Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 258756, 10 pages
doi:10.1155/2009/258756

Research Article

Optimality Conditions and Duality for DC Programming in Locally Convex Spaces

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Received 10 February 2009; Revised 25 June 2009; Accepted 25 September 2009

Recommended by K. L. Teo

Consider the DC programming problem $(P_A)\inf_{x\in X}\{f(x)-g(Ax)\}$, where f and g are proper convex functions defined on locally convex Hausdorff topological vector spaces X and Y, respectively, and A is a linear operator from X to Y. By using the properties of the epigraph of the conjugate functions, the optimality conditions and strong duality of (P_A) are obtained.

1. Introduction

Let X and Y be real locally convex Hausdorff topological vector spaces, whose respective dual spaces, X^* and Y^*, are endowed with the weak*-topologies $w^*(X^*,X)$ and $w^*(Y^*,Y)$. Let $f : X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$, $g : Y \to \overline{\mathbb{R}}$ be proper convex functions, and let $A : X \to Y$ be a linear operator such that $A(\operatorname{dom} f)\cap \operatorname{dom} g \neq \emptyset$. We consider the primal DC (difference of convex) programming problem

$$(P_A) \quad \inf_{x\in X}\{f(x)-g(Ax)\}, \tag{1.1}$$

and its associated dual problem

$$(D_A) \quad \inf_{y^*\in Y_A^*}\{g^*(y^*)-f^*(A^*y^*)\}, \tag{1.2}$$

where f^* and g^* are the Fenchel conjugates of f and g, respectively, and $A^* : Y_A^* \to X^*$ stands for the adjoint operator, where Y_A^* is the subspace of Y^* such that $y^*\in Y_A^*$ if and only if A^*y^*

defined by $\langle A^*y^*, \cdot\rangle = \langle y^*, A(\cdot)\rangle$ is continuous on X. Note that, in general, Y_A^* is not the whole space Y^* because A is not necessarily continuous.

Problems of DC programming are highly important from both viewpoints of optimization theory and applications. They have been extensively studied in the literature; see, for example, [1–6] and the references therein. On one hand, such problems being heavily nonconvex can be considered as a special class in nondifferentiable programming (in particular, quasidifferentiable programming [7]) and thus are suitable for applying advanced techniques of variational analysis and generalized differentiation developed, for example, in [7–10]. On the other hand, the special convex structure of both plus function f and minus function $g \circ A$ in the objective of (1.1) offers the possibility to use powerful tools of convex analysis in the study of DC Programming.

DC programming of type (1.1) (when A is an identity operator) has been considered in the $\mathbb{R}^n$ space in paper [5], where the authors obtained some necessary optimality conditions for local minimizers to (1.1) by using refined techniques and results of convex analysis. In this paper, we extend these results to DC programming in topological vector spaces and also derive some new necessary and/or sufficient conditions for local minimizers to (1.1). Finally, we consider the strong duality of problem (1.1); that is, there is no duality gap between the problem (P_A) and the dual problem (D_A) and (D_A) has at least an optimal solution.

In this paper we study the optimality conditions and the strong duality between (P_A) and (D_A) in the most general setting, namely, when f and g are proper convex functions (not necessarily lower semicontinuous) and A is a linear operator (not necessarily continuous). The rest of the paper is organized as follows. In Section 2 we present some basic definitions and preliminary results. The optimality conditions are derived in Section 3, and the strong duality of DC programming is obtained in Section 4.

2. Notations and Preliminary Results

The notation used in the present paper is standard (cf. [11]). In particular, we assume throughout the paper that X and Y are real locally convex Hausdorff topological vector spaces, and let X^* denote the dual space, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$ we will denote the value of the functional $x^* \in X^*$ at $x \in X$, that is, $\langle x^*, x\rangle = x^*(x)$. The zero of each of the involved spaces will be indistinctly represented by 0.

Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. The effective domain and the epigraph of f are the nonempty sets defined by

$$\begin{aligned} \operatorname{dom} f &:= \{x \in X : f(x) < +\infty\}, \\ \operatorname{epi} f &:= \{(x, r) \in X \times \mathbb{R} : \quad f(x) \le r\}. \end{aligned} \tag{2.1}$$

The conjugate function of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}. \tag{2.2}$$

If f is lower semicontinuous, then the following equality holds:

$$f^{**} = f. \tag{2.3}$$

Let $x \in \operatorname{dom} f$. For each $\epsilon \geq 0$, the ϵ-subdifferential of f at x is the convex set defined by

$$\partial_\epsilon f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle - \epsilon \leq f(y) - f(x) \text{ for each } y \in X\}. \tag{2.4}$$

When $x \notin \operatorname{dom} f$, we put $\partial_\epsilon f(x) := \emptyset$. If $\epsilon = 0$ in (2.4), the set $\partial f(x) := \partial_0 f(x)$ is the classical subdifferential of convex analysis, that is,

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \text{ for each } y \in X\}. \tag{2.5}$$

Let $\epsilon > 0$, the following inequality holds (cf. [11, Theorem 2.4.2(ii)]):

$$f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \epsilon \Longleftrightarrow x^* \in \partial_\epsilon f(x). \tag{2.6}$$

Following [12],

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geq 0} \{(x^*, \langle x^*, x\rangle - f(x) + \epsilon) : x^* \in \partial_\epsilon f(x)\}. \tag{2.7}$$

The Young equality holds

$$f(x) + f^*(x^*) = \langle x^*, x\rangle \Longleftrightarrow x^* \in \partial f(x). \tag{2.8}$$

As a consequence of that,

$$(x^*, \langle x^*, x\rangle - f(x)) \in \operatorname{epi} f^* \quad \forall x^* \in \partial f(x). \tag{2.9}$$

The following notion of Cartesian product map is used in [13].

Definition 2.1. Let M_1, M_2, N_1, N_2 be nonempty sets and consider maps $F : M_1 \to M_2$ and $G : N_1 \to N_2$. We denote by $F \times G : M_1 \times N_1 \to M_2 \times N_2$ the map defined by

$$(F \times G)(x, y) := (F(x), G(y)). \tag{2.10}$$

3. Optimality Conditions

Let $\mathrm{id}_{\mathbb{R}}$ denote the identity map on $\mathbb{R}$. We consider the image set $(A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi} g^*)$ of $\operatorname{epi} g^*$ through the map $A^* \times \mathrm{id}_{\mathbb{R}} : Y_A^* \times \mathbb{R} \to X^* \times \mathbb{R}$, that is,

$$(x^*, r) \in (A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi} g^*) \Longleftrightarrow \exists y^* \in Y_A^* \text{ such that } (y^*, r) \in \operatorname{epi} g^* \text{ and } A^* y^* = x^*. \tag{3.1}$$

By [14, Proposition 4.1] and the well-known characterization of optimal solution to DC problem, we obtain the following lemma.

Lemma 3.1. *Let ϕ_1, ϕ_2 be proper convex fucntions on X, and let $\phi = \phi_1 - \phi_2$. Then x_0 is a local minimizer of ϕ if and only if, for each $\epsilon \geq 0$*

$$\partial_\epsilon \phi_2(x_0) \subseteq \partial_\epsilon \phi_1(x_0). \tag{3.2}$$

Especially, if x_0 is a local minimizer of ϕ, then

$$\partial \phi_2(x_0) \subseteq \partial \phi_1(x_0). \tag{3.3}$$

Theorem 3.2. *The following statements are equivalent:*

(i) $\operatorname{epi}(g \circ A)^* = (A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi} g^*)$,

(ii) *For each $x_0 \in A^{-1}(\operatorname{dom} g)$ and each $\epsilon \geq 0$,*

$$\partial_\epsilon (g \circ A)(x_0) = A^* \partial_\epsilon g(Ax_0). \tag{3.4}$$

Moreover, x_0 is a local optimal solution to problem (P_A) if and only if for each $\epsilon \geq 0$,

$$A^* \partial_\epsilon g(Ax_0) \subseteq \partial_\epsilon (g \circ A)(x_0) \subseteq \partial_\epsilon f(x_0). \tag{3.5}$$

Proof. (i)⇒(ii). Suppose that (i) holds. Let $x_0 \in A^{-1}(\operatorname{dom} g), \epsilon \geq 0$, and $u \in Y_A^* \cap \partial_\epsilon g(Ax_0)$, then for each $x \in X$,

$$\langle A^* u, x - x_0 \rangle = \langle u, Ax - Ax_0 \rangle \leq g(Ax) - g(Ax_0) + \epsilon. \tag{3.6}$$

Therefore, $A^* u \in \partial_\epsilon (g \circ A)(x_0)$. Hence, $A^* \partial_\epsilon g(Ax_0) \subseteq \partial_\epsilon (g \circ A)(x_0)$.

Conversely, let $v \in Y_A^* \cap \partial_\epsilon (g \circ A)(x_0)$. Then $(v, \langle v, x_0 \rangle - (g \circ A)(x_0) + \epsilon) \in \operatorname{epi}(g \circ A)^*$. By (i),

$$(v, \langle v, x_0 \rangle - (g \circ A)(x_0) + \epsilon) \in (A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi} g^*). \tag{3.7}$$

Therefore, there exists $w \in Y_A^*$ such that $A^* w = v$ and $g^*(w) \leq \langle v, x_0 \rangle - g(Ax_0) + \epsilon$. Noting that $\langle A^* w, x_0 \rangle = \langle w, Ax_0 \rangle$, then

$$0 \leq \langle v, x_0 \rangle - g(Ax_0) - g^*(w) + \epsilon = \langle w, Ax_0 \rangle - g(Ax_0) - g^*(w) + \epsilon. \tag{3.8}$$

This implies $w \in \partial g(Ax_0)$ thanks to (2.6). Thus, $v = A^* w \in A^* \partial_\epsilon g(Ax_0)$ and $\partial_\epsilon (g \circ A)(x_0) \subseteq A^* \partial_\epsilon g(Ax_0)$. Hence, (3.4) is seen to hold.

(ii)⇒(i). Suppose that (ii) holds. To show (i), it suffices to show that $\operatorname{epi}(g \circ A)^* \subseteq (A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi} g^*)$. To do this, let $(x^*, r) \in \operatorname{epi}\ (g \circ A)^*$ and $x_0 \in A^{-1}(\operatorname{dom} g)$. By (2.7), there exists $\epsilon \geq 0$ such that $x^* \in \partial_\epsilon (g \circ A)(x_0)$ and $r = \langle x^*, x_0 \rangle - g(Ax_0) + \epsilon$. From (3.4), there exists $y^* \in \partial_\epsilon g(Ax_0)$ such that $x^* = A^* y^*$. Since $y^* \in \partial_\epsilon g(Ax_0)$, it follows from (2.6) that

$$g^*(y^*) + g(Ax_0) \leq \langle y^*, Ax_0 \rangle + \epsilon = \langle x^*, x_0 \rangle + \epsilon, \tag{3.9}$$

that is $g^*(y^*) \leq \langle x^*, x_0\rangle - g(Ax_0) + \epsilon = r$. Hence, $(x^*, r) \in (A^* \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, g^*)$ and so $\mathrm{epi}\,(g \circ A)^* \subseteq (A^* \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, g^*)$.

By the well-known characterization of optimal solution to DC problem (see Lemma 3.1), x_0 is a local optimal solution to problem (P_A) if and only if, for each $\epsilon \geq 0$,

$$\partial_\epsilon (g \circ A)(x_0) \subseteq \partial_\epsilon f(x_0). \tag{3.10}$$

Obviously, $A^* \partial_\epsilon g(Ax_0) \subseteq \partial_\epsilon (g \circ A)(x_0)$ holds automatically. The proof is complete. □

Let $p \in Y$. Define

$$(f^* \circ A^*)^*_A(p) := \sup_{y^* \in Y^*_A} \{\langle p, y^*\rangle - f^*(A^* y^*)\}. \tag{3.11}$$

Theorem 3.3. *The following statements are equivalent:*

(i) $\mathrm{epi}(f^* \circ A^*)^*_A = (A \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, f)$,

(ii) *For each* $\epsilon \geq 0$ *and each* $y^* \in Y^*_A \cap A^*(\mathrm{dom} f^*)$,

$$\partial_\epsilon (f^* \circ A^*)(y^*) = A \partial_\epsilon f^*(A^* y^*). \tag{3.12}$$

Moreover, y^* *is a local optimal solution to problem* (D_A) *if and only if, for each* $\epsilon \geq 0$,

$$A \partial_\epsilon f^*(A^* y^*) \subseteq \partial_\epsilon (f^* \circ A^*)(y^*) \subseteq \partial_\epsilon g^*(y^*). \tag{3.13}$$

Proof. (i)⇒(ii). Suppose that (i) holds. Let $\epsilon \geq 0$, $y^* \in Y^*_A \cap A^*(\mathrm{dom}\, f^*)$ and $y \in \partial_\epsilon (f^* \circ A^*)(y^*)$. Then one has

$$(f^* \circ A^*)^*_A(y) + (f^* \circ A^*)(y^*) \leq \langle y, y^*\rangle + \epsilon. \tag{3.14}$$

Hence, $(y, \langle y, y^*\rangle - f^*(A^* y^*) + \epsilon) \in \mathrm{epi}(f^* \circ A^*)^*_A$. By the given assumption,

$$(y, \langle y, y^*\rangle - f^*(A^* y^*) + \epsilon) \in (A \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, f). \tag{3.15}$$

Therefore, there exists $x \in X$ such that $Ax = y$ and $(x, \langle y, y^*\rangle - f^*(A^* y^*) + \epsilon) \in \mathrm{epi}\, f$. Hence, $f(x) \leq \langle y, y^*\rangle - f^*(A^* y^*) + \epsilon$, this means $x \in \partial_\epsilon f^*(A^* y^*)$ and so $Ax \in A \partial_\epsilon f^*(A^* y^*)$. Consequently, $\partial_\epsilon (f^* \circ A^*)(y^*) \subseteq A \partial_\epsilon f^*(A^* y^*)$. This completes the proof because the converse inclusion holds automatically.

(ii)⇒(i). Suppose that (ii) holds. To show (i), it suffice to show that epi $(f^* \circ A^*)^*_A \subseteq (A \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, f)$. To do this, let $(y, r) \in \mathrm{epi}(f^* \circ A^*)^*_A$ and $y^* \in Y^*_A \cap A^*(\mathrm{dom}\ f^*)$. By (2.7), there exists $\epsilon \geq 0$ such that $y \in \partial_\epsilon (f^* \circ A^*)(y^*)$ and $r = \langle y^*, y\rangle - f^*(A^* y^*) + \epsilon$. From (3.12), there exists $x \in \partial_\epsilon f^*(A^* y^*)$ such that $y = Ax$. Since $x \in \partial_\epsilon f^*(A^* y^*)$, it follows from (2.6) that

$$f^*(A^* y^*) + f(x) \leq \langle y^*, y\rangle + \epsilon, \tag{3.16}$$

that is $f(x) \le \langle y^*, y\rangle + \epsilon - f^*(A^*y^*) = r$. Hence, $(y,r) \in (A\times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\ f)$ and so $\mathrm{epi}\,(f^* \circ A^*)^*_A \subseteq (A \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}\, f)$.

Similar to the proof of (3.5), one has that (3.13) holds. □

4. Duality in DC Programming

This section is devoted to study the strong duality between the primal problem and its Toland dual, namely, the property that both optimal values coincide and the dual problem has at least an optimal solution.

Given $p \in X^*$, we consider the DC programming problem given in the form

$$\left(P_{(A,p)}\right) \inf_{x\in X}\{f(x) - g(Ax) - \langle p, x\rangle\}, \tag{4.1}$$

and the corresponding dual problem

$$\left(D_{(A,p)}\right) \inf_{y^*\in Y_A^*}\{g^*(y^*) - f^*(p + A^*y^*)\}. \tag{4.2}$$

Let $v(P_{(A,p)}), v(D_{(A,p)})$ denote the optimal values of problems $(P_{(A,p)})$ and $(D_{(A,p)})$, respectively, that is

$$\begin{aligned} v\left(P_{(A,p)}\right) &= \inf_{x\in X}\{f(x) - g(Ax) - \langle p, x\rangle\}, \\ v\left(D_{(A,p)}\right) &= \inf_{y^*\in Y_A^*}\{g^*(y^*) - f^*(p + A^*y^*)\}. \end{aligned} \tag{4.3}$$

In the special case when $p = 0$, problems $(P_{(A,p)})$ and $(D_{(A,p)})$ are just the problem (P_A) and (D_A).

Before establishing the relationship between problems $(P_{(A,p)})$ and $(D_{(A,p)})$, we give useful formula for computing the values of conjugate functions. The formula is an extension of a well-known result, called Toland duality, for DC problems. In this section, we always assume that g and f^* are everywhere subdifferentible.

Proposition 4.1. *Let $h = f - g \circ A$. Then the conjugate function h^* of h is given by*

$$h^*(x^*) = \sup_{y^*\in Y_A^*}\{f^*(x^* + A^*y^*) - g^*(y^*)\}. \tag{4.4}$$

Proof. By the definition of conjugate function, it follows that

$$\begin{aligned} h^*(x^*) &= \sup_{x\in X}\{\langle x^*, x\rangle - (f - g\circ A)(x)\} \\ &= \sup_{x\in X}\{\langle x^*, x\rangle - f(x) + g(Ax)\} \\ &= \sup_{x\in X}\{\langle x^*, x\rangle + \langle A^*y^*, x\rangle - f(x) - \langle A^*y^*, x\rangle + g(Ax)\} \quad \forall y^*\in Y_A^* \\ &\geq \sup_{x\in X}\{\langle x^* + A^*y^*, x\rangle - f(x)\} - \sup_{x\in X}\{\langle y^*, Ax\rangle - g(Ax)\} \quad \forall y^*\in Y_A^* \\ &\geq \sup_{y^*\in Y_A^*}\{f^*(x^* + A^*y^*) - g^*(y^*)\}. \end{aligned} \tag{4.5}$$

Next, we prove that

$$h^*(x^*) \leq \sup_{y^*\in Y_A^*}\{f^*(x^* + A^*y^*) - g^*(y^*)\}. \tag{4.6}$$

Suppose on the contrary that $h^*(x^*) > \sup_{y^*\in Y_A^*}\{f^*(x^* + A^*y^*) - g^*(y^*)\}$, that is, there exists $x_0 \in X$ such that

$$\langle x^*, x_0\rangle - f(x_0) + g(Ax_0) > \sup_{y^*\in Y_A^*}\{f^*(x^* + A^*y^*) - g^*(y^*)\}. \tag{4.7}$$

Let $y_0 = Ax_0$ and $y_0^* \in \partial g(y_0)$, then

$$g^*(y_0^*) = \langle y_0^*, y_0\rangle - g(y_0). \tag{4.8}$$

From this, it follows that

$$\begin{aligned} \langle x^*, x_0\rangle - f(x_0) + g(Ax_0) &= \langle x^* + A^*y^*, x_0\rangle - f(x_0) - (\langle A^*y^*, x_0\rangle - g^*(y_0^*)) \\ &\leq f^*(x^* + A^*y^*) - g^*(y_0)^*, \end{aligned} \tag{4.9}$$

which is contradiction to (4.7), and so (4.4) holds. □

Following from Proposition 4.1, we obtain the following proposition.

Proposition 4.2. *For each $p \in X^*$,*

$$v\left(P_{(A,p)}\right) = v\left(D_{(A,p)}\right). \tag{4.10}$$

Proof. Let $p \in X^*$. Since $\inf_{x\in X}\{f(x) - g(Ax) - \langle p, x\rangle\} = -(f - g \circ A)^*(p)$, it follows from (4.4) that

$$\begin{aligned} v\left(P_{(A,p)}\right) &= \inf_{x\in X}\{f(x) - g(Ax) - \langle p, x\rangle\} \\ &= -\sup_{y^*\in Y_A^*}\{f^*(p + A^*y^*) - g^*(y^*)\} \\ &= \inf_{y^*\in Y_A^*}\{g^*(y^*) - f^*(p + A^*y^*)\} \\ &= v\left(D_{(A,p)}\right). \end{aligned} \tag{4.11}$$

□

Remark 4.3. In the special case when $p = 0$ and $A = 0$, formula (4.10) was first given by Pshenichnyi (see [10]) and related results on duality can be found in [15–17].

Proposition 4.4. *For each $p \in X^*$,*

(i) *if x_0 is an optimal solution to problem $(P_{(A,p)})$, then $y_0^* \in Y_A^* \cap \partial g(Ax_0)$ is an optimal solution to problem $(D_{(A,p)})$;*

(ii) *suppose that f and g are lower semicontinuous. If y_0^* is an optimal solution to problem $(D_{(A,p)})$, then $x_0 \in \partial f^*(A^*y_0^*)$ is an optimal solution to problem $(P_{(A,p)})$.*

Proof. (i) Let x_0 be an optimal solution to problem $(P_{(A,p)})$ and let $y_0^* \in Y_A^* \cap \partial g(Ax_0)$. Then $A^*y_0^* \in A^*\partial g(Ax_0)$. It follows from (3.5) that $A^*y_0^* \in \partial f(x_0)$. By the Young equality, we have

$$\begin{aligned} \left\langle A^*y_0^*, x_0\right\rangle = \left\langle A^*y_0^* + p, x_0\right\rangle - \langle p, x_0\rangle &= f^*(p + A^*y_0^*) + f(x_0) - \langle p, x_0\rangle, \\ \left\langle y_0^*, Ax_0\right\rangle &= g^*(y_0^*) + g(Ax_0). \end{aligned} \tag{4.12}$$

Therefore,

$$g^*(y_0^*) - f^*(p + A^*y_0^*) = f(x_0) - g(Ax_0) - \langle p, x_0\rangle. \tag{4.13}$$

By (4.10), y_0^* is an optimal solution to problem $(D_{(A,p)})$.

(ii) Let y_0^* be an optimal solution to problem $(D_{(A,p)})$ and $x_0 \in \partial f^*(A^*y_0^*)$. Then $Ax_0 \in A\partial f^*(A^*y_0^*)$ and hence $Ax_0 \in \partial g^*(y_0^*)$ thanks to Theorem 3.3. By the Young equality, we have

$$\begin{aligned} \left\langle A^*y_0^*, x_0\right\rangle = \left\langle A^*y_0^* + p, x_0\right\rangle - \langle p, x_0\rangle &= f^{**}(x_0) + f^*(p + A^*y_0^*) - \langle p, x_0\rangle, \\ \left\langle y_0^*, Ax_0\right\rangle &= g^{**}(Ax_0) + g^*(y_0^*). \end{aligned} \tag{4.14}$$

Since the functions f and g are lower semicontinuous, it follows from (2.3) that $f^{**} = f$ and $g^{**} = g$. Hence, by the above two equalities, one has

$$g^*(y_0^*) - f^*(p + A^*y_0^*) = f(x_0) - g(Ax_0) - \langle p, x_0\rangle. \tag{4.15}$$

By (4.10), x_0 is an optimal solution to problem $(P_{(A,p)})$. □

Obviously, if A is continuous, then $Y_A^* = Y^*$ and so $Y_A^* \cap \partial g(Ax) \neq \emptyset$ for each $x \in X$. By Propositions 4.2 and 4.4, we get the following strong duality theorem straightforwardly.

Theorem 4.5. *For each $p \in X^*$,*

(i) *suppose that A is continuous. If the problem $(P_{(A,p)})$ has an optimal solution, then $v(P_{(A,p)}) = v(D_{(A,p)})$ and $(D_{(A,p)})$ has an optimal solution;*

(ii) *suppose that f and g are lower semicontinuous. If the problem $(D_{(A,p)})$ has an optimal solution, then $v(P_{(A,p)}) = v(D_{(A,p)})$ and $(P_{(A,p)})$ has an optimal solution.*

Corollary 4.6. *(i) If the problem (P_A) has an optimal solution, then $v(P_A) = v(D_A)$ and (D_A) has an optimal solution.*

(ii)Suppose that f and g are lower semicontinuous. If the problem (D_A) has an optimal solution, then $v(P_A) = v(D_A)$ and (P_A) has an optimal solution.

Remark 4.7. As in [13], if $v(P_A) = v(D_A)$ and (P_A) has an optimal solution, then we say the converse duality holds between (P_A) and (D_A).

Example 4.8. Let $X = Y = \mathbb{R}$ and let $A = \mathrm{id}$. Define $f, g : X \to \mathbb{R}$ by

$$f(x) = x^4, \qquad g(x) = 2x^2. \tag{4.16}$$

Then the conjugate functions f^* and g^* are

$$f^*(p) = p\left(\frac{p}{4}\right)^{1/3} - \left(\frac{p}{4}\right)^{4/3}, \quad g^*(p) = \frac{p^2}{8}, \quad p \in \mathbb{R}. \tag{4.17}$$

Obviously, $v(P_A) := \inf_{x\in\mathbb{R}}\{f(x) - g(x)\} = -1$ and (P_A) attained the infimun at ± 1, $v(D_A) = \inf_{p\in\mathbb{R}}\{g^*(p) - f^*(p)\} = -1$ and (D_A) attained the infimum at ± 4. Hence, $v(P) = v(D)$. It is easy to see that $\partial g(1) = \{4\}, \partial g(-1) = \{-4\}$ and $\partial f^*(4) = \{1\}, \partial f^*(-4) = \{-1\}$. Therefore, Proposition 4.4 is seen to hold and Theorem 4.5 is applicable.

Acknowledgment

The author wish to thank the referees for careful reading of this paper and many valuable comments, which helped to improve the quality of the paper.

References

[1] L. T. H. An and P. D. Tao, "The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems," *Annals of Operations Research*, vol. 133, pp. 23–46, 2005.

[2] R. I. Boţ and G. Wanka, "Duality for multiobjective optimization problems with convex objective functions and D.C. constraints," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 2, pp. 526–543, 2006.

[3] N. Dinh, T. T. A. Nghia, and G. Vallet, "A closedness condition and its applications to DC programs with convex constraints," *Optimization*, vol. 1, pp. 235–262, 2008.

[4] N. Dinh, G. Vallet, and T. T. A. Nghia, "Farkas-type results and duality for DC programs with convex constraints," *Journal of Convex Analysis*, vol. 15, no. 2, pp. 235–262, 2008.

[5] R. Horst and N. V. Thoai, “DC programming: overview,” *Journal of Optimization Theory and Applications*, vol. 103, no. 1, pp. 1–43, 1999.
[6] J.-E. Martínez-Legaz and M. Volle, “Duality in D.C. programming: the case of several D.C. constraints,” *Journal of Mathematical Analysis and Applications*, vol. 237, no. 2, pp. 657–671, 1999.
[7] V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, vol. 7 of *Approximation & Optimization*, Peter Lang, Frankfurt, Germany, 1995.
[8] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I: Basic Theory*, vol. 330 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 2006.
[9] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. II: Application*, vol. 331 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 2006.
[10] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, vol. 317 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1998.
[11] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ, USA, 2002.
[12] V. Jeyakumar, G. M. Lee, and N. Dinh, “New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs,” *SIAM Journal on Optimization*, vol. 14, no. 2, pp. 534–547, 2003.
[13] R. I. Boţ and G. Wanka, “A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 12, pp. 2787–2804, 2006.
[14] B. S. Mordukhovich, N. M. Nam, and N. D. Yen, “Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming,” *Optimization*, vol. 55, no. 5-6, pp. 685–708, 2006.
[15] I. Singer, “A general theory of dual optimization problems,” *Journal of Mathematical Analysis and Applications*, vol. 116, no. 1, pp. 77–130, 1986.
[16] I. Singer, “Some further duality theorems for optimization problems with reverse convex constraint sets,” *Journal of Mathematical Analysis and Applications*, vol. 171, no. 1, pp. 205–219, 1992.
[17] M. Volle, “Concave duality: application to problems dealing with difference of functions,” *Mathematical Programming*, vol. 41, no. 2, pp. 261–278, 1988.