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Research Article

On a Multiple Hilbert-Type Integral Operator and Applications

Qiliang Huang and Bicheng Yang

Department of Mathematics, Guangdong Institute of Education, Guangzhou, Guangdong 510303, China

Correspondence should be addressed to Qiliang Huang, qlhuang@yeah.net

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By using the way of weight functions and the technic of real analysis, a multiple Hilbert-type integral operator with the homogeneous kernel of $-\lambda$-degree ($\lambda \in \mathbf{R}$) and its norm are considered. As for applications, two equivalent inequalities with the best constant factors, the reverses, and some particular norms are obtained.

1. Introduction

If $p > 1, 1/p + 1/q = 1$, $f(\geq 0) \in L^p(0,\infty), g(\geq 0) \in L^q(0,\infty), \|f\|_p = \{\int_0^\infty f^p(x)dx\}^{1/p} > 0, \|g\|_q > 0$, then we have the following famous Hardy-Hilbert's integral inequality and its equivalent form (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \tag{1.1}$$

$$\left\{\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy\right\}^{1/p} < \frac{\pi}{\sin(\pi/p)} \|f\|_p, \tag{1.2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Define the Hardy-Hilbert's integral operator $T : L^p(0,\infty) \to L^p(0,\infty)$ as follows: for $f \in L^p(0,\infty), Tf(y) := \int_0^\infty (1/(x+y))f(x)dx$ $(y \in (0,\infty))$. Then in view of (1.2), it follows that $\|Tf\|_p < \pi/\sin(\pi/p)\|f\|_p$ and $\|T\| \leq (\pi/\sin(\pi/p))$. Since the constant factor in (1.2) is the best possible, we find that (cf.[2]) $\|T\| = \pi/\sin(\pi/p)$.

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [3]). In 2002, reference [4] considered the property of Hardy-Hilbert's integral operator and gave an improvement of (1.1) (for $p = q = 2$). In 2004-2005, introducing another pair of conjugate exponents $(r,s)(r > 1, 1/r + 1/s = 1)$ and an independent parameter $\lambda > 0$, [5, 6] gave two best extensions of (1.1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx\, dy < \frac{\pi}{\lambda \sin(\pi/r)} \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\, dy < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{1.4}$$

where $B(u,v)$ is the Beta function ($\phi(x) = x^{p(1-\lambda/r)-1}$, $\psi(x) = x^{q(1-\lambda/s)-1}, \|f\|_{p,\phi} := \{\int_0^\infty \phi(x) f^p(x) dx\}^{1/p} > 0, \|g\|_{q,\psi} > 0$). In 2009, [7, Theorem 9.1.1] gave the following multiple Hilbert-type integral inequality: suppose that $n \in \mathbf{N} \setminus \{1\}, p_i > 1, \sum_{i=1}^n (1/p_i) = 1, \lambda > 0$, then $k_\lambda(x_1, \ldots, x_n) \geq 0$ is a measurable function of $-\lambda$-degree in $\mathbf{R}_+^n$ and for any $(r_1, \ldots, r_n)(r_i > 1)$ satisfies $\sum_{i=1}^n (1/r_i) = 1$ and

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{(\lambda/r_j)-1} du_1 \cdots du_{n-1} > 0. \tag{1.5}$$

If $\phi_i(x) = x^{p_i(1-\lambda/r_i)-1}, f_i(\geq 0) \in L_{\phi_i}^{p_i}(0,\infty), \|f\|_{p_i,\phi_i} > 0 (i = 1, \ldots, n)$, then we have the following inequality:

$$\int_{\mathbf{R}_+^n} k_\lambda(x_1, \ldots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < k_\lambda \prod_{i=1}^n \|f_i\|_{p_i,\phi_i}, \tag{1.6}$$

where the constant factor k_λ is the best possible. For $n = 2, k_\lambda(x,y) = 1/(x^\lambda + y^\lambda)$, and $1/(x+y)^\lambda$ in (1.6), we obtain (1.3) and (1.4). Inequality (1.6) is some extensions of the results in [6, 8–11]. In 2006, reference [12] also considered a multiple Hilbert-type integral operator with the homogeneous kernel of $-n+1$-degree and its inequality with the norm, which is the best extension of (1.2).

In this paper, by using the way of weight functions and the technic of real analysis, a new multiple Hilbert-type integral operator with the norm is considered, which is an extension of the result in [12]. As for applications, an extended multiple Hilbert-type integral inequality and the equivalent form, the reverses, and some particular norms are obtained.

2. Some Lemmas

Lemma 2.1. *If $n \in \mathbf{N} \setminus \{1\}, p_i \in \mathbf{R} \setminus \{0,1\}, \lambda_i \in \mathbf{R}$ $(i = 1, \ldots, n), \sum_{i=1}^n 1/p_i = 1$, then*

$$A := \prod_{i=1}^n \left[x_i^{(\lambda_i - 1)(1-p_i)} \prod_{j=1 (j \neq i)}^n x_j^{\lambda_j - 1} \right]^{1/p_i} = 1. \tag{2.1}$$

Proof. We find that

$$A := \prod_{i=1}^{n}\left[x_i^{(\lambda_i-1)(1-p_i)+1-\lambda_i}\prod_{j=1}^{n}x_j^{\lambda_j-1}\right]^{1/p_i}$$
$$= \prod_{i=1}^{n}\left[x_i^{(1-\lambda_i)p_i}\prod_{j=1}^{n}x_j^{\lambda_j-1}\right]^{1/p_i} = \prod_{i=1}^{n}x_i^{1-\lambda_i}\left(\prod_{j=1}^{n}x_j^{\lambda_j-1}\right)^{\sum_{i=1}^{n}(1/p_i)}, \tag{2.2}$$

and then (2.1) is valid. □

Definition 2.2. If $n \in \mathbf{N}, \mathbf{R}_+^n := \{(x_1,\ldots,x_n)x_i > 0\ (i = 1,\ldots,n)\}, \lambda \in \mathbf{R}$, and $k_\lambda(x_1,\ldots,x_n)$ is a measurable function in $\mathbf{R}_+^n$ such that for any $u > 0$ and $(x_1,\ldots,x_n) \in \mathbf{R}_+^n, k_\lambda(ux_1,\ldots,ux_n) = u^{-\lambda}k_\lambda(x_1,\ldots,x_n)$, then call $k_\lambda(x_1,\ldots,x_n)$ the homogeneous function of $-\lambda$-degree in $\mathbf{R}_+^n$.

Lemma 2.3. *As for the assumption of Lemma 2.1, if $\sum_{i=1}^{n}\lambda_i = \lambda, k_\lambda(x_1,\ldots,x_n) \geq 0$ is a homogeneous function of $-\lambda$-degree in $\mathbf{R}_+^n$,*

$$H(i) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{i-1},1,u_{i+1},\ldots,u_n)\prod_{j=1(j\neq i)}^{n}u_j^{\lambda_j-1}du_1\cdots du_{i-1}du_{i+1}\cdots du_n \tag{2.3}$$

$(i = 1,\ldots,n)$, and $H(n) = k_\lambda \in \mathbf{R}$, then each $H(i) = H(n) = k_\lambda (i = 1,\ldots,n)$, and for any $i = 1,\ldots,n$, it follows that

$$\omega_i(x_i) := x_i^{\lambda_i}\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\prod_{j=1(j\neq i)}^{n}x_j^{\lambda_j-1}dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n = k_\lambda. \tag{2.4}$$

Proof. Setting $u_j = u_n v_j\ (j \neq i, n)$ in the integral $H(i)$, we find that

$$H(i) = \int_{\mathbf{R}_+^{n-1}} k_\lambda\left(v_1,\ldots,v_{i-1},u_n^{-1},v_{i+1},\ldots,v_{n-1},1\right)\prod_{j=1(j\neq i)}^{n-1}v_j^{\lambda_j-1}u_n^{-1-\lambda_i}dv_1\cdots dv_{i-1}dv_{i+1}\cdots dv_{n-1}du_n. \tag{2.5}$$

Setting $v_i = u_n^{-1}$ in the above integral, we obtain $H(i) = H(n)$. Setting $u_j = x_j/x_i\ (j \neq i)$ in (2.4), we find that $\omega_i(x_i) = H(i) = H(n) = k_\lambda$. □

Lemma 2.4. *As for the assumption of Lemma 2.3, setting*

$$k\left(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{n-1}\right) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1}u_j^{\tilde{\lambda}_j-1}du_1\cdots du_{n-1}, \tag{2.6}$$

then there exist $\delta_0 > 0$ *and* $I = \{(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1}) \mid \widetilde{\lambda}_i = \lambda_i + \delta_i, |\delta_i| \le \delta_0\ (i = 1,\ldots,n-1)\}$, *such that for any* $(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1}) \in I, k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1}) \in \mathbf{R}$, *if and only if* $k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ *is continuous at* $(\lambda_1,\ldots,\lambda_{n-1})$.

Proof. The sufficiency property is obvious. We prove the necessary property of the condition by mathematical induction. For $n = 2$, since

$$
\begin{gathered}
k(\lambda_1+\delta_1) = \int_0^1 k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1}du_1 + \int_1^\infty k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1}du_1,\\
k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1} \le k_\lambda(u_1,1)u_1^{\lambda_1-\delta_0-1}du_1, \quad u_1 \in (0,1],\\
k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1} \le k_\lambda(u_1,1)u_1^{\lambda_1+\delta_0-1}du_1, \quad u_1 \in (1,\infty),
\end{gathered}
\tag{2.7}
$$

and $k(\lambda_1-\delta_0)+k(\lambda_1+\delta_0) < \infty$, then by Lebesgue control convergence theorem (cf. [13]), it follows that $k(\lambda_1+\delta_1) = k(\lambda_1)+o(1)(\delta_1 \to 0)$. Assuming that for $n(\ge 2), k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ is continuous at $(\lambda_1,\ldots,\lambda_{n-1})$, then for $n+1$, in view of the result for $n = 2$, we have that

$$
\begin{aligned}
&\lim_{\delta_n\to 0} k(\lambda_1+\delta_1,\ldots,\lambda_n+\delta_n)\\
&\quad= \lim_{\delta_n\to 0}\int_0^\infty\left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_n,1)\prod_{j=1}^{n-1}u_j^{\lambda_j+\delta_j-1}du_1\cdots du_{n-1}\right)u_n^{\lambda_n+\delta_n-1}du_n\\
&\quad= \int_0^\infty\left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_n,1)\prod_{j=1}^{n-1}u_j^{\lambda_j+\delta_j-1}du_1\cdots du_{n-1}\right)u_n^{\lambda_n-1}du_n\\
&\quad= \int_{\mathbf{R}_+^{n-1}}\left(\int_0^\infty k_\lambda(u_1,\ldots,u_n,1)u_n^{\lambda_n-1}du_n\right)\prod_{j=1}^{n-1}u_j^{\lambda_j+\delta_j-1}du_1\cdots du_{n-1},
\end{aligned}
\tag{2.8}
$$

then by the assumption for n, it follows that

$$
\lim_{\delta_n\to 0} k(\lambda_1+\delta_1,\ldots,\lambda_n+\delta_n) = k(\lambda_1,\ldots,\lambda_n)+o(1) \quad (\delta_i \longrightarrow 0, i = 1,\ldots,n-1).
\tag{2.9}
$$

By mathematical induction, we prove that for $n \in \mathbf{N}\setminus\{1\}, k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ is continuous at $(\lambda_1,\ldots,\lambda_{n-1})$. □

Lemma 2.5. *As for the assumption of Lemma 2.4, if* $0 < \varepsilon < \min_{1\le i\le n}\{|p_i|\}\delta_0$, *then for* $\varepsilon \to 0^+$,

$$
I_\varepsilon := \varepsilon\int_1^\infty\cdots\int_1^\infty k_\lambda(x_1,\ldots,x_n)\prod_{j=1}^{n}x_j^{\lambda_j-\varepsilon/p_j-1}dx_1\cdots dx_n = k_\lambda + o(1).
\tag{2.10}
$$

Proof. Setting $u_j = x_j/x_n$ $(j = 1, \ldots, n-1)$, we find that

$$I_\varepsilon = \varepsilon\int_1^\infty x_n^{-1-\varepsilon}\left[\int_{x_n^{-1}}^\infty \cdots \int_{x_n^{-1}}^\infty k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1}\right] dx_n. \tag{2.11}$$

Setting $D_j := \{(u_1, \ldots, u_{n-1}) \mid u_j \in (0, x_n^{-1}), u_k \in (0, \infty)\ (k \neq j)\}$ and

$$A_j(x_n) := \int \cdots \int_{D_j} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1}, \tag{2.12}$$

then by (2.11), it follows that

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} - \varepsilon\sum_{j=1}^{n-1}\int_1^\infty x_n^{-1} A_j(x_n) dx_n. \tag{2.13}$$

Without loses of generality, we estimate that $\int_1^\infty x_n^{-1} A_{n-1}(x_n) dx_n = O(1)$. In fact, setting $\alpha > 0$ such that $|\varepsilon/(p_{n-1}) + \alpha| < \delta_0$, since $-u_{n-1}^\alpha \ln u_{n-1} \to 0$ $(u_{n-1} \to 0^+)$, there exists $M > 0$, such that $-u_{n-1}^\alpha \ln u_{n-1} \leq M(u_{n-1} \in (0, 1])$, and then by Fubini theorem, it follows that

$$\begin{aligned}
0 &\leq \int_1^\infty x_n^{-1} A_{n-1}(x_n) dx_n \\
&= \int_1^\infty x_n^{-1}\left[\int_{\mathbf{R}_+^{n-2}}\int_0^{x_n^{-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_{n-1} du_1 \cdots du_{n-2}\right] dx_n \\
&= \int_0^1\int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1}\left(\int_1^{u_{n-1}^{-1}} x_n^{-1} dx_n\right) du_1 \cdots du_{n-1} \\
&= \int_0^1\int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1}(-\ln u_{n-1}) du_1 \cdots du_{n-1} \\
&\leq M\int_0^1\int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-2} u_j^{\lambda_j - \varepsilon/p_j - 1} u_{n-1}^{\lambda_{n-1} - (\varepsilon/p_{n-1} + \alpha) - 1} du_1 \cdots du_{n-1} \\
&\leq M\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1)\prod_{j=1}^{n-2} u_j^{\lambda_j - \varepsilon/p_j - 1} u_{n-1}^{\lambda_{n-1} - (\varepsilon/p_{n-1} + \alpha) - 1} du_1 \cdots du_{n-1} \\
&= M \cdot k\left(\lambda_1 - \frac{\varepsilon}{p_1}, \ldots, \lambda_{n-2} - \frac{\varepsilon}{p_{n-2}}, \lambda_{n-1} - \left(\frac{\varepsilon}{p_{n-1}} + \alpha\right)\right) < \infty.
\end{aligned} \tag{2.14}$$

Hence by (2.13), we have that

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-\varepsilon/p_j-1}du_1\cdots du_{n-1} - o_1(1). \tag{2.15}$$

By Lemma 2.4, we find that

$$\begin{aligned}
I_\varepsilon &\leq \varepsilon\int_1^\infty x_n^{-1-\varepsilon}\left[\int_0^\infty\cdots\int_0^\infty k_\lambda(u_1,\ldots,u_{n-1},1)\right.\\
&\qquad\left.\times\prod_{j=1}^{n-1} u_j^{\lambda_j-\varepsilon/p_j-1}du_1\cdots du_{n-1}\right]dx_n\\
&= \int_0^\infty\cdots\int_0^\infty k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-\varepsilon/p_j-1}du_1\cdots du_{n-1}\\
&= k\left(\lambda_1-\frac{\varepsilon}{p_1},\ldots,\lambda_{n-1}-\frac{\varepsilon}{p_{n-1}}\right) = k_\lambda + o_2(1),
\end{aligned} \tag{2.16}$$

Then by combination with (2.15), we have (2.10). □

Lemma 2.6. *Suppose that* $n \in \mathbf{N}\setminus\{1\}, p_1 \in \mathbf{R}_+\setminus\{1\}, \sum_{i=1}^{n}(1/p_i) = 1, 1/q_n = 1-1/p_n, (\lambda_1,\ldots,\lambda_n) \in \mathbf{R}^n, \sum_{i=1}^{n}\lambda_i = \lambda$, *then* $k_\lambda(x_1,\ldots,x_n) \geq 0$ *is a measurable function of* $-\lambda$*-degree in* $\mathbf{R}_+^n$ *such that*

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-1}du_1\cdots du_{n-1} \in \mathbf{R}. \tag{2.17}$$

If $f_i \geq 0$ *are measurable functions in* $\mathbf{R}_+ (i = 1,\ldots,n-1)$, *then (1) for* $p_i > 1\ (i = 1,\ldots,n)$,

$$\begin{aligned}
J &:= \left\{\int_0^\infty x_n^{q_n\lambda_n-1}\left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)dx_1\cdots dx_{n-1}\right]^{q_n}dx_n\right\}^{1/q_n}\\
&\leq k_\lambda\prod_{i=1}^{n-1}\left\{\int_0^\infty x^{p_i(1-\lambda_i)-1}f^{p_i}(x)dx\right\}^{1/p_i},
\end{aligned} \tag{2.18}$$

(2) for $0 < p_1 < 1, p_i < 0\ (i = 2,\ldots,n)$, *the reverse of* (2.18) *is obtained.*

Proof. (1) For $p_i > 1$ $(i = 1, \ldots, n)$, by Hölder's inequality (cf. [14]) and (2.4), it follows that

$$\begin{aligned}
&\left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)\,dx_1\cdots dx_{n-1}\right]^{q_n} \\
&= \left\{\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\right]^{1/p_i} f_i(x_i)\right. \\
&\qquad\left.\times\left[x_n^{(\lambda_n-1)(1-p_n)}\prod_{j=1}^{n-1} x_j^{\lambda_j-1}\right]^{1/p_n} dx_1\cdots dx_{n-1}\right\}^{q_n} \\
&\le \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\right]^{q_n/p_i} \\
&\qquad\times f_i^{q_n}(x_i)\,dx_1\cdots dx_{n-1} \\
&\qquad\times\left\{\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)x_n^{(\lambda_n-1)(1-p_n)}\prod_{j=1}^{n-1} x_j^{\lambda_j-1}\,dx_1\cdots dx_{n-1}\right\}^{q_n-1} \\
&= (k_\lambda)^{q_n-1} x_n^{1-q_n\lambda_n}\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n) \\
&\qquad\times\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\right]^{q_n/p_i} f_i^{q_n}(x_i)\,dx_1\cdots dx_{n-1},
\end{aligned}\tag{2.19}$$

$$\begin{aligned}
J &\le (k_\lambda)^{1/p_n}\left\{\int_0^\infty\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n)\right. \\
&\qquad\left.\times\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\right]^{q_n/p_i} f_i^{q_n}(x_i)\,dx_1\cdots dx_{n-1}dx_n\right\}^{1/q_n} \\
&= (k_\lambda)^{1/p_n}\left\{\int_{\mathbf{R}_+^{n-1}}\left(\int_0^\infty k_\lambda(x_1,\ldots,x_n)x_n^{\lambda_n-1}\,dx_n\right)\right. \\
&\qquad\left.\times\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1}\right]^{q_n/p_i} f_i^{q_n}(x_i)\,dx_1\cdots dx_{n-1}\right\}^{1/q_n}.
\end{aligned}\tag{2.20}$$

For $n \geq 3$, by Hölder's inequality again, it follows that

$$J \leq (k_\lambda)^{1/p_n} \left\{ \prod_{i=1}^{n-1} \left[\int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty k_\lambda(x_1,\ldots,x_n) x_n^{\lambda_n - 1} dx_n \right) \right.\right.$$
$$\left.\left. \times\, x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1} f_i^{p_i}(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n/p_i} \right\}^{1/q_n}$$
$$= (k_\lambda)^{1/p_n} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n) \right.\right.$$
$$\left.\left. \times\, x_i^{\lambda_i} \prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \right] x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i}$$
$$= (k_\lambda)^{1/p_n} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \omega_i(x_i) x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i}. \tag{2.21}$$

Then by (2.4), we have (2.18) (note that for $n = 2$, we do not use Hölder's inequality again).
(2) For $0 < p_1 < 1, p_i < 0$ $(i = 2,\ldots,n)$, by the reverse Hölder's inequality and the same way, we obtain the reverses of (2.18). □

3. Main Results and Applications

As for the assumption of Lemma 2.6, setting $\phi_i(x) := x^{p_i(1-\lambda_i)-1} (x \in (0,\infty); i = 1,\ldots,n)$, then we find that $\phi_n^{1/(1-p_n)}(x) = x^{q_n\lambda_n - 1}$. If $p_i > 1 (i = 1,\ldots,n)$, then define the following real function spaces:

$$L_{\phi_i}^{p_i}(0,\infty) := \left\{ f; \|f\|_{p_i,\phi_i} = \left\{ \int_0^\infty \phi_i(x) |f(x)|^{p_i} dx \right\}^{1/p_i} < \infty \right\} \quad (i = 1,\ldots,n),$$
$$\prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty) := \left\{ (f_1,\ldots,f_{n-1}); f_i \in L_{\phi_i}^{p_i}(0,\infty), i = 1,\ldots,n-1 \right\}, \tag{3.1}$$

and a multiple Hilbert-type integral operator $T : \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty) \to L^{q_n}_{\phi_n^{1/(1-p_n)}}$ as follows: for $f = (f_1,\ldots,f_{n-1}) \in \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty)$,

$$(Tf)(x_n) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1,\ldots,x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1}, \quad x_n \in (0,\infty). \tag{3.2}$$

Then by (2.18), it follows that $Tf \in L^{q_n}_{\phi_n^{1/(1-p_n)}}$, T is bounded, $\|Tf\|_{q_n,\phi_n^{1/(1-p_n)}} \le k_\lambda \prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}$, and $\|T\| \le k_\lambda$, where

$$\|T\| := \sup_{f\in\prod_{i=1}^{n-1}L^{p_i}_{\phi_i}(0,\infty)(f_i\neq\theta, i=1,\ldots,n-1)} \frac{\|Tf\|_{q_n,\phi_n^{1/(1-p_n)}}}{\prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}}. \tag{3.3}$$

Define the formal inner product of $T(f_1,\ldots,f_{n-1})$ and f_n as

$$(T(f_1,\ldots,f_{n-1}), f_n) := \int_{\mathbf{R}_+^n} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n} f_i(x_i)dx_1\cdots dx_n. \tag{3.4}$$

Theorem 3.1. *Suppose that $n \in \mathbf{N}\setminus\{1\}$, $p_1 \in \mathbf{R}_+\setminus\{1\}$, $\sum_{i=1}^n(1/p_i) = 1$, $1/q_n = 1-1/p_n$, then $k_\lambda(x_1,\ldots,x_n) \ge 0$ is a measurable function of $-\lambda$-degree in $\mathbf{R}_+^n$, and for any $(\lambda_1,\ldots,\lambda_n)\in\mathbf{R}^n$, it satisfies $\sum_{i=1}^n \lambda_i = \lambda$ and*

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-1}du_1\cdots du_{n-1} > 0. \tag{3.5}$$

If $f_i(\ge 0) \in L^{p_i}_{\phi_i}(0,\infty)$, $\|f\|_{p_i,\phi_i} > 0 (i = 1,\ldots,n)$, then (1) for $p_i > 1$ $(i=1,\ldots,n)$, $\|T\| = k_\lambda$ and the following equivalent inequalities are obtained:

$$\|T(f_1,\ldots,f_{n-1})\|_{q_n,\phi_n^{1/(1-p_n)}} < k_\lambda\prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}, \tag{3.6}$$

$$(T(f_1,\ldots,f_{n-1}), f_n) < k_\lambda\prod_{i=1}^{n}\|f_i\|_{p_i,\phi_i}, \tag{3.7}$$

where the constant factor k_λ is the best possible; (2) for $0<p_1<1$, $p_i<0$ $(i=2,\ldots,n)$, using the formal symbols of the case in $p_i > 1$ $(i=1,\ldots,n)$, the equivalent reverses of (3.6) *and* (3.7) *with the best constant factor are given.*

Proof. (1) For $p_i > 1$ $(i=1,\ldots,n)$, if (2.18) takes the form of equality, then for $n\ge 3$ in (2.21), there exist constants C_i and C_k $(i\neq k)$ such that they are not all zero and

$$\begin{aligned} &C_i x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1} f_i^{p_i}(x_i) \\ &\quad= C_k x_k^{(\lambda_k-1)(1-p_k)} \prod_{j=1(j\neq k)}^{n-1} x_j^{\lambda_j-1} f_k^{p_k}(x_k) \ a.e. \text{ in } \mathbf{R}_+^n, \end{aligned} \tag{3.8}$$

viz. $C_i x_i^{p_i(1-\lambda_i)} f_i^{p_i}(x_i) = C_k x_k^{p_k(1-\lambda_k)} f_k^{p_k}(x_k) = C$ *a.e.* in $\mathbf{R}_+^n$. Assuming that $C_i > 0$, then $x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) = C/(C_i x_i)$, which contradicts $\|f\|_{p_i,\phi_i} > 0$. (Note that for $n=2$, we only consider (2.19) for $f_k^{p_k}(x_k) = 1$ in the above). Hence we have (3.6). By Hölder's inequality, it follows that

$$(Tf, f_n) = \int_0^\infty \left(x_n^{\lambda_n - 1/q_n} \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right) \times \left(x_n^{1/q_n - \lambda_n} f_n(x_n) \right) dx_n \leq \| T(f_1, \ldots, f_{n-1}) \|_{q_n, \phi_n^{1/(1-p_n)}} \| f_n \|_{p_n, \phi_n}, \tag{3.9}$$

and then by (3.6), we have (3.7). Assuming that (3.7) is valid, setting

$$f_n(x_n) := x_n^{q_n \lambda_n - 1} \left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n - 1}, \tag{3.10}$$

then $J = \left\{ \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n \right\}^{1/q_n}$. By (2.18), it follows that $J < \infty$. If $J = 0$, then (3.6) is naturally valid. Assuming that $0 < J < \infty$, by (3.7), it follows that

$$\begin{gathered} \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n = J^{q_n} = (Tf, f_n) < k_\lambda \prod_{i=1}^n \| f_i \|_{p_i, \phi_i}, \\ \left\{ \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n \right\}^{1/q_n} = J < k_\lambda \prod_{i=1}^{n-1} \| f_i \|_{p_i, \phi_i}, \end{gathered} \tag{3.11}$$

and then (3.6) is valid, which is equivalent to (3.7).

For $\varepsilon > 0$ small enough, setting $\widetilde{f}_i(x)$as follows: $\widetilde{f}_i(x) = 0, x \in (0,1)$; $\widetilde{f}_i(x) = x^{\lambda_i - \varepsilon/p_i - 1}, x \in [1, \infty)$ $(i = 1, \ldots, n)$, if there exists $k \leq k_\lambda$, such that (3.7) is still valid as we replace k_λ by k, then in particular, by Lemma 2.5, we have that

$$k_\lambda + o(1) = I_\varepsilon = \varepsilon\left(T\left(\widetilde{f}_1, \ldots, \widetilde{f}_{n-1} \right), \widetilde{f}_n \right) < \varepsilon k \prod_{i=1}^n \left\| \widetilde{f}_i \right\|_{p_i, \phi_i} = k, \tag{3.12}$$

and $k_\lambda \leq k$ $(\varepsilon \to 0^+)$. Hence $k = k_\lambda$ is the best value of (3.7). We conform that the constant factor k_λ in (3.6) is the best possible; otherwise, we can get a contradiction by (3.9) that the constant factor in (3.7) is not the best possible. Therefore $\|T\| = k_\lambda$.

(2) For $0 < p_1 < 1, p_i < 0$ $(i = 2, \ldots, n)$, by using the reverse Hölder's inequality and the same way, we have the equivalent reverses of (3.6) and (3.7) with the same best constant factor. □

Example 3.2. For $\lambda > 0, \lambda_i = (\lambda/r_i)$ $(i = 1, \ldots, n), \sum_{i=1}^{n}(1/r_i) = 1, k_\lambda(x_1, \ldots, x_n) = 1/\left(\sum_{i=1}^{n} x_i\right)^\lambda$, we obtain $k_\lambda = (1/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$ (cf. [7, (9.1.19)]. By Theorem 3.1, it follows that $\|T\| = k_\lambda = (1/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$, and then by (3.7), we find that

$$\begin{aligned}&\int_{\mathbf{R}_+^n} \frac{1}{\left(\sum_{i=1}^{n} x_i\right)^\lambda} \prod_{i=1}^{n} F_i(x_i)dx_1 \cdots dx_n \\ &\quad < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{r_i}\right) \left\{\int_0^\infty x_i^{p_i(1-\lambda/r_i)-1} F_i^{p_i}(x_i)dx_i\right\}^{1/p_i}.\end{aligned} \tag{3.13}$$

Setting $F_i(x_i) = x_i^{\beta/n} f_i(x_i)$ and $\lambda_i = \lambda/r_i - \beta/n(i = 1, \ldots, n)$ in (3.13), we obtain $\sum_{i=1}^{n} \lambda_i = \lambda - \beta, \min_{1\le i\le n}\{\lambda_i\} > -\beta/n$ and

$$\begin{aligned}&\int_{\mathbf{R}_+^n} \frac{\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^\beta}{\left(\sum_{i=1}^{n} x_i\right)^\lambda} \prod_{i=1}^{n} f_i(x_i)dx_1 \cdots dx_n \\ &\quad < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\lambda_i + \frac{\beta}{n}\right) \left\{\int_0^\infty x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i)dx_i\right\}^{1/p_i}.\end{aligned} \tag{3.14}$$

It is obvious that (3.13) and (3.14) are equivalent in which the constant factors are all the best possible. Hence for $k_{\lambda-\beta}(x_1, \ldots, x_n) = \left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^\beta / \left(\sum_{i=1}^{n} x_i\right)^\lambda$ $(\lambda > 0, \min_{1\le i\le n}\{\lambda_i\} > -\beta/n)$, we can show that $\|T\| = k_{\lambda-\beta} = (1/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda_i + \beta/n)$.

Example 3.3. For $\lambda > 0, \lambda_i = \lambda/r_i$ $(i = 1, \ldots, n), \sum_{i=1}^{n}(1/r_i) = 1, k_\lambda(x_1, \ldots, x_n) = 1/(\max_{1\le i\le n}\{x_i\})^\lambda$, we obtain $k_\lambda = (1/\lambda^{n-1})\prod_{i=1}^{n} r_i$ (cf. [7, (9.1.24)]. By Theorem 3.1, it follows that $\|T\| = k_\lambda = (1/\lambda^{n-1})\prod_{i=1}^{n} r_i$, and then by (3.7), we find that

$$\begin{aligned}&\int_{\mathbf{R}_+^n} \frac{1}{(\max_{1\le i\le n}\{x_i\})^\lambda} \prod_{i=1}^{n} F_i(x_i)dx_1 \cdots dx_n \\ &\quad < \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left\{\int_0^\infty x_i^{p_i(1-\lambda/r_i)-1} F_i^{p_i}(x_i)dx_i\right\}^{1/p_i}.\end{aligned} \tag{3.15}$$

Setting $F_i(x_i) = x_i^{\beta/n} f_i(x_i)$ and $\lambda_i = \lambda/r_i - \beta/n(i = 1, \ldots, n)$ in (3.15), we obtain $\sum_{i=1}^{n} \lambda_i = \lambda - \beta, \min_{1\le i\le n}\{\lambda_i\} > -\beta/n$ and

$$\begin{aligned}&\int_{\mathbf{R}_+^n} \frac{\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^\beta}{(\max_{1\le i\le n}\{x_i\})^\lambda} \prod_{i=1}^{n} f_i(x_i)dx_1 \cdots dx_n \\ &\quad < \lambda \prod_{i=1}^{n} \frac{1}{\lambda_i + \beta/n} \left\{\int_0^\infty x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i)dx_i\right\}^{1/p_i}.\end{aligned} \tag{3.16}$$

Hence for $k_{\lambda-\beta}(x_1,\ldots,x_n) = \left(\sqrt[n]{\prod_{i=1}^n x_i}\right)^{\beta} / (\max_{1\le i\le n}\{x_i\})^{\lambda}$ ($\lambda > 0, \min_{1\le i\le n}\{\lambda_i\} > -\beta/n$), we can show that $\|T\| = k_{\lambda-\beta} = \lambda\prod_{i=1}^n (1/(\lambda_i + \beta/n))$.

Example 3.4. For $\lambda > 0, \lambda_i = -\lambda/r_i, \sum_{i=1}^n (1/r_i) = 1$ $(i = 1,\ldots,n), k_{-\lambda}(x_1,\ldots,x_n) = (\min_{1\le i\le n}\{x_i\})^{\lambda}$, by mathematical induction, we can show that

$$k_{-\lambda} = \int_{R_+^{n-1}} (\min\{u_1,\ldots,u_{n-1},1\})^{\lambda} \prod_{j=1}^{n-1} u_j^{-\lambda/r_j - 1} du_1 \cdots du_{n-1} = \frac{\prod_{i=1}^n r_i}{\lambda^{n-1}}. \tag{3.17}$$

In fact, for $n = 2$, we obtain

$$k_{-\lambda} = \int_0^1 u_1^{\lambda/r_2 - 1} du_1 + \int_1^{\infty} u_1^{-\lambda/r_1 - 1} du_1 = \frac{1}{\lambda} r_1 r_2. \tag{3.18}$$

Assuming that for $n(\ge 2)$ (3.17) is valid, then for $n+1$, it follows that

$$\begin{aligned}
k_{-\lambda} &= \int_{R_+^{n-1}} \prod_{j=2}^{n} u_j^{-\lambda/r_j - 1} \left[\int_0^{\infty} (\min\{u_1,\ldots,u_n,1\})^{\lambda} u_1^{-\lambda/r_1 - 1} du_1\right] du_2 \cdots du_n \\
&= \int_{R_+^{n-1}} \prod_{j=2}^{n} u_j^{-\lambda/r_j - 1} \left[\int_0^{\min\{u_2,\ldots,u_n,1\}} u_1^{\lambda} u_1^{-\lambda/r_1 - 1} du_1 \right.\\
&\qquad \left. + \int_{\min\{u_2,\ldots,u_n,1\}}^{\infty} (\min\{u_2,\ldots,u_n,1\})^{\lambda} u_1^{-\lambda/r_1 - 1} du_1\right] du_2 \cdots du_n \\
&= \frac{r_1^2}{\lambda(r_1 - 1)} \int_{R_+^{n-1}} (\min\{u_2,\ldots,u_n,1\})^{\lambda(1-1/r_1)} \prod_{j=2}^{n} u_j^{-\lambda(1-1/r_1)/(1-1/r_1)r_j - 1} du_2 \cdots du_n \\
&= \frac{r_1^2}{\lambda(r_1 - 1)} \frac{1}{[\lambda(1-1/r_1)]^{n-1}} \prod_{i=2}^{n+1} \left(1 - \frac{1}{r_1}\right) r_i = \frac{1}{\lambda^n} \prod_{i=1}^{n+1} r_i.
\end{aligned} \tag{3.19}$$

Then by mathematical induction, (3.17) is valid for $n \in \mathbf{N} \setminus \{1\}$.

By Theorem 3.1, it follows that $\|T\| = k_{-\lambda} = (1/\lambda^{n-1})\prod_{i=1}^n r_i$, and by (3.7), we find that

$$\begin{aligned}
&\int_{\mathbf{R}_+^n} \left(\min_{1\le i\le n}\{x_i\}\right)^{\lambda} \prod_{i=1}^n F_i(x_i) dx_1 \cdots dx_n \\
&\quad < \frac{1}{\lambda^{n-1}} \prod_{i=1}^n r_i \left\{\int_0^{\infty} x_i^{p_i(1+\lambda/r_i) - 1} F_i^{p_i}(x_i) dx_i\right\}^{1/p_i}.
\end{aligned} \tag{3.20}$$

Setting $F_i(x_i) = x_i^{-\beta/n} f_i(x_i)$ and $\lambda_i = -\lambda/r_i + \beta/n$ $(i = 1, \ldots, n)$ in (3.20), we obtain $\sum_{i=1}^{n} \lambda_i = \beta - \lambda$, $\max_{1\le i\le n}\{\lambda_i\} < \beta/n$ and

$$\int_{\mathbf{R}_+^n} \frac{(\min_{1\le i\le n}\{x_i\})^{\lambda}}{\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^{\beta}} \prod_{i=1}^{n} f_i(x_i)\,dx_1 \cdots dx_n < \lambda \prod_{i=1}^{n} \frac{1}{\beta/n - \lambda_i} \left\{ \int_0^{\infty} x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i)\,dx_i \right\}^{1/p_i}. \tag{3.21}$$

Hence for $k_{\beta-\lambda}(x_1, \ldots, x_n) = (\min_{1\le i\le n}\{x_i\})^{\lambda} / \left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^{\beta}$ $(\lambda > 0, \max_{1\le i\le n}\{\lambda_i\} < \beta/n)$, we can show that $\|T\| = k_{\beta-\lambda} = \lambda \prod_{i=1}^{n} (1/(\beta/n - \lambda_i))$.

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