(\pi-2x)\le\sin x\le\frac{2}{\pi}x+\frac{2}{\pi^2}x(\pi-2x),\tag{2.9}$$

where the constants $(\pi-2)/\pi^3$, $4/\pi^3$, $(\pi-2)/\pi^2$, and $2/\pi^2$ are the best possible.

In [47], by the method used in [45, 46, 48], the following inequalities were deduced: for $x\in[0,\pi/2]$,

$$\sin x\ge\frac{2}{\pi}x+\frac{2}{\pi^4}x^2\left(\pi^2-4x^2\right),\tag{2.10}$$

$$\sin x\ge\frac{2}{\pi}x+\frac{8}{\pi^4}x^3(\pi-2x),\tag{2.11}$$

$$\frac{2}{3\pi^4}x\left(\pi^3-8x^3\right)\le\sin x-\frac{2}{\pi}x\le\frac{\pi-2}{\pi^4}x\left(\pi^3-8x^3\right).\tag{2.12}$$

Remark 2.3. The inequality (2.9) may be rewritten as (2.5). Therefore, inequalities (2.4) and (2.9) are equivalent to each other.

Remark 2.4. Combination of (1.14) and (2.7) leads to

$$\frac{3}{\pi}x-\frac{4}{\pi^3}x^3\le\sin x\le x-\frac{4(\pi-2)}{\pi^3}x^3,\quad x\in\left[0,\frac{\pi}{2}\right].\tag{2.13}$$

Inequalities (2.5) and (2.13) are not included each other on $[0,\pi/2]$. Inequality (2.8) is weaker than the left-hand side inequality in (2.13) and cannot compare with the left-hand side inequality of (2.5).

Remark 2.5. In [49], by constructing suitable auxiliary functions as above, inequality (2.7) or the right-hand side inequality in (2.13), the double inequality (2.9) or (2.5), the inequality

(2.8), the double inequality (2.2) or (2.4), the double inequality (2.3), and their sharpness are verified again. Employing these inequalities, it was derived in [49] that

$$\frac{4}{3} < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi+1}{3}, \qquad \frac{1}{2} < \int_0^{\pi/2} \frac{1-\cos x}{x} dx < \frac{6-\pi}{4}. \tag{2.14}$$

Remark 2.6. In [50], inequalities (1.14) and (2.7) or their variant (2.13) and the inequality (2.2) or (2.4) were proved once again by considering suitable auxiliary functions as above. From (2.13) and the symmetry and period of $\sin x$, it was deduced in [50] that

$$\frac{4}{\pi^3}x^3 - \frac{12}{\pi^2}x^2 + \frac{9}{\pi}x - 1 \le \sin x \le \frac{4(\pi-2)}{\pi^3}x^3 - \frac{12(\pi-2)}{\pi^2}x^2 + \frac{11\pi-24}{\pi}x + 8 - 3\pi \tag{2.15}$$

on $[\pi/2, \pi]$ and

$$\frac{7}{6} - \ln 2 < \int_{\pi/2}^{\pi} \frac{\sin x}{x} dx < \frac{13\pi-32}{6} + (8-3\pi)\ln 2. \tag{2.16}$$

Remark 2.7. The method used in [45, 46, 49–53] was reused to construct inequalities involving the sine and cosine functions in [54] and obtained the following one-sided inequalities:

$$\begin{aligned} &\sin x \ge \frac{2}{\pi}x + \frac{\pi-2}{\pi}x\cos x, \quad x \in \left[0, \frac{\pi}{2}\right]; \\ &\sin x \ge \frac{4-\pi}{\left(\sqrt{2}-1\right)\pi}x + \frac{4\left(\pi-2\sqrt{2}\right)}{\pi\left(2-2\sqrt{2}\right)}x\cos x, \quad x \in \left[0, \frac{\pi}{4}\right]. \end{aligned} \tag{2.17}$$

The first inequality above refines the left-hand side inequality in (2.9).

Remark 2.8. In [22], among other things, a lot of inequalities and integrals related to $\sin x/x$ and similar to inequalities in (2.14) and (2.16) are constructed by using the famous Tchebysheff's integral inequality [4, page 39, Theorem 8]. For examples,

$$\begin{aligned} &\left(\frac{\sin t}{t}\right)^2 + 2\left(\frac{\sin t}{t}\right) \ge 4\left(\frac{1-\cos t}{t^2}\right) + \cos t, \quad t \in [0, \pi], \\ &\int_0^t \left(\frac{x}{\sin x}\right)^2 dx < 2\tan\left(\frac{t}{2}\right) + \frac{2}{3}\tan^3\left(\frac{t}{2}\right), \quad t \in \left(0, \frac{\pi}{2}\right]. \end{aligned} \tag{2.18}$$

2.2. Refinements of Jordan's Inequality by L'Hôspital's Rule

2.2.1. L'Hôspital's Rule

The following monotonic form of the famous L'Hôspital's rule was put forward in [55, Theorem 1.25].

Lemma 2.9. *Let f and g be continuous on $[a,b]$ and differentiable on (a,b) such that $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (or decreasing) on (a,b), then so are the functions $(f(x)-f(b))/(g(x)-g(b))$ and $(f(x)-f(a))/(g(x)-g(a))$ on (a,b).*

2.2.2. Zhang-Wang-Chu's Recoveries

In [8], by using Lemma 2.9, inequalities (1.14), (2.2), (2.3), (2.7), and (2.9) were recovered once more.

2.3. Li's Power Series Expansion and Refinements of Jordan's Inequality

In [56], a power series expansion was established as follows: for $x>0$,

$$\frac{\sin x}{x}=\frac{2}{\pi}+\sum_{k=1}^{\infty}(-1)^k\frac{R_k(\pi/2)}{k!\pi^{2k}}\left(\pi^2-4x^2\right)^k, \tag{2.19}$$

where

$$R_k(x)=\sum_{n=k}^{\infty}\frac{(-1)^n n!}{(2n+1)!(n-k)!}x^{2n} \tag{2.20}$$

satisfy $(-1)^k R_k(\pi/2)>0$ and

$$R_1(x)=\frac{x}{2}\left(\frac{\sin x}{x}\right)', \qquad R_{k+1}(x)=-kR_k(x)+\frac{x}{2}R_k'(x) \tag{2.21}$$

for $k\in\mathbb{N}$.

As a direct consequence of the above identity, the following lower bound for the function $\sin x/x$ was established in [56]:

$$\begin{aligned}\frac{\sin x}{x}\geq{}&\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{12-\pi^2}{16\pi^5}\left(\pi^2-4x^2\right)^2+\frac{10-\pi^2}{16\pi^7}\left(\pi^2-4x^2\right)^3\\&+\frac{\pi^4-180\pi^2+1680}{3072\pi^9}\left(\pi^2-4x^2\right)^4, \quad 0<x<\frac{\pi}{2}.\end{aligned} \tag{2.22}$$

Equality in (2.22) is valid if and only if $x=\pi/2$. The constants $1/\pi^3$, $(12-\pi^2)/16\pi^5$, $(10-\pi^2)/16\pi^7$, and $(\pi^4-180\pi^2+1680)/3072\pi^9$ are the best possible.

Moreover, by employing

$$\frac{x}{\sin x}=\sum_{k=0}^{\infty}(-1)^{k+1}B_{2k}\frac{2^{2k}-2}{(2k)!}x^{2k} \tag{2.23}$$

for $|x| < \pi$, where B_{2k} for $0 \le k < \infty$ is the well-known Bernoulli's numbers, it was presented in [56] that

$$\frac{x}{\sin x} \le 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6, \quad |x| < \pi. \tag{2.24}$$

2.4. Li-Li's Refinements and Generalizations

In [12], two seemingly general but not much significant results for refining or generalizing Jordan's inequality (1.1) were discovered.

The first result may be stated as follows: if the function $g : [0, \pi/2] \to [0, 1]$ is continuous and

$$\frac{\sin x}{x} \ge g(x) \tag{2.25}$$

for $x \in [0, \pi/2]$, then the double inequality

$$\frac{2}{\pi} - h\left(\frac{\pi}{2}\right) + h(x) \le \frac{\sin x}{x} \le 1 + h(x) \tag{2.26}$$

for $x \in [0, \pi/2]$ holds with equality if and only if $x = \pi/2$, where

$$h(x) = -\int_0^x \frac{1}{u^2} \int_0^u v^2 g(v) dv\, du, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2.27}$$

Since $g(x)$ is positive, it is clear that the function $h(x)$ is decreasing and negative, and therefore, the double inequality (2.26) refines Jordan's inequality (1.1).

Remark 2.10. It is remarked that the upper bound in (2.26) was not considered in [12], although it is implied in the arguments. On the other hand, if inequality (2.25) is reversed, then so is inequality (2.26).

Remark 2.11. Upon taking $g(x) = 0$ in (2.25) and (2.27), Jordan's inequality (1.1) is derived from (2.26). If letting $g(x) = 2/\pi$, then inequalities (1.12) and

$$\frac{\sin x}{x} \le 1 - \frac{1}{3\pi}x^2, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{2.28}$$

are deduced from (2.26). If choosing $g(x)$ as the function in the right-hand side of (1.12), then the inequality

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{60 + \pi^2}{720\pi}\left(\pi^2 - 4x^2\right) + \frac{1}{960\pi}\left(\pi^2 - 4x^2\right)^2, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{2.29}$$

follows from the left-hand side of (2.26). These three examples given in [12] seemly show that, by using some lower bound for $\sin x/x$ on $(0, \pi/2]$, a corresponding stronger lower bound may be derived from the left-hand side inequality in (2.26). Actually, this is not always valid. By taking $g(x)$ as the function in the right-hand side of (1.14) or the one in the left-hand side of (2.13), it was obtained that

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{60\pi}\left(\pi^2 - 4x^2\right) + \frac{1}{80\pi^3}\left(\pi^2 - 4x^2\right)^2, \quad x \in \left(0, \frac{\pi}{2}\right]. \tag{2.30}$$

Unluckily, inequality (2.30) is worse than both inequality (1.14) and the left-hand side inequality in (2.13). This tells us that the inequality

$$\frac{2}{\pi} - h\left(\frac{\pi}{2}\right) + h(x) > g(x), \quad x \in \left(0, \frac{\pi}{2}\right] \tag{2.31}$$

is not always sound. Therefore, Theorem 2.1 in [12], one of the main results in [12], is not always significant and meaningful. This reminds us of proposing a question: under what conditions on $0 < g(x) < 1$ for $x \in (0, \pi/2]$ the inequality (2.31) holds?

The second result in [12] is procured basing on Lemma 2.9. It can be summarized as follows: if the function $f(x) \in C^2[0, \pi/2]$ satisfies $f'(x) > 0$ and $[x^2 f'(x)]' \neq 0$ for $x \in [0, \pi/2]$, then the double inequality

$$\begin{aligned} &\lim_{x \to 0^+} \frac{\sin x/x - 2/\pi}{f(x) - f(\pi/2)}\left[f(x) - f\left(\frac{\pi}{2}\right)\right] \\ &\quad \leq \frac{\sin x}{x} - \frac{2}{\pi} \leq \lim_{x \to (\pi/2)^-} \frac{\sin x/x - 2/\pi}{f(x) - f(\pi/2)}\left[f(x) - f\left(\frac{\pi}{2}\right)\right], \quad x \in \left(0, \frac{\pi}{2}\right] \end{aligned} \tag{2.32}$$

is sharp in the sense that the limits before brackets in (2.32) cannot be replaced by larger or smaller numbers. If $f'(x) < 0$ and $[x^2 f'(x)]' \neq 0$, then the inequality (2.32) is reversed.

As an application, by taking $f(x) = x^n$ for $n \in \mathbb{N}$ in (2.32), the inequality (2.9) and

$$\frac{2}{\pi} + \frac{2}{n\pi^{n+1}}\left[\pi^n - (2x)^n\right] \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^{n+1}}\left[\pi^n - (2x)^n\right], \quad n \geq 2 \tag{2.33}$$

were showed in [12, Theorem 3.2], where the equalities hold if and only if $x = \pi/2$ and the constants $2/n\pi^{n+1}$ and $(\pi - 2)/\pi^{n+1}$ in (2.33) are the best possible.

If taking $n = 2$ in (2.33), then the inequalities (1.14) and (2.7) are recovered. The inequality (2.33) for $n = 3$ and $n = 4$, respectively, implies (2.12) and (3.10).

Remark 2.12. What essentially established in [12, Section 3] are sufficient conditions for the function $(\sin x/x - 2/\pi)/(f(x) - f(\pi/2))$ to be monotonic on $[0, \pi/2]$. Generally, more new sufficient conditions may be further found.

2.5. Some Generalizations of Redheffer-Williams's Inequality

2.5.1. Chen-Zhao-Qi's Results

In [57, 58], the following three inequalities similar to (1.5) were established: if $|x| \leq 1/2$, then

$$\cos(\pi x) \geq \frac{1-4x^2}{1+4x^2}, \qquad \cosh(\pi x) \leq \frac{1+4x^2}{1-4x^2}. \tag{2.34}$$

If $0 < |x| < 1$, then

$$\frac{\sinh(\pi x)}{\pi x} \leq \frac{1+x^2}{1-x^2}. \tag{2.35}$$

2.5.2. Zhu-Sun's Results

In [59], by using Lemma 2.9 and other techniques, the above three inequalities are sharpened, and some new results were demonstrated as follows.

(1) The double inequality

$$\left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\beta} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\alpha} \tag{2.36}$$

holds for $0 < x < \pi$ if and only if $\alpha \leq \pi^2/12$ and $\beta \geq 1$.

(2) The double inequality

$$\left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\beta} \leq \cos x \leq \left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\alpha} \tag{2.37}$$

holds for $0 \leq x \leq \pi/2$ if and only if $\alpha \leq \pi^2/16$ and $\beta \geq 1$.

(3) The double inequality

$$\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\alpha} \leq \frac{\tan x}{x} \leq \left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\beta} \tag{2.38}$$

holds for $0 < x < \pi/2$ if and only if $\alpha \leq \pi^2/24$ and $\beta \geq 1$.

(4) The double inequality

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \frac{\sinh x}{x} \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta} \tag{2.39}$$

holds for $0 < x < r$ if and only if $\alpha \leq 0$ and $\beta \geq r^2/12$.

(5) The double inequality

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \cosh x \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta} \tag{2.40}$$

holds for $0 \leq x < r$ if and only if $\alpha \leq 0$ and $\beta \geq r^2/4$.

(6) The double inequality

$$\left(\frac{r^2-x^2}{r^2+x^2}\right)^{\beta} \leq \frac{\tanh x}{x} \leq \left(\frac{r^2-x^2}{r^2+x^2}\right)^{\alpha} \tag{2.41}$$

holds for $0 < x < r$ if and only if $\alpha \leq 0$ and $\beta \geq r^2/6$.

2.5.3. Zhu's Sharp Inequalities

In [60], using Lemma 2.9 and other techniques, some sharp inequalities for the sine, cosine, and tangent functions are presented as follows.

(1) For $x \in (0, \pi]$, the double inequality

$$\left(\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}\right)^{\alpha} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}\right)^{\beta} \tag{2.42}$$

holds if and only if $\alpha \geq \pi^2/6$ and $\beta \leq 1$.

(2) The following inequalities are valid:

$$\begin{aligned}
&\left(\frac{\pi^2-4x^2}{\sqrt{\pi^4+48x^4}}\right)^{\pi^2/6} \leq \cos x \leq \left(\frac{\pi^2-4x^2}{\sqrt{\pi^4+48x^4}}\right)^{3/4}, \quad 0 < x \leq \frac{\pi}{2}; \\
&\left(\frac{\sqrt{\pi^4+48x^4}}{\pi^2-4x^2}\right)^{1/2} \leq \frac{\tan x}{x} \leq \left(\frac{\sqrt{\pi^4+48x^4}}{\pi^2-4x^2}\right)^{\pi^2/6}, \quad 0 < x < \frac{\pi}{2}.
\end{aligned} \tag{2.43}$$

(3) For $0 < x \leq \pi/2$, the double inequality

$$\left[\frac{\sin(2x)}{2x}\right]^{\alpha} \leq \cos x \leq \left[\frac{\sin(2x)}{2x}\right]^{\beta} \tag{2.44}$$

holds if and only if $\alpha \geq 1$ and $\beta \leq 3/4$.

(4) For $0 < x < \pi/2$, the double inequality

$$\left[\frac{2x}{\sin(2x)}\right]^{\alpha} \leq \frac{\tan x}{x} \leq \left[\frac{2x}{\sin(2x)}\right]^{\beta} \tag{2.45}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq 1$.

2.5.4. Baricz-Wu's Generalization

For detailed information, please refer to [61].

2.6. Some Generalizations and Related Results

In [62], it was obtained that

$$\frac{2}{\pi} \le \left|\frac{\sin(\lambda y)}{\lambda x \sin(\pi y/2x)}\right| \le \left|\frac{\sin(\lambda x)}{\lambda x}\right| \le \left|\frac{\sin(\lambda y)}{\lambda y}\right| < 1 \tag{2.46}$$

for $0 < |y| < |x|$ and $0 < |\lambda x| \le \pi/2$.

In [63, 64], by considering the logarithmic concavity of $\sin x/x$ and the logarithmic convexity of $\tan x/x$ and by using Jensen's inequality for convex functions, it was obtained that

$$\begin{aligned} \left|\prod_{i=1}^{n} \tan x_i\right| &\ge \left|\prod_{i=1}^{n} x_i \left[\frac{\tan(\sum_{i=1}^{n}|x_i|/n)}{\sum_{i=1}^{n}|x_i|/n}\right]^n\right| \\ &> \left|\prod_{i=1}^{n} x_i\right| \\ &> \left|\prod_{i=1}^{n} x_i \left[\frac{\sin(\sum_{i=1}^{n}|x_i|/n)}{\sum_{i=1}^{n}|x_i|/n}\right]^n\right| \\ &\ge \left|\prod_{i=1}^{n} \sin x_i\right| \end{aligned} \tag{2.47}$$

for $0 < |x_i| < \pi/2$, $1 \le i \le n$ and $n \in \mathbb{N}$ and that

$$\frac{|\tan(\alpha x)|}{\alpha|x|} > \frac{|\tan(\beta x)|}{\beta|x|} > 1 > \frac{|\sin(\beta x)|}{\beta|x|} > \frac{|\sin(\alpha x)|}{\alpha|x|} > \frac{|\sin(\beta x)|}{\alpha|x|} \csc\frac{\beta\pi}{2\alpha} \tag{2.48}$$

for $0 < \beta < \alpha$ and $0 < |\alpha x| < \pi/2$.

In [65], it was proved that a positive and concave function is logarithmically concave and that the function $\sin x/x$ for $0 < x < \pi/2$ is a concave function. As a corollary, the following inequality was obtained:

$$\frac{\sin x}{x} \ge 1 + \frac{2(2-\pi)}{\pi^2}x \ge \frac{2}{\pi}, \quad 0 < x \le \frac{\pi}{2}. \tag{2.49}$$

This inequality is better than (1.16) and it is not included or includes (1.14).

Remark 2.13. In passing it is pointed out that the above relationship between concave functions and logarithmically concave functions was also verified much simply in [66, page 85].

Recently, the following double inequalities and others were established in [67, 68]:

$$\frac{x^2}{\sinh^2 x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \quad x \in \left(0, \frac{\pi}{2}\right);$$
$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}, \quad x \in (0,1). \tag{2.50}$$

Some results obtained in [69–71] and the related references therein may be also interesting.

In [6, 7], [3, pages 269–288], and [4, pages 235–265], a large amount of inequalities involving trigonometric functions are collected.

3. Refinements of Jordan's Inequality and Yang's Inequality

3.1. Yang's Inequality

In [72, pages 116–118], an inequality states that

$$\cos^2(\lambda A) + \cos^2(\lambda B) - 2\cos(\lambda A)\cos(\lambda B)\cos(\lambda\pi) \geq \sin^2(\lambda\pi) \tag{3.1}$$

is valid for $0 \leq \lambda \leq 1$, $A > 0$ and $B > 0$ with $A + B \leq \pi$, where the equality holds if and only if $\lambda = 0$ or $A + B = \pi$.

Remark 3.1. The inequality (3.1) has been generalized in [73, 74] and related references therein.

3.2. Zhao's Result

In [75, Theorems 1 and 2], by using inequalities (1.1) and (1.5), respectively, it was concluded that

$$4\binom{n}{2}\lambda^2\cos^2\left(\frac{\pi}{2}\lambda\right) \leq \sum_{1\leq i<j\leq n} H_{ij} \leq \binom{n}{2}\pi^2\lambda^2,$$
$$\binom{n}{2}\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^2 \leq \sum_{1\leq i<j\leq n} H_{ij} \leq \binom{n}{2}\pi^2\lambda^2, \tag{3.2}$$

where

$$H_{ij} = \cos^2(\lambda A_i) + \cos^2(\lambda A_j) - 2\cos(\lambda A_i)\cos(\lambda A_j)\cos(\lambda\pi) \tag{3.3}$$

for $0 \leq \lambda \leq 1$ and $A_i > 0$ with $\sum_{i=1}^{n} A_i \leq \pi$ for $n \geq 2$.

This generalizes Yang's inequality (3.1).

3.3. Debnath-Zhao's Result

In [76], inequalities (1.12) and (1.14) or the left-hand side inequality in (2.13) were recovered once again. However, it seems that the authors of the paper [76] did not compare (1.12) and (1.14) explicitly.

As an application of (1.14), with the help of

$$\sin^2(\lambda\pi) \le H_{ij} \le 4\sin^2\left(\frac{\lambda}{2}\pi\right) \tag{3.4}$$

in [73] and [74, (2.13)], Yang's inequality (3.1) was generalized in [76] to

$$\binom{n}{2}\lambda^2\left(3-\lambda^2\right)^2\cos^2\left(\frac{\lambda}{2}\pi\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\lambda^2\pi^2. \tag{3.5}$$

3.4. Özban's Result

In [77], a new refined form of Jordan's inequality was given for $0 < x \le \pi/2$ as follows:

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \frac{\pi-3}{\pi^3}(2x-\pi)^2 \tag{3.6}$$

with equality if and only if $x = \pi/2$. As an application of (3.6) as in [76], the lower bound in (3.5) was refined as

$$\sum_{1\le i<j\le n} H_{ij} \ge \binom{n}{2}\lambda^2\left[\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right). \tag{3.7}$$

3.5. Jiang-Hua's Results

Motivated by the papers [47, 49], it was procured in [78] that

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{8x}{\pi^3}\left(\frac{\pi}{2} - x\right) + \frac{4(\pi-2)}{\pi^3}\left(\frac{\pi}{2} - x\right)^2 \tag{3.8}$$

for $x \in (0, \pi/2]$. Equality in (3.8) holds if and only if $x = \pi/2$.

As an application of (3.8), Yang's inequality (3.1) is generalized and refined as

$$4\binom{n}{2}\lambda^2\left[\frac{\pi-2}{2}(\lambda-1)^2 + \lambda(1-\lambda) + 1\right]^2\cos^2\left(\frac{\pi\lambda}{2}\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\pi^2\lambda^2. \tag{3.9}$$

In [79], by Lemma 2.9, the inequality

$$\frac{1}{2\pi^5}\left(\pi^4 - 16x^4\right) \le \frac{\sin x}{x} - \frac{2}{\pi} \le \frac{\pi-2}{\pi^5}\left(\pi^4 - 16x^4\right) \tag{3.10}$$

for $0 < x \le \pi/2$, a refinement of Jordan's inequality (1.1), was presented. Meanwhile, Yang's inequality was refined as

$$\binom{n}{2}\frac{\lambda^2(5-\lambda^4)^2}{4}\cos^2\left(\frac{\lambda}{2}\pi\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\lambda^2\left[\pi + (2-\pi)\lambda^4\right]^2. \tag{3.11}$$

3.6. Agarwal-Kim-Sen's Result

In [80], inequalities (3.6) and (3.19) were refined as follows: for $0 < x \le \pi/2$, the double inequality

$$1 + B_1x - B_2x^2 + B_3x^3 \le \frac{\sin x}{x} \le 1 + C_1x - C_2x^2 + B_3x^3 \tag{3.12}$$

holds with equalities if and only if $x = \pi/2$ and

$$\begin{gathered} B_1 = \frac{4}{\pi^2}\left(66 - 43\pi + 7\pi^2\right), \qquad B_2 = \frac{4}{\pi^3}\left(124 - 83\pi + 14\pi^2\right), \qquad B_3 = \frac{16}{\pi^4}(\pi - 3), \\ C_1 = \frac{4}{\pi^2}\left(75 - 49\pi + 8\pi^2\right), \qquad C_2 = \frac{4}{\pi^3}\left(142 - 95\pi + 16\pi^2\right). \end{gathered} \tag{3.13}$$

By using (3.12), Yang's inequality was refined in [80, Theorem 3.1] as

$$U(\lambda) \le \sum_{1\le i<j\le n} \sin^2(\lambda\pi) \le \sum_{1\le i<j\le n} H_{ij} \le \sum_{1\le i<j\le n} \lambda^2\pi^2, \tag{3.14}$$

where

$$U(\lambda) = \frac{n(n-1)}{2}\lambda^2[B(\lambda;\pi)]^2\cos^2\left(\frac{\lambda}{2}\pi\right) \tag{3.15}$$

with

$$B(\lambda;\pi) = \pi + 2\left(66 - 43\pi + 7\pi^2\right)\lambda - \left(124 - 83\pi + 14\pi^2\right)\lambda^2 + 2(\pi - 3)\lambda^3. \tag{3.16}$$

3.7. Zhu's Results

In [81], inequalities (1.14) and (2.7), equivalently, the double inequality (2.13), and their sharpness were recovered once more by using Lemma 2.9.

As an application of (2.7), the upper bound in (3.5) was refined as

$$\sum_{1\le i<j\le n} H_{ij} \le 4\binom{n}{2}\left[\lambda^3 + \frac{\lambda(1-\lambda^2)\pi}{2}\right]^2. \tag{3.17}$$

In [82], by using Lemma 2.9, the inequality (3.6) and the following two refined forms of Jordan's inequality were established:

$$\frac{12-\pi^2}{16\pi^5}\left(\pi^2-4x^2\right)^2 \le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{1}{\pi^3}\left(\pi^2-4x^2\right) \le \frac{\pi-3}{\pi^5}\left(\pi^2-4x^2\right)^2, \tag{3.18}$$

$$\frac{\sin x}{x} \le \frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{12-\pi^2}{\pi^3}\left(x-\frac{\pi}{2}\right)^2. \tag{3.19}$$

The inequality (3.18) and the right-hand side inequality in (3.19) were also applied to obtain

$$N_3(\lambda) \le \sum_{1\le i<j\le n} H_{ij} \le \min\{M_3(\lambda), M_3'(\lambda)\}, \tag{3.20}$$

where

$$\begin{aligned}
N_3(\lambda) &= \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{16}\left(1-\lambda^2\right)^2\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right),\\
M_3(\lambda) &= \binom{n}{2}\lambda^2\left[3-\lambda^2+(\pi-3)\left(1-\lambda^2\right)^2\right]^2,\\
M_3'(\lambda) &= \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{4}(1-\lambda)^2\right]^2.
\end{aligned} \tag{3.21}$$

In [83], a general refinement of Jordan's inequality (1.1) was presented by a different approach from that used in [84, 85] as follows: for $0<x\le\pi/2$ and any nonnegative integer $n\ge 0$, the inequality

$$a_{n+1}\left(\pi^2-4x^2\right)^{n+1} \le \frac{\sin x}{x}-P_{2n}(x) \le \frac{1-\sum_{k=0}^{n}a_k\pi^{2k}}{\pi^{2(n+1)}}\left(\pi^2-4x^2\right)^{n+1} \tag{3.22}$$

is valid with the equalities if and only if $x=\pi/2$, where

$$P_{2n}(x)=\sum_{k=0}^{n}a_k\left(\pi^2-4x^2\right)^k, \tag{3.23}$$

and a_k satisfies the recurrent formula

$$a_0=\frac{2}{\pi}, \qquad a_1=\frac{1}{\pi^3}, \qquad a_{k+1}=\frac{2k+1}{2(k+1)\pi^2}a_k-\frac{1}{16k(k+1)\pi^2}a_{k-1} \tag{3.24}$$

for $k\in\mathbb{N}$. Furthermore, the constants a_{n+1} and $(1-\sum_{k=0}^{n}a_k\pi^{2k})/\pi^{2(n+1)}$ in (3.22) are the best possible.

Moreover, the following series expansion for $\sin x/x$ was also deduced in [83]: for $0 < x \le \pi/2$ and $n \ge 0$, we have

$$\frac{\sin x}{x} = P_{2n}(x) + Q_{2n+2}, \tag{3.25}$$

where the reminder term is

$$Q_{2n+2} = \frac{1}{2^{3(n+1)}(n+1)!(2n+3)!!} \cdot \frac{\sin \eta}{\eta}\left(\pi^2 - 4x^2\right)^{n+1}, \quad 0 < \eta < \frac{\pi}{2}. \tag{3.26}$$

If taking $n \to \infty$ in (3.25), since $\lim_{n\to\infty} Q_{2n+2} = 0$, then

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} a_k \left(\pi^2 - 4x^2\right)^k, \quad 0 < |x| \le \frac{\pi}{2}, \tag{3.27}$$

which implies $\sum_{k=0}^{\infty} a_k \pi^{2k} = 1$.

As an application of (3.22), a general improvement of Yang's inequality (3.1) was deduced in [83] as

$$\begin{aligned} &\binom{n}{2}(\lambda\pi)^2\left[P_{2n}\left(\frac{\lambda}{2}\pi\right) + a_{n+1}\pi^{2(n+1)}\left(1-\lambda^2\right)^{n+1}\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right) \\ &\quad \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}(\lambda\pi)^2\left[P_{2n}\left(\frac{\lambda}{2}\pi\right) + \left(1 - \sum_{k=0}^{n} a_k \pi^{2k}\right)\left(1-\lambda^2\right)^{n+1}\right]^2. \end{aligned} \tag{3.28}$$

3.8. Niu-Huo-Cao-Qi's Result

In [84, 85], the following general refinement of Jordan's inequality was presented: for $0 < x \le \pi/2$ and $n \in \mathbb{N}$, the inequality

$$\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \frac{\sin x}{x} \le \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k \left(\pi^2 - 4x^2\right)^k \tag{3.29}$$

holds with the equalities if and only if $x = \pi/2$, where the constants

$$\alpha_k = \frac{(-1)^k}{(4\pi)^k k!} \sum_{i=1}^{k+1} \left(\frac{2}{\pi}\right)^i a_{i-1}^k \sin\left(\frac{k+i}{2}\pi\right), \tag{3.30}$$

$$\beta_k = \begin{cases} \dfrac{1 - 2/\pi - \sum_{i=1}^{n-1} \alpha_i \pi^{2i}}{\pi^{2n}}, & k = n, \\ \alpha_k, & 1 \le k < n \end{cases} \tag{3.31}$$

with

$$a_i^k = \begin{cases} (i+k-1)a_{i-1}^{k-1} + a_i^{k-1}, & 0 < i \le k, \\ 1, & i = 0, \\ 0, & i > k \end{cases} \tag{3.32}$$

in (3.29) are the best possible.

As an application of inequality (3.29), a refinement and generalization of Yang's inequality (3.1) is obtained: for $0 \le \lambda \le 1$ and $A_i > 0$ such that $\sum_{i=1}^{n} A_i \le \pi$, if $m \in \mathbb{N}$ and $n \ge 2$, then

$$L_m(n,\lambda) \le \sum_{1\le i<j\le n} H_{ij} \le R_m(n,\lambda), \tag{3.33}$$

where

$$\begin{aligned} L_m(n,\lambda) &= \binom{n}{2}\lambda^2\left[2 + \sum_{k=1}^{m}\alpha_k\pi^{2k+1}\left(1-\lambda^2\right)^k\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \\ R_m(n,\lambda) &= \binom{n}{2}\lambda^2\left[2 + \sum_{k=1}^{m}\beta_k\pi^{2k+1}\left(1-\lambda^2\right)^k\right]^2. \end{aligned} \tag{3.34}$$

4. Generalizations of Jordan's Inequality and Applications

4.1. Qi-Niu-Cao's Generalization and Application

In [84, 86], a general generalization of Jordan's inequality was established: for $0 < x \le \theta < \pi$, $n \in \mathbb{N}$ and $t \ge 2$, the inequality

$$\sum_{k=1}^{n}\mu_k\left(\theta^t - x^t\right)^k \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} \le \sum_{k=1}^{n}\omega_k\left(\theta^t - x^t\right)^k \tag{4.1}$$

holds with the equalities if and only if $x = \theta$, where the constants

$$\mu_k = \frac{(-1)^k}{k!t^k}\sum_{i=1}^{k+1} a_{i-1}^k \theta^{k-i-kt}\sin\left(\theta + \frac{k+i-1}{2}\pi\right), \tag{4.2}$$

$$\omega_k = \begin{cases} \dfrac{1-(\sin\theta)/\theta - \sum_{i=1}^{n-1}\mu_i\theta^{ti}}{\theta^{tn}}, & k = n, \\ \mu_k, & 1 \le k < n \end{cases} \tag{4.3}$$

with

$$a_i^k = \begin{cases} a_i^{k-1} + [i + (k-1)(t-1)]a_{i-1}^{k-1}, & 0 < i \le k, \\ 1, & i = 0, \\ 0, & i > k \end{cases} \tag{4.4}$$

in (4.1) are the best possible.

As an application of inequality (4.1), Yang's inequality was refined as follows: let $0 \le \lambda \le 1$, $0 < x \le \theta < \pi$, $t \ge 2$, and $A_i > 0$ with $\sum_{i=1}^{n} A_i \le \pi$ for $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n \ge 2$, then

$$L_m(n,\lambda) \le \sum_{1 \le i < j \le n} H_{ij} \le R_m(n,\lambda), \tag{4.5}$$

where

$$\begin{aligned} L_m(n,\lambda) &= \binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt}\mu_k\left(2^t\theta^t - \lambda^t\pi^t\right)^k\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \\ R_m(n,\lambda) &= \binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt}\omega_k\left(2^t\theta^t - \lambda^t\pi^t\right)^k\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \end{aligned} \tag{4.6}$$

and μ_k and ω_k are defined by (4.2).

4.2. Zhu's Generalizations and Applications

In [87], by making use of Lemma 2.9, the author obtained the following generalization of Jordan's inequality: if $0 < x \le r \le \pi/2$, then

$$\frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}\left(r^2 - x^2\right) \le \frac{\sin x}{x} \le \frac{\sin r}{r} + \frac{r - \sin r}{r^3}\left(r^2 - x^2\right). \tag{4.7}$$

As an application of (4.7), in virtue of (3.4), Yang's inequality (3.1) was sharpened and generalized as

$$\begin{aligned} &4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r} + \frac{\sin r - r\cos r}{2r^3}\left(\frac{\lambda\pi r^2}{2} - \frac{(\lambda\pi)^3}{8}\right)\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right) \\ &\quad \le \sum_{1 \le i < j \le n} H_{ij} \le 4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r} + \frac{r - \sin r}{r^3}\left(\frac{\lambda\pi r^2}{2} - \frac{(\lambda\pi)^3}{8}\right)\right]^2. \end{aligned} \tag{4.8}$$

In [88], the double inequality (3.22) was extended by using the method in [83] as

$$A_{2n,r}(x)+\alpha_{n,r}\left(r^2-x^2\right)^{n+1}\le\frac{\sin x}{x}\le A_{2n,r}(x)+\beta_{n,r}\left(r^2-x^2\right)^{n+1},\tag{4.9}$$

$$A_{2m,r}(x)+\mu_{m,r}(r-x)^{m+1}\le\frac{\sin x}{x}\le A_{2m,r}(x)+\nu_{m,r}(r-x)^{m+1}\tag{4.10}$$

with the equalities in (4.9) and (4.10) if and only if $x=r$, where $0<x\le r\le\pi/2$, $n\ge 0$, $m\in\mathbb{N}$ and

$$A_{2n,r}(x)=\sum_{k=0}^{n}a_{k,r}\left(r^2-x^2\right)^k\tag{4.11}$$

with

$$\begin{gathered}a_{0,r}=\frac{\sin r}{r},\qquad a_{1,r}=\frac{\sin r-r\cos r}{2r^3},\\ a_{k+1,r}=\frac{2k+1}{2(k+1)r^2}a_{k,r}-\frac{1}{4k(k+1)r^2}a_{k-1,r},\quad k\in\mathbb{N}.\end{gathered}\tag{4.12}$$

The constants $\alpha_{n,r}=a_{n+1}$ and

$$\beta_{n,r}=\frac{1-\sum_{k=0}^{n}a_kr^{2k}}{r^{2(n+1)}}\tag{4.13}$$

in (4.9) and the constants

$$\mu_{m,r}=\frac{1-\sum_{k=0}^{m}a_{k,r}r^{2k}}{r^{n+1}}\tag{4.14}$$

and $\nu_{m,r}=(2r)^{m+1}a_{m+1}$ in (4.10) are the best possible.

As an application of inequalities in (4.9), Yang's inequality (3.1) was extended or generalized as follows: if $A_i>0$ for $i\in\mathbb{N}$ with $\sum_{i=1}^{n}A_i\le r$ for $0<r\le\pi$ and $n\ge 2$, then

$$\begin{aligned}\max\{L_1(r),L_2(r)\}&\le(n-1)\sum_{k=1}^{n}\cos^2A_k-2\cos r\sum_{1\le i<j\le n}\cos A_i\cos A_j\\&\le\min\{R_1(r),R_2(r)\},\end{aligned}\tag{4.15}$$

where

$$
\begin{aligned}
L_1(r) &= \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + a_{n+1}\left(\pi^2 - r^2\right)^{n+1} \right]^2 \cos^2\frac{r}{2}, \\
L_2(r) &= \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + \frac{1-\sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}} \left(\frac{\pi - r}{2}\right)^{n+1} \right]^2 \cos^2\frac{r}{2}, \\
R_1(r) &= \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + \frac{1-\sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}} \left(\pi^2 - r^2\right)^{n+1} \right]^2, \\
R_2(r) &= \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + a_{n+1}\left(\frac{\pi - r}{2}\right)^{n+1} \right]^2.
\end{aligned}
\tag{4.16}
$$

In [89], the double inequality (4.9) was recovered by a similar method as in [83, 88].

The series expansion (3.25) was generalized in [89, Theorem 8] as follows: if $0 < x \le r \le \pi/2$ and $n \ge 0$, then

$$
\frac{\sin x}{x} = S_{2n}(x) + R_{2n+2}, \tag{4.17}
$$

where $S_{2n}(x) = \sum_{k=0}^{n} a_k (r^2 - x^2)^k$ and

$$
R_{2n+2} = \frac{1}{2^{n+1}(n+1)!(2n+3)!!} \cdot \frac{\sin \eta}{\eta} (r^2 - x^2)^{n+1}, \quad 0 < \eta < r \le \frac{\pi}{2} \tag{4.18}
$$

with

$$
\begin{gathered}
a_0 = \frac{\sin r}{r}, \qquad a_1 = \frac{\sin r - r\cos r}{2r^3}, \\
a_{k+1} = \frac{2k+1}{2(n+1)r^2} a_k - \frac{1}{4k(k+1)r^2} a_{k-1}, \quad k \in \mathbb{N}.
\end{gathered}
\tag{4.19}
$$

The series expansion (3.27) was also generalized in [89, Theorem 9]: if $0 < |x| \le r \le \pi/2$, then

$$
\frac{\sin x}{x} = \sum_{k=0}^{\infty} a_k \left(r^2 - x^2\right)^k, \tag{4.20}
$$

where a_k for $k \ge 0$ are defined by (4.19).

As applications of the above inequalities, the following general improvement of Yang's inequality was established in [89, Theorem 11]:

$$\begin{aligned}
&\binom{n}{2}(\lambda\pi)^2\left[S_{2n}\left(\frac{\pi\lambda}{2}\right)+a_{n+1}\left(r^2-\frac{1}{4}\pi^2\lambda^2\right)^{n+1}\right]^2\cos^2\left(\frac{\pi\lambda}{2}\right)\\
&\quad\le\sum_{1\le i<j\le n}H_{ij}\le\binom{n}{2}(\lambda\pi)^2\left[S_{2n}\left(\frac{\pi\lambda}{2}\right)+\frac{1-\sum_{k=0}^{n}a_k r^{2k}}{r^{2(n+1)}}\left(r^2-\frac{1}{4}\pi^2\lambda^2\right)^{n+1}\right]^2
\end{aligned}\tag{4.21}$$

for $n\ge 2$ and $0<r\le\pi/2$.

4.3. Wu's Generalization and Applications

In [90], Jordan's inequality (1.1) was generalized as

$$\begin{aligned}
&\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\left(1-\frac{x^\lambda}{\theta^\lambda}\right)+\left[1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right]\left(1-\frac{x}{\theta}\right)^\lambda\\
&\quad\le\frac{\sin x}{x}-\frac{\sin\theta}{\theta}\le\left(1-\frac{\sin\theta}{\theta}\right)\left(1-\frac{x^\lambda}{\theta^\lambda}\right),
\end{aligned}\tag{4.22}$$

where $0<x\le\theta\le\pi$ and $\lambda\ge 2$.

As an application of (4.22), Yang's inequality (3.1) was generalized as follows: if $A_i\ge 0$ for $1\le i\le n$ and $n\ge 2$ satisfy $\sum_{i=1}^{n}A_i\le\theta\in[0,\pi]$, then

$$\begin{aligned}
&\binom{n}{2}\left[\left(\pi-2-\frac{2}{\lambda}\right)\left(1-\frac{\theta}{\pi}\right)^\lambda-\frac{2}{\lambda}\left(\frac{\theta}{\pi}\right)^\lambda+\frac{2}{\lambda}+2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2\\
&\quad\le(n-1)\sum_{k=1}^{n}\cos^2 A_k-2\cos\theta\sum_{1\le i<j\le n}\cos A_i\cos A_j\\
&\quad\le\binom{n}{2}\left[2\left(\frac{\theta}{\pi}\right)^{\lambda+1}-\theta\left(\frac{\theta}{\pi}\right)^\lambda+\theta\right],\quad\lambda\ge 2.
\end{aligned}\tag{4.23}$$

Remark 4.1. The right-hand side inequality in (2.14) was recovered, and the left-hand side inequality in (2.14) was improved in [90].

4.4. Wu-Debnath's Generalizations and Applications

In [91], the following generalizations of Jordan's inequality were established:

$$\begin{aligned}
&\max\left\{\frac{3}{2}\varphi_1(\theta)\left(1-\frac{x}{\theta}\right)^2, \frac{3}{8}\varphi_2(\theta)\left(1-\frac{x^2}{\theta^2}\right)^2\right\} \\
&\quad\le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{1}{2}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1-\frac{x^2}{\theta^2}\right) \\
&\quad\le \min\left\{\frac{3}{2}\varphi_2(\theta)\left(1-\frac{x}{\theta}\right)^2, \frac{3}{2}\varphi_1(\theta)\left(1-\frac{x^2}{\theta^2}\right)^2\right\}
\end{aligned} \tag{4.24}$$

for $0 < x \le \theta$ and $\theta \in (0, \pi]$, where

$$\varphi_1(\theta) = \frac{2}{3} + \frac{\cos\theta}{3} - \frac{\sin\theta}{\theta}, \qquad \varphi_2(\theta) = \frac{\sin\theta}{\theta} - \frac{1}{3}\theta\sin\theta - \cos\theta. \tag{4.25}$$

The equalities in (4.24) hold if and only if $x = \theta$, and the coefficients of the factors $(1 - x/\theta)^2$ and $(1 - x^2/\theta^2)^2$ are the best possible.

If taking $\theta = \pi/2$, then inequalities (3.18) and (3.19) are deduced from (4.24).

Integrating on both sides of (4.24) yields

$$\begin{aligned}
&\max\left\{\frac{5\sin\theta - \theta\cos\theta + 2\theta}{6}, \frac{23\sin\theta - 8\theta\cos\theta - \theta^2\sin\theta}{15}\right\} \\
&\quad< \int_0^\theta \frac{\sin x}{x}dx < \min\left\{\frac{11\sin\theta - 5\theta\cos\theta - \theta^2\sin\theta}{6}, \frac{8\sin\theta - \theta\cos\theta + 8\theta}{15}\right\}.
\end{aligned} \tag{4.26}$$

If taking $\theta = \pi/2$ in (4.26), then

$$\frac{92-\pi^2}{60} < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{8+4\pi}{15}, \tag{4.27}$$

which is better than (2.14).

The basic tool for proving (4.24) is also Lemma 2.9.

As another application of (4.24), a generalization of Yang's inequality (3.1) was obtained: if $A_i > 0$ for $1 \le i \le n$ and $n \ge 2$ such that $\sum_{i=1}^{n} A_i \le \theta \in [0, \pi]$, then

$$\begin{aligned} \max\{N_1(\theta), N_2(\theta)\} &\le \binom{n}{2}\sin^2\theta \\ &\le (n-1)\sum_{k=1}^{n}\cos^2 A_k - 2\cos\theta \sum_{1\le i<j\le n}\cos A_i \cos A_j \\ &\le 4\binom{n}{2}\sin^2\frac{\theta}{2} \le \min\{M_1(\theta), M_2(\theta)\}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} N_1(\theta) &= \binom{n}{2}\left[3 - \frac{\theta^2}{\pi^2} + (\pi - 3)\left(1 - \frac{\theta}{\pi}\right)^2\right]^2 \left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \\ N_2(\theta) &= \binom{n}{2}\left[3 - \frac{\theta^2}{\pi^2} + \frac{12 - \pi^2}{16}\left(1 - \frac{\theta^2}{\pi^2}\right)^2\right]^2 \left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \\ M_1(\theta) &= \binom{n}{2}\left[3 - \frac{\theta^2}{\pi^2} + \frac{12 - \pi^2}{4}\left(1 - \frac{\theta}{\pi}\right)^2\right]^2 \left(\frac{\theta}{\pi}\right)^2, \\ M_2(\theta) &= \binom{n}{2}\left[3 - \frac{\theta^2}{\pi^2} + (\pi - 3)\left(1 - \frac{\theta^2}{\pi^2}\right)^2\right]^2 \left(\frac{\theta}{\pi}\right)^2. \end{aligned} \tag{4.29}$$

If substituting A_i by λA_i and θ by $\lambda\pi$ in (4.28), then inequalities (3.7) and (3.20) are deduced.

In [92], as a generalization of inequality (4.24), the following sharp inequality

$$\begin{aligned} &\frac{1}{2\tau^2}\left[(1+\lambda)\left(\frac{\sin\theta}{\theta} - \cos\theta\right) - \theta\sin\theta\right]\left(1 - \frac{x^\tau}{\theta^\tau}\right)^2 \\ &\quad\le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) \\ &\quad\le \left[1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\right]\left(1 - \frac{x^\tau}{\theta^\tau}\right)^2 \end{aligned} \tag{4.30}$$

was obtained for $0 < x \le \theta \in (0, \pi/2]$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$ by employing Lemma 2.9. The equalities in (4.30) holds if and only if $x = \theta$. The coefficients of the term $(1 - x^\tau/\theta^\tau)^2$

are the best possible. If $1 \le \tau \le 5/3$ and either $\lambda \neq 0$ or $\lambda \ge 2\tau$, then the inequality (4.30) is reversed. Specially, when $\theta = \pi/2$, the inequality (4.30) becomes

$$\frac{4\lambda+4-\pi^2}{4\tau^2\pi^{2\tau+1}}(\pi^\tau - 2^\tau x^\tau)^2 \le \frac{\sin x}{x} - \frac{2}{\pi} - \frac{2}{\lambda\pi^{\lambda+1}}\left(\pi^\lambda - 2^\lambda x^\lambda\right) \le \frac{\lambda\pi - 2\lambda - 2}{\lambda\pi^{2\tau+1}}(\pi^\tau - 2^\tau x^\tau)^2 \tag{4.31}$$

for $0 < x \le \pi/2$, $\tau \ge 2$, and $\tau \le \lambda \le 2\tau$. If $1 \le \tau \le 5/3$ and either $\lambda \neq 0$ or $\lambda \ge 2\tau$, then the inequality (4.31) is reversed.

If taking $(\tau, \lambda) = (2, 2)$ and $(\tau, \lambda) = (1, 2)$, then inequalities (3.6), (3.18), and (3.19) are derived.

If $\lambda \ge 2$ and $A_i \ge 0$ with $\sum_{i=1}^n A_i \le \theta \in [0, \pi]$ for $n \ge 2$, then the following generalization of Yang's inequality was obtained by using the inequality (4.30) in [92]:

$$\begin{aligned} \max\{K_1(\lambda,\theta), K_2(\lambda,\theta)\} &\le (n-1)\sum_{k=1}^n \cos^2 A_k - 2\cos\theta \sum_{1\le i<j\le n} \cos A_i \cos A_j \\ &\le \min\{Q_1(\lambda,\theta), Q_2(\lambda,\theta)\}, \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} K_1(\lambda,\theta) &= \binom{n}{2}\left\{\left[\lambda + 1 - \frac{\theta^\lambda}{\pi^\lambda} + \frac{\lambda\pi - 2\lambda - 2}{2}\left(1 - \frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \\ K_2(\lambda,\theta) &= \binom{n}{2}\left\{\left[\lambda + 1 - \frac{\theta^\lambda}{\pi^\lambda} + \frac{4\lambda\pi + 4 - \pi^2}{8\lambda}\left(1 - \frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \\ Q_1(\lambda,\theta) &= \binom{n}{2}\left\{\left[\lambda + 1 - \frac{\theta^\lambda}{\pi^\lambda} + \frac{4\lambda + 4\lambda^2 - \lambda\pi^2}{8}\left(1 - \frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2, \\ Q_2(\lambda,\theta) &= \binom{n}{2}\left\{\left[\lambda + 1 - \frac{\theta^\lambda}{\pi^\lambda} + \frac{\lambda\pi - 2\lambda - 2}{2}\left(1 - \frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2. \end{aligned} \tag{4.33}$$

Note that inequalities (3.7), (3.20), and (4.28) can be deduced from (4.32).

By analytic techniques, the following inequalities are presented in [93].

(1) If $0 < x \le \theta \le \pi$, then

$$\frac{\sin x}{x} \ge \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x - \theta). \tag{4.34}$$

(2) If $0 < x \le \pi$ and $0 < \theta \le \pi/2$, then

$$\frac{\sin x}{x} \le \frac{\sin\theta}{\theta} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x - \theta). \tag{4.35}$$

(3) Equalities in (4.34) and (4.35) hold if and only if $x = \theta$.

These two inequalities extend the double inequality obtained by applying $n = 1$ to the inequality (4.41).

As applications of inequalities in (4.34) and (4.35), the following double inequalities were gained: if $x_i > 0$ for $1 \le i \le n$ and $n \ge 2$ satisfying $\sum_{i=1}^{n} x_i = \theta$ for $0 < \theta \le \pi$, then

$$\frac{\sin\theta}{\theta} + n - 1 < \sum_{i=1}^{n} \frac{\sin x_i}{x_i} \le \frac{n^2}{\theta} \sin\frac{\theta}{n}, \tag{4.36}$$

$$\sum_{i=1}^{n} \frac{\sin x_i}{\theta - x_i} > 1 + \frac{1}{n-1} \cdot \frac{\sin\theta}{\theta} \tag{4.37}$$

$$\begin{aligned} 1 + (n-1)\left(\frac{\sin\theta}{\theta}\right) &< \sum_{i=1}^{n} \frac{\sin(\theta - x_i)}{\theta - x_i} \\ &< \left(n^2 - 3n + 1\right) \cos\frac{\theta}{n-1} \\ &\quad - \frac{(n-1)\left(n^2 - 4n + 1\right)}{\theta} \sin\frac{\theta}{n-1}, \quad n \ge 3. \end{aligned} \tag{4.38}$$

The equality in (4.36) holds if and only if $x_i = \theta/n$ for all $1 \le i \le n$.

The inequality (4.36) generalizes Janous-Klamkin's inequality [94, 95]:

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi}, \tag{4.39}$$

where $A > 0$, $B > 0$, and $C > 0$ satisfy $A + B + C = \pi$. Meanwhile, the inequalities (4.37) and (4.38) generalize and improve Tsintsifas-Murty-Henderson's double inequality [96, 97]:

$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}, \tag{4.40}$$

where $0 < A < \pi/2$, $0 < B < \pi/2$, and $0 < C < \pi/2$ satisfy $A + B + C = \pi$.

4.5. Wu-Srivastava's Generalizations and Applications

By using Lemma 2.9 and other techniques, a double inequality was obtained in [98], which can be simplified as follows: let i be a nonnegative integer and $0 < x \le \theta \le \pi/2$.

(1) For $n = 4i + 1$ or $n = 4i + 2$,

$$\frac{(\theta - x)^n}{\theta^n}\left[1 - \sum_{k=0}^{n-1}\sum_{\ell=0}^{k}\frac{(-1)^\ell\theta^{\ell-1}}{\ell!}\sin\left(\theta + \frac{\ell\pi}{2}\right)\right] \le \frac{\sin x}{x} - \sum_{k=0}^{n-1}\sum_{\ell=0}^{k}\frac{(-1)^{k+\ell}(x-\theta)^k}{\ell!\theta^{k-\ell+1}}\sin\left(\theta + \frac{\ell\pi}{2}\right) \le \frac{(\theta - x)^n}{\theta^{n+1}}\sum_{\ell=0}^{n}\frac{(-1)^\ell\theta^\ell}{\ell!}\sin\left(\theta + \frac{\ell\pi}{2}\right). \tag{4.41}$$

(2) For $n = 4i + 3$ or $n = 4i + 4$, the inequality (4.41) is reversed.

(3) The equalities in (4.41) hold true if and only if $x = \theta$.

Upon letting $n = 2$ in (4.41), the following inequality is derived:

$$\frac{\theta - 2\sin\theta + \theta\cos\theta}{\theta^3}(x-\theta)^2 \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta) \le \frac{2\sin\theta - 2\theta\cos\theta - \theta^2\sin\theta}{2\theta^3}(x-\theta)^2 \tag{4.42}$$

for $0 < x \le \theta \le \pi$.

Upon taking $n = 2$ and $\theta = \pi/2$, the inequality (3.19) follows.

As a consequence of (4.41), a double inequality for estimating the definite integral $\int_0^{\pi/2}(\sin x/x)dx$ was established in [98], which refines the double inequality (2.14).

Finally, the inequality (4.41) for $n = 5$ and $\theta = \pi/2$ was applied to refine and generalize Yang's inequality (3.1).

4.6. Wu-Debnath's General Generalizations and Applications

In [99], the inequality (4.41) was generalized to a general form which can be recited as follows: let f be a real-valued $(n+1)$-time differentiable function on $[0, \theta]$ with $f(0) = 0$.

(1) If n is either a positive even number such that $f^{(n+1)}$ is increasing on $[0, \theta]$ or a positive odd number such that $f^{(n+1)}$ is decreasing on $[0, \theta]$, then the following double inequality is valid for $x \in (0, \theta]$:

$$\frac{(-1)^n}{\theta^n}\left[f'(0^+) + \sum_{k=0}^{n-1}\sum_{i=0}^{k}\frac{(-1)^{i-1}\theta^{i-1}}{i!}f^{(i)}(\theta)\right] \le \frac{f(x)}{x} - \sum_{k=0}^{n-1}\sum_{i=0}^{k}\frac{(-1)^i(\theta - x)^k}{i!\theta^{k-i+1}}f^{(i)}(\theta) \le \sum_{i=0}^{n}\frac{(-1)^i(\theta - x)^n}{i!\theta^{n-i+1}}f^{(i)}(\theta). \tag{4.43}$$

(2) If n is either a positive even number such that $f^{(n+1)}$ is decreasing on $[0,\theta]$ or a positive odd number such that $f^{(n+1)}$ is increasing on $[0,\theta]$, then the inequality (4.43) is reversed.

(3) The equalities in (4.43) hold if and only if $x=\theta$.

Upon taking $f(x)=\sin x$, the inequality (4.41) follows straightforwardly.

The tool of the paper [99] is Lemma 2.9. The authors also used their techniques to present similar inequalities for the functions

$$\frac{\sinh x}{x}, \qquad \frac{\ln(1+x)}{x}. \tag{4.44}$$

As consequences of the above inequalities, a double inequality for bounding the definite integral $\int_0^a (\ln(1+x)/x)dx$ for $a>0$ and some known inequalities were derived.

4.7. Wu-Srivastava-Debnath's Generalization and Applications

In virtue of Lemma 2.9, the following conclusion for bounding the function $\sin x/x$ was gained in [100]: for $n\in\mathbb{N}$, $0<x\le\theta\le\pi$ and $f(x)=\sin\sqrt{x}/\sqrt{x}$, we have

$$\begin{aligned}\frac{f^{(n)}(\theta^2)}{n!}\left(x^2-\theta^2\right)^n &\le \frac{\sin x}{x}-\sum_{k=0}^{n-1}\frac{f^{(k)}(\theta^2)}{k!}\left(x^2-\theta^2\right)^k\\ &\le \frac{1}{\theta^{2n}}\left[1-\sum_{k=0}^{n-1}\frac{(-1)^k\theta^{2k}f^{(k)}(\theta^2)}{k!}\right]\left(\theta^2-x^2\right)^n.\end{aligned} \tag{4.45}$$

The equalities in (4.45) hold true if and only if $x=\theta$.

In [100, Lemma 3], the function $f(x)=\sin\sqrt{x}/\sqrt{x}$ was proved to be completely monotonic on $(0,\pi^2]$. For detailed information on the class of completely monotonic functions, please see the survey paper [101] and related references therein.

In the final of [100], Yang's inequality (3.1) was generalized by virtue of the inequality (4.45) for $n=4$ and $\theta=\pi/2$.

5. Refinements of Kober's Inequality

5.1. Niu's Results

As a direct consequence of (3.29), the following general refinements of Kober's inequality was obtained in [84]: for $0<x\le\pi/2$, $k\in\mathbb{N}$, and $n\in\mathbb{N}$, inequalities

$$\begin{aligned}&\left(\frac{\pi}{2}-x\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k(4x)^k(\pi-x)^k\right]\\ &\quad\le\cos x\le\left(\frac{\pi}{2}-x\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\beta_k(4x)^k(\pi-x)^k\right],\end{aligned} \tag{5.1}$$

which may be deduced by replacing x with $x-\pi/2$ in (3.29), and

$$\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^{i}\binom{k}{i}\alpha_k\pi^{2k-2i}}{2i+2}x^{2i+2} \le 1-\cos x-\frac{x^2}{\pi} \le \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^{i}\binom{k}{i}\beta_k\pi^{2k-2i}}{2i+2}x^{2i+2}, \tag{5.2}$$

which follows from integrating (3.29) from 0 to $x\in[0,\pi/2]$, hold with constants α_k and β_k defined by (3.30) and (3.31), respectively.

5.2. Zhu's Result

By a utilization of the inequality (3.22) and a simple transformation of variables, the following Kober type inequality was deduced in [89, Theorem 13]: let

$$R(u)=\sum_{k=0}^{n}\frac{a_k}{2}\pi^{2k+1}(1-u)u^{k}(2-u)^{k},\qquad S(u)=\frac{1}{2}\pi^{2n+3}(1-u)u^{n+1}(2-u)^{n+1} \tag{5.3}$$

for $n\ge 0$, where a_k for $k\ge 0$ are defined by (3.24). Then the inequality

$$R(u)+\lambda S(u)\le\cos\left(\frac{\pi u}{2}\right)\le R(u)+\mu S(u) \tag{5.4}$$

holds if either $0\le u\le 1$, $\lambda=a_{n+1}$ and $\mu=\left(1-\sum_{k=0}^{n}a_k\pi^{2k}\right)/\pi^{2(n+1)}$ or $1\le u\le 2$ and $\lambda=\left(1-\sum_{k=0}^{n}a_k\pi^{2k}\right)/\pi^{2(n+1)}$ and $\mu=a_{n+1}$.

6. Niu's Applications and Analysis of Coefficients

6.1. An Application to the Gamma Function

In [84], combining

$$\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z} \tag{6.1}$$

with (3.29) yields that if $0<x<\pi/2$ and $n\in\mathbb{N}$, then

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\left(\pi^2-4x^2\right)^{k}\le\frac{1}{\Gamma(1+x/\pi)\Gamma(1-x/\pi)}\le\frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\left(\pi^2-4x^2\right)^{k}, \tag{6.2}$$

where $\Gamma(x)$ is the classical Euler gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \mathrm{d}t. \tag{6.3}$$

6.2. Applications to Definite Integrals

In [84], as applications of (3.29), the following conclusions were also obtained.

(1) For $0 < x \le \pi/2$ and $k, n \in \mathbb{N}$,

$$\begin{aligned} &\frac{2}{\pi}x + \sum_{k=1}^{n}\sum_{i=0}^{k} \frac{(-4)^i \binom{k}{i} \alpha_k \pi^{2k-2i}}{2i+1} x^{2i+1} \\ &\quad \le \int_0^x \frac{\sin t}{t} \mathrm{d}t \le \frac{2}{\pi}x + \sum_{k=1}^{n}\sum_{i=0}^{k} \frac{(-4)^i \binom{k}{i} \beta_k \pi^{2k-2i}}{2i+1} x^{2i+1}. \end{aligned} \tag{6.4}$$

(2) Let $f(x)$ be continuous on $[a, b]$ such that $f(x) \not\equiv 0$ and $0 \le f(x) \le M$. If $0 < b-a < \pi$ and $n \in \mathbb{N}$, then

$$\begin{aligned} 0 &< \left(\int_a^b f(x) dx\right)^2 - \left(\int_a^b f(x) \cos x \, dx\right)^2 - \left(\int_a^b f(x) \sin x \, dx\right)^2 \\ &\le M^2 (b-a)^2 \left\{ 1 - \left[\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - a^2 - b^2 + 2ab\right)^k \right]^2 \right\}. \end{aligned} \tag{6.5}$$

6.3. Analysis of Coefficients

The coefficients α_k and β_k defined by (3.30) and (3.31) were estimated in [84] as follows: for $k > 1$,

$$\begin{aligned} -\frac{\sqrt{\pi}}{\pi^{2k}\sqrt{4k+1}} &< \alpha_k < \frac{1}{\pi^{2k}\sqrt{4k+1}}, \\ \beta_k &< \frac{1 - 2/\pi + \sqrt{\pi}\left(\sqrt{k-1} - 1/2\right)}{\pi^{2k}}, \\ 0 \le \beta_k - \alpha_k &< \frac{1 - 2/\pi + \sqrt{\pi}\left(\sqrt{k} - 1/2\right)}{\pi^{2k}}. \end{aligned} \tag{6.6}$$

Recently, some more accurate estimates of the coefficients α_k and β_k are carried out in [102].

6.4. A Power Series

The inequality (3.29) can be rearranged as

$$0 \le \frac{\sin x}{x} - \frac{2}{\pi} - \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \sum_{k=1}^{n} (\beta_k - \alpha_k)\left(\pi^2 - 4x^2\right)^k \longrightarrow 0 \tag{6.7}$$

as $n \to \infty$, this implies that

$$\sin x = \frac{2}{\pi} x - \sum_{k=1}^{\infty} \alpha_k x \left(\pi^2 - 4x^2\right)^k. \tag{6.8}$$

This gives an alternative power series expansion similar to (2.19) and (3.27).

6.5. A Remark

It is natural to consider that the series (2.19), (3.27), and (6.8) should be the same one, although they seem to have different expressions. This was affirmed in [102], among other things.

7. Generalizations of Jordan's Inequality to Bessel Functions

For $x \in \mathbb{R}$, some Bessel functions are defined by

$$\begin{aligned}
J_p(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \\
I_p(x) &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \\
\lambda_p(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n c^n \Gamma(p+(b+1)/2)}{n!\Gamma(p+(b+1)/2+n)} \left(\frac{x}{2}\right)^{2n}, \\
\mathcal{J}_p(x) &= 2^p \Gamma(p+1) x^{-p} J_p(x), \\
\mathcal{I}_p(x) &= 2^p \Gamma(p+1) x^{-p} I_p(x).
\end{aligned} \tag{7.1}$$

It is well known that

$$\begin{aligned}
\mathcal{J}_{-1/2}(x) &= \cos x, \qquad \mathcal{I}_{-1/2}(x) = \cosh x, \\
\mathcal{J}_{1/2}(x) &= \frac{\sin x}{x}, \qquad \mathcal{I}_{1/2}(x) = \frac{\sinh x}{x}.
\end{aligned} \tag{7.2}$$

7.1. Neuman's Generalizations of Jordan's Inequality

In [103], it was established for $p \ge 1/2$ and $|x| \le \pi/2$ that

$$\frac{1}{3(p+1)}\left[2p+1+(p+2)\cos\left(\sqrt{\frac{3}{2(p+2)}}\,x\right)\right] \ge \mathcal{J}_p(x) \ge \cos\left(\frac{x}{\sqrt{2(p+1)}}\right). \tag{7.3}$$

When $p = -1/2$, equality in (7.3) validates.

Taking in (7.3) $p = 1/2$ leads to

$$\frac{2}{9}\left[2+\frac{5}{2}\cos\left(\sqrt{\frac{3}{5}}\,x\right)\right] \ge \frac{\sin x}{x} \ge \cos\left(\frac{x}{\sqrt{3}}\right), \quad x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{7.4}$$

By employing Lemma 2.9, inequalities (2.2), and (2.13) are generalized in [104] as

$$\left[1-\lambda_p\left(\frac{\pi}{2}\right)\right]\frac{\pi-2x}{\pi} \le \lambda_p(x)-\lambda_p\left(\frac{\pi}{2}\right) \le \left[\left(\frac{c\pi}{2k}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right]\frac{\pi-2x}{\pi} \tag{7.5}$$

for $k \ge 1/2$ and $0 \le c \le 1$ and

$$\left[\left(\frac{c}{4k}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right]\frac{\pi^2-4x^2}{4} \le \lambda_p(x)-\lambda_p\left(\frac{\pi}{2}\right) \le \left[1-\lambda_p\left(\frac{\pi}{2}\right)\right]\frac{\pi^2-4x^2}{\pi^2} \tag{7.6}$$

for $k \ge 0$ and $0 \le c \le 1$.

In [105], inequalities (7.5) and (7.6) were further improved.

7.2. Niu-Huo-Cao-Qi's Generalizations of Jordan's Inequality

In [84, 102], the following two conclusions were established.

(1) For $n \in \mathbb{N}$ and $x \in (0,\pi/2]$, if $k \ge 1/2$ and $0 \le c \le 1$, then

$$\sum_{i=0}^{n}\gamma_i\left(\pi^2-4x^2\right)^i \le \lambda_p(x) \le \sum_{i=0}^{n}\eta_i\left(\pi^2-4x^2\right)^i, \tag{7.7}$$

where

$$\begin{aligned}
\gamma_i &= \left(\frac{c}{16}\right)^i\frac{\Gamma(k)}{i!\,\Gamma(k+i)}\lambda_{i+p}\left(\frac{\pi}{2}\right), \quad 0 \le i \le n,\\
\eta_i &= \begin{cases}\gamma_i, & 0 \le i \le n-1,\\ \dfrac{1-\sum_{\ell=0}^{n-1}\gamma_\ell\pi^{2\ell}}{\pi^{2n}}, & i = n\end{cases}
\end{aligned} \tag{7.8}$$

are the best possible. For $k > 0$, $c \leq 0$ and $x \in (0, \pi/2]$, when n is odd, the inequality (7.7) holds; when n is even, the inequality (7.7) is reversed.

(2) For $n \in \mathbb{N}$ and $0 < x \leq \theta \leq \pi/2$, if $k \geq 1/2$ and $0 \leq c \leq 1$, then

$$\sum_{i=0}^{n} \sigma_i \left(\theta^2 - x^2\right)^i \leq \lambda_p(x) \leq \sum_{i=0}^{n} \nu_i \left(\theta^2 - x^2\right)^i, \tag{7.9}$$

where

$$\begin{aligned} \sigma_i &= \left(\frac{c}{4}\right)^i \frac{\Gamma(k)}{i!\Gamma(k+i)} \lambda_{i+p}(\theta), \quad 0 \leq i \leq n \\ \nu_i &= \begin{cases} \sigma_i, & 0 \leq i \leq n-1, \\ \dfrac{1 - \sum_{\ell=0}^{n-1} \sigma_\ell \theta^{2\ell}}{\theta^{2n}}, & i = n \end{cases} \end{aligned} \tag{7.10}$$

are the best possible. For $k > 0$, $c \leq 0$, and $0 < x \leq \theta < \infty$, if n is odd, the inequality (7.9) holds true; if n is even, the inequality (7.9) is reversed.

We remark that for $c \in [0, 1]$ the conditions on x and k can be relaxed, as it was stated in [106, pages 123–124].

7.3. Baricz's Generalizations of Cusa-Huygens's Inequality

The inequality (1.24) was generalized in [104] to

$$\frac{1 + 2ak\lambda_p(x)}{a(2k-1) + \pi/2} \leq \lambda_{p+1}(x) \leq \frac{1 + 2ak\lambda_p(x)}{a + 1 + a(2k-1)}, \tag{7.11}$$

where $|x\sqrt{c}| \leq \pi/2$, $a \in (0, 1/2]$, $c \geq 0$, and $k \geq 1/2$.

By making use of the inequality (1.22) and (1.23), the inequality (7.11) was further strengthened as

$$\frac{1 + 2k\lambda_p(x)}{2k+1} \leq \lambda_{p+1}(x) \leq \frac{1 + k\lambda_p(x)}{k+1}. \tag{7.12}$$

7.4. Baricz's Generalizations of Redheffer-Williams's Inequality

In [107], inequalities (1.5), (2.34), and (2.35) were generalized to the case of Bessel functions. The motivation of the paper [107] comes from [22, 50, 57, 58, 108] and other related references.

7.5. Lazarević's Inequality and Generalizations

An inequality due to [109] states that

$$\left(\frac{\sinh t}{t}\right)^{3} > \cosh t \tag{7.13}$$

for $t \neq 0$. The exponent 3 in (7.13) is the best possible. See also [1, page 131], [3, page 300], and [4, page 270].

In [110, pages 808–809], among other things, it was proved that the function

$$\frac{\ln((\sinh x)/x)}{\ln \cosh x} \tag{7.14}$$

is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with range $(1/3, 1)$. From this, the following double inequality was inferred:

$$\frac{\sinh x}{x} < \cosh x < \left(\frac{\sinh x}{x}\right)^{3}, \quad x \neq 0. \tag{7.15}$$

It was also mentioned that the inequality (7.13) can be proved directly by applying Lemma 2.9 for $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$ to the function

$$\frac{(\cosh x)^{-1/3} \sinh x}{x}. \tag{7.16}$$

The inequality (7.13) was recovered in [111, Lemma 3].

The inequality (7.13) was refined in [23] as follows: for $x \neq 0$, the inequality

$$\left(\frac{\sinh x}{x}\right)^{\lambda} > \frac{\lambda}{3} \cosh x - \frac{\lambda}{3} + 1 \tag{7.17}$$

holds if and only if $\lambda < 0$ or $\lambda \geq 7/5$ and reverses if and only if $0 < \lambda \leq 1$.

The inequality (7.13) was generalized in [112] to modified Bessel functions.

Remark 7.1. In [113], it was proved that the inequality

$$\left(\frac{\sin x}{x}\right)^{3} > \cos x \tag{7.18}$$

is valid for $x \in (0, \pi/2)$, and the exponent 3 is the best possible. See also [4, pages 238–240].

In [110, pages 806–807], it was pointed out that the inequality (7.18) can also be proved directly by Lemma 2.9 for $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$ by considering the quotient

$$\frac{(\cos x)^{-1/3} \sin x}{x}, \tag{7.19}$$

and that the inequality (7.18) is the special case $g(x) > g(0) = 3$ for x on $(0, \pi/2)$, where

$$g(x) = \frac{\ln \cos x}{\ln((\sin x)/x)}. \tag{7.20}$$

The inequality (7.18) was refined in [23] as follows: for $x \in (0, \pi/2)$, the inequality

$$\left(\frac{\sin x}{x}\right)^{\lambda} > \frac{\lambda}{3}\cos x - \frac{\lambda}{3} + 1 \tag{7.21}$$

holds if and only if $\lambda < 0$ or $\lambda \geq \lambda_0 = 1.420\ldots$ and reverses if and only if $0 < \lambda \leq 7/5$, where λ_0 satisfies $\lambda/3 + (2/\pi)^{\lambda} - 1 = 0$.

7.6. Oppenheim's Problem

Considering inequalities stated in Sections 1.7 and 7.3, it is natural to ask the following problems.

(1) What are the best possible positive constants a, b, c, r and α, β, γ, λ such that

$$\alpha + \beta \cos^{\gamma}(\lambda x) \leq \frac{\sin x}{x} \leq a + b\cos^{c}(rx) \tag{7.22}$$

for $-\pi/2 \leq x \leq \pi/2$ with $x \neq 0$ and

$$\alpha + \beta \cosh^{\gamma}(\lambda x) \leq \frac{\sinh x}{x} \leq a + b\cosh^{c}(rx) \tag{7.23}$$

for $-\infty < x < \infty$ with $x \neq 0$ hold, respectively?

(2) What about the analogues of Bessel functions or other special functions?

These problems are similar to Oppenheim's problem which has been investigated in [24, 30, 36, 104, 112].

7.7. Some Inequalities of Bessel Functions

For more information on inequalities of Bessel functions and some other special functions, please refer to [106, 114–117] and related references therein.

8. Wilker-Anglesio's Inequality and Its Generalizations

8.1. Wilker's Inequality and Generalizations

In [118], Wilker proved

$$\left(\frac{\sin x}{x}\right)^{2} + \frac{\tan x}{x} > 2 \tag{8.1}$$

and proposed that there exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+cx^3\tan x \tag{8.2}$$

for $0<x<\pi/2$.

In recent years, Wilker's inequality (8.1) has been proved once and again in papers such as [110, 119–125].

In [126], the inequality (8.1) was generalized as follows: if $q>0$ or $q\le\min\{-1,-\lambda/\mu\}$, then

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q>1 \tag{8.3}$$

holds for $0<x<\pi/2$, where $\lambda>0$, $\mu>0$ and $p\le 2q\mu/\lambda$. As an application of the inequality (8.3), an inequality posed as an open problem in [21] was solved and improved.

In [111], the inequality (8.1) was generalized as

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2 \tag{8.4}$$

for $x\ne 0$, which together with (8.1) was further extended and refined in [23, 127] as

$$\left(\frac{\sinh x}{x}\right)^{2p}+\left(\frac{\tanh x}{x}\right)^{p}>\left(\frac{x}{\sinh x}\right)^{2p}+\left(\frac{x}{\tanh x}\right)^{p}>2,\quad x\ne 0, \tag{8.5}$$

$$\left(\frac{\sin x}{x}\right)^{2p}+\left(\frac{\tan x}{x}\right)^{p}>\left(\frac{x}{\sin x}\right)^{2p}+\left(\frac{x}{\tan x}\right)^{p}>2,\quad 0<x<\frac{\pi}{2} \tag{8.6}$$

for $p\ge 1$.

Note that the right-hand side inequality in (8.6) is a special case of (8.3).

In [112], inequalities (8.1) and (8.4) were generalized and extended naturally to the cases of Bessel function. Recently, the inequality (8.3) and all results in [126] were extended in [128] to Bessel functions.

8.2. Wilker-Anglesio's Inequality

In [129], the best constant c in (8.2) was found, and it was proved that

$$2+\frac{8}{45}x^3\tan x>\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+\left(\frac{2}{\pi}\right)^4x^3\tan x \tag{8.7}$$

for $0<x<\pi/2$. The constants $8/45$ and $(2/\pi)^4$ in the inequality (8.7) are the best possible.

In [130–133], several proofs of Wilker-Anglesio's inequality (8.7) were given.

In [134], a new proof of the inequality (8.7) was provided by using Lemma 2.9 and compared with [132].

In [124, 135, 136], three lower bounds for $(\sin x/x)^2 + \tan x/x - 2$ were presented, but they are weaker than $(2/\pi)^4 x^3 \tan x$ in (8.7).

In [137, 138], the following Wilker type inequality was obtained:

$$2 + \frac{2}{45}x^3 \sin x < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \left(\frac{2}{\pi} - \frac{16}{\pi^3}\right)x^3 \sin x \tag{8.8}$$

for $x \in (0, \pi/2)$. The constants $2/45$ and $2/\pi - 16/\pi^3$ in (8.8) are the best possible.

In [139, Theorem 3], by Lemma 2.9 for $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$, the inequality (8.7) was recovered and the double inequality

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(-1)^k 2^{2k+4}[1 - (4k+10)B_{2k+4}]}{(2k+5)!} x^{2k+3} \tan x \\
&\quad < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 \\
&\quad < \sum_{k=0}^{n-1} \frac{(-1)^k 2^{2k+4}[1 - (4k+10)B_{2k+4}]}{(2k+5)!} x^{2k+3} \tan x \\
&\qquad + \left(\frac{2}{\pi}\right)^{2n+4} \left\{1 - \sum_{k=0}^{n-1} \frac{(-1)^k \pi^{2k+4}[1 - (4k+10)B_{2k+4}]}{(2k+5)!}\right\} x^{2n+3} \tan x
\end{aligned} \tag{8.9}$$

for $0 < x < \pi/2$ was procured, where B_i for $i \in \mathbb{N}$ are defined by (1.31).

8.3. An Open Problem

It is clear that to generalize Wilker-Anglesio's inequality (8.7) is more significant than to generalize Wilker's inequality (8.1).

We conjecture that Wilker-Anglesio's inequality (8.7) may be generalized as follows: let α, β, λ, and μ be positive real numbers satisfying $\alpha\lambda = 2\beta\mu$, then

$$\begin{aligned}
\frac{16\mu}{\pi^4} x^4 \left(\frac{\tan x}{x}\right)^\beta &< \lambda \left(\frac{\sin x}{x}\right)^\alpha + \mu \left(\frac{\tan x}{x}\right)^\beta - (\lambda + \mu) \\
&< \frac{\lambda\alpha[5\lambda\alpha + \mu(12 + 5\alpha)]}{360\mu} x^4 \left(\frac{\tan x}{x}\right)^\beta
\end{aligned} \tag{8.10}$$

holds for $0 < x < \pi/2$.

9. Applications of a Method of Auxiliary Functions

In Section 2.1 of this paper, a method constructing auxiliary functions to refine Jordan's inequality (1.1) in [45, 46, 49, 50, 140] is introduced. Now the aim of this section is to summarize some other applications of this method, including estimation of some complete elliptic integrals and construction of inequalities for the exponential function e^x.

The complete elliptic integrals are classified into three kinds and defined for $0 < k < 1$ as

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$
$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta, \tag{9.1}$$
$$II(k,h) = \int_0^{\pi/2} \frac{d\theta}{\left(1 + h\sin^2\theta\right)\sqrt{1 - k^2\sin^2\theta}}.$$

9.1. Estimates for a Concrete Complete Elliptic Integral

In [141], it was posed that

$$\frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} dx < \frac{\pi\sqrt{2}}{8}. \tag{9.2}$$

In [142], the inequality (9.2) was verified by using $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$ on the unit interval $[0, 1]$.

In [48], by considering monotonicity and convexity of the function

$$\frac{1}{\sqrt{4 - x^2 - x^3}} - \frac{1}{2} + \frac{1 - \sqrt{2}}{2} x^4 + \alpha x^3 (1 - x) \tag{9.3}$$

on $[0, 1]$ for undetermined constant $\alpha \geq 0$, the inequality

$$\frac{1}{\sqrt{4 - x^2 - x^3}} \geq \frac{1}{2} + \frac{\sqrt{2} - 1}{2} x^4 + \left(\frac{11\sqrt{2}}{8} - 2\right)(1 - x)x^3 \tag{9.4}$$

for $x \in [0, 1]$ was established, and then the lower bound in (9.2) was improved to

$$\int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} dx > \frac{3}{10} + \frac{27\sqrt{2}}{160}. \tag{9.5}$$

It was also remarked in [48] that if discussing the auxiliary functions

$$\frac{1}{\sqrt{4 - x^2 - x^3}} - \frac{1}{2} + \frac{1 - \sqrt{2}}{2} x^2 + \beta(1 - x)x^2,$$
$$\frac{1}{\sqrt{4 - x^2 - x^3}} - \frac{1}{2} + \frac{1 - \sqrt{2}}{2} x^4 + \theta\left(1 - x^3\right)x \tag{9.6}$$

on $[0,1]$, then inequalities

$$\begin{aligned}
\frac{1}{\sqrt{4-x^2-x^3}} &\geq \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\left(\frac{3\sqrt{2}}{8}-1\right)(1-x)x^2 \\
\frac{1}{\sqrt{4-x^2-x^3}} &\geq \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^4+\left(\frac{2}{3}-\frac{11\sqrt{2}}{24}\right)\left(x^3-1\right)x
\end{aligned} \tag{9.7}$$

can be obtained, and then, by integrating on both sides of above two inequalities, the lower bound in (9.2) may be improved to

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\mathrm{d}x &> \frac{1}{4}+\frac{19\sqrt{2}}{96} \\
\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\mathrm{d}x &> \frac{1}{5}+\frac{19\sqrt{2}}{80}.
\end{aligned} \tag{9.8}$$

Numerical computation shows that the lower bound in (9.5) is better than that in (9.8).

In [53], by directly proving the inequality (9.4) and

$$\frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\frac{5-4\sqrt{2}}{8}x^2(1-x)\left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\right), \tag{9.9}$$

the inequality (9.5) and an improved upper bound in (9.2),

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\mathrm{d}x < \frac{79}{192}+\frac{\sqrt{2}}{10}, \tag{9.10}$$

were obtained.

In [52], by considering an auxiliary function

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^2+\alpha x^2(1-x)\left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\right) \tag{9.11}$$

on $[0,1]$, inequalities (9.9) and

$$\frac{1}{\sqrt{4-x^2-x^3}} \geq \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2-\frac{1137\left(4\sqrt{2}-5\right)}{64\left(64-39\sqrt{2}\right)}(1-x)\left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\right) \tag{9.12}$$

were demonstrated to be sharp, and then, by integrating on both sides of (9.9), the inequality (9.10) was recovered.

9.2. Estimates for the Second Kind of Complete Elliptic Integrals

In [51], by discussing

$$\sqrt{1+k^2\cos^2 t}-\sqrt{1+k^2}+\frac{4}{\pi^2}\left(\sqrt{1+k^2}-1\right)t^2+\theta\left(\frac{\pi}{2}-t\right)t \tag{9.13}$$

or

$$\sqrt{1+k^2\cos^2 t}-\sqrt{1+k^2}+\frac{2}{\pi}\left(\sqrt{1+k^2}-1\right)t+\beta\left(\frac{\pi}{2}-t\right)t \tag{9.14}$$

on $[0,\pi/2]$, where θ and β are undetermined constants, the inequality

$$-\frac{8}{\pi^2}\left(\sqrt{1+k^2}-1\right)t\left(\frac{\pi}{2}-t\right)\le\sqrt{1+k^2\cos^2 t}-\left[\sqrt{1+k^2}-\frac{4}{\pi^2}\left(\sqrt{1+k^2}-1\right)t^2\right]\le 0 \tag{9.15}$$

for $t\in[0,\pi/2]$ was obtained, where $k^2=b^2/a^2-1$ and $a,b>0$. Integrating (9.15) yields

$$\frac{\pi}{6}(2a+b)<\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,\mathrm{d}t\le\frac{\pi}{6}(a+2b) \tag{9.16}$$

for $b>a$. When $b\ge 7a$, the right-hand side of the inequality (9.16) is stronger than the well-known result

$$\frac{\pi}{4}(a+b)\le\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,\mathrm{d}t\le\frac{\pi}{4}\sqrt{2(a^2+b^2)}, \tag{9.17}$$

which can be obtained by using some properties of definite integral or by applying the well-known Hermite-Hadamard double integral inequality for convex functions to the integral in question.

Remark 9.1. By employing Lemma 2.9, some inequalities for complete elliptic integrals, including the tighter upper bound for the elliptic integral of the second kind, were obtained in [143].

Remark 9.2. It is worthwhile to point out that some inequalities for bounding complete elliptic integrals of the first and second kinds are presented in [144].

9.3. Inequalities for the Remainder of Power Series Expansion of e^x

In [140, 145], by considering the auxiliary function

$$e^x-S_n(x)-\alpha_n x^{n+1}+\theta(b-x)x^{n+1} \tag{9.18}$$

for $0 \le x \le b \in (0,\infty)$, where $\alpha_{-1} = e^b$ and $\alpha_n = (1/b)(\alpha_{n-1} - 1/n!)$, the following inequalities of the reminder

$$R_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \tag{9.19}$$

for $n \ge 0$ and $x \in [0,\infty)$ were established:

$$\begin{gathered}
\frac{n+2-(n+1)x}{(n+2)!} x^{n+1} e^x \le R_n(x) \le \frac{n+1+e^x}{(n+2)!} x^{n+1} \le \frac{e^x}{(n+1)!} x^{n+1}, \\
\frac{(n+2)!}{(n-k+2)!} R_n(x) \le x^k R_{n-k}(x) + \frac{k}{(n-k+2)!} x^{n+1}, \quad 0 \le k \le n,
\end{gathered} \tag{9.20}$$

and, for $n \ge k \ge 1$,

$$x^k R_{n-k}(x) \le \frac{k x^{n+1} e^x}{(n+1)(n-k+2)!} - \frac{n! - (n-k+2)(n+1)!}{(n-k+2)!} R_n(x). \tag{9.21}$$

10. Estimates and Inequalities for Complete Elliptic Integrals

In this section, we continue to recite some estimates and inequalities for complete elliptic integrals and their new developments in recent years.

10.1. Inequalities between Three Kinds of Complete Elliptic Integrals

By using Tchebycheff's integral inequality [4, page 39, Theorem 9], the following inequalities between three kinds of complete elliptic integrals were derived in [146]:

$$\frac{\pi \arcsin k}{2k} < K(k) < \frac{\pi \ln((1+k)/(1-k))}{4k}; \tag{10.1}$$

$$E(k) < \frac{16 - 4k^2 - 3k^4}{4(4+k^2)} K(k); \tag{10.2}$$

$$K(k) < \left(1 + \frac{h}{2}\right) II(k,h), \quad -1 < h < 0 \text{ or } h > \frac{k^2}{2-3k^2} > 0; \tag{10.3}$$

$$II(k,h)E(k) > \frac{\pi^2}{4\sqrt{1+h}}, \quad -2 < 2h < k^2; \tag{10.4}$$

$$E(k) \ge \frac{16 - 28k^2 + 9k^4}{4(4-5k^2)} K(k), \quad k^2 \le \frac{2}{3}. \tag{10.5}$$

For $0 < 2h < k^2$, the inequality (10.3) is reversed. For $h > k^2/(2-3k^2) > 0$, the inequality (10.4) is reversed.

As concrete examples, the following estimates of the complete elliptic integrals are also deduced in [146]:

$$\frac{\pi^2}{4\sqrt{2}} < \int_0^{\pi/2} \left(1 - \frac{\sin^2 x}{2}\right)^{-1/2} \mathrm{d}x < \frac{\pi \ln\left(1+\sqrt{2}\right)}{\sqrt{2}}, \tag{10.6}$$

$$\int_0^{\pi/2} \left(1 + \frac{\cos x}{2}\right)^{-1} \mathrm{d}x < \frac{\pi(\ln 3 - \ln 2)}{2}, \tag{10.7}$$

$$\int_0^{\pi/2} \left(1 - \frac{\sin x}{2}\right)^{-1} \mathrm{d}x = \int_{\pi/2}^{\pi} \left(1 + \frac{\cos x}{2}\right)^{-1} \mathrm{d}x > \frac{\pi \ln 2}{2}. \tag{10.8}$$

These results are better than those in [7, page 607].

10.2. Carlson-Vuorinen's Inequality

In [147], the following inequality was proposed:

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \le \frac{\ln b - \ln a}{b - a}. \tag{10.9}$$

Equality holds if and only if $a = b$.

The inequality (10.9) was recovered in [148, Theorem 4].

There are two natural questions on bounding the complete elliptic integral in (10.9) to ask.

(1) What are the best constants $\beta > \alpha > 0$ such that the inequality

$$\left(\frac{\ln b - \ln a}{b - a}\right)^{\alpha} \le \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \le \left(\frac{\ln b - \ln a}{b - a}\right)^{\beta} \tag{10.10}$$

holds for all positive numbers a and b with $a \ne b$?

(2) Is the lower bound for (10.9) the reciprocal of the identric mean

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{10.11}$$

for positive numbers a and b with $a \ne b$?

Since the complete elliptic integral in (10.9) tends to infinity as the ratio b/a for $a > b > 0$ tends to zero, so we think that the former question is more significant.

For more information on the origin, refinements, extensions, and generalizations of the inequality (10.9), please refer to [149–151] and closely related references therein.

10.3. Some Recent Results of Elliptic Integrals

The double inequality (10.1) was strengthened in [152, Theorem 4.1] and [153, (1.13)].

It was pointed in [154] that the right-hand side inequality in (10.1) is a recovery of [155, Theorem 3.10]. In [154], the inequality (10.1) was also generalized to the case of generalized complete elliptic integrals by the same method as in [22, 146].

Some tighter inequalities than inequalities (10.2) and (10.5) were contained in [153, (3.21)].

The elliptic integral appeared in (10.6) is $K(1/\sqrt{2})$ which can be found in [156].

In [157], some of the results in [154] were further improved.

Addendum

This article is a revised and updated version of the papers [108, 158].

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