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Research Article

Some Strong Limit Theorems for Weighted Product Sums of $\widetilde{\rho}$-Mixing Sequences of Random Variables

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We study almost sure convergence for $\widetilde{\rho}$-mixing sequences of random variables. Many of the previous results are our special cases. For example, the authors extend and improve the corresponding results of Chen et al. (1996) and Wu and Jiang (2008). We extend the classical Jamison convergence theorem and the Marcinkiewicz strong law of large numbers for independent sequences of random variables to $\widetilde{\rho}$-mixing sequences of random variables without necessarily adding any extra conditions.

1. Introduction and Lemmas

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The random variables we deal with are all defined on $(\Omega, \mathcal{F}, P)$. Let $\{X_n; n \geq 1\}$ be a sequence of random variables. For each nonempty set $S \subset N$, and write $\mathcal{F}_S = \sigma(X_i, i \in S)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup\{|\operatorname{corr}(X, Y)|; X \in L_2(\mathcal{B}),\ Y \in L_2(\mathcal{R})\}, \tag{1.1}$$

where $\operatorname{corr}(X, Y) = (EXY - EXEY)/\sqrt{\operatorname{Var} X \operatorname{Var} Y}$. Define the $\widetilde{\rho}$-mixing coefficients by

$$\widetilde{\rho}(n) = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T);\ \text{finite subsets } S, T \subset N \text{ such that } \operatorname{dist}(S, T) \geq n\}, \quad n \geq 0. \tag{1.2}$$

Obviously $0 \leq \widetilde{\rho}(n+1) \leq \widetilde{\rho}(n) \leq 1,\ n \geq 0$, and $\widetilde{\rho}(0) = 1$ except in the trivial case where all of the random variables X_i are degenerate.

Definition 1.1. A random variables sequence $\{X_n; n \geq 1\}$ is said to be a $\widetilde{\rho}$-mixing random variables sequence if there exists $k \in N$ such that $\widetilde{\rho}(k) < 1$.

$\tilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. A number of writers have studied $\tilde{\rho}$-mixing random variables sequences and a series of useful results have been established. We refer to Bradley [1] (which assumes $\tilde{\rho}(k) \to 0$ in the central limit theorem), Bryc and Smoleński [2], Goldie and Greenwood [3] (which assumes $\sum_{k=1}^{\infty} \tilde{\rho}(2^k) < \infty$), and Yang [4] for moment inequalities and the strong law of large numbers, Wu [5, 6], Wu and Jiang [7], Peligrad and Gut [8], and Gan [9] for almost sure convergence and Utev and Peligrad [10] for maximal inequalities and the invariance principle. When these are compared with the corresponding results of independent random variables sequences, there still remains much to be desired.

Lemma 1.2 (see [7, Theorem 1]). *Let $\{X_n; n \geq 1\}$ be a $\tilde{\rho}$-mixing sequence of random variables which satisfies*

$$\sum_{n=1}^{\infty} \operatorname{Var} X_n < \infty. \tag{1.3}$$

Then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely (a.s.) and in quadratic mean.

Lemma 1.3 (see [11, Lemma 2.4]). *For each positive integer m, let $G(m)$ denote the set of all vectors $(\vec{r}, \vec{l}) := ((r_1, r_2, \ldots, r_m), (l_1, l_2, \ldots, l_m)) \in \{0,1,2,\ldots,m\}^m \times \{0,1,2,\ldots,m\}^m$ such that $\sum_{j=1}^{m} r_j l_j = m$.*

Then for each positive integer m, there exists a function $A^{(m)} : G(m) \to \mathrm{R}$ such that the following holds.

For any integer $n \geq m$ and any choice of real numbers $x_1, x_2, \ldots, x_n$, one has that

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \left(\prod_{1 \leq j \leq m} x_{i_j} \right) = \sum_{(\vec{r}, \vec{l}) \in G(m)} \left[A^{(m)}\left(\left(\vec{r}, \vec{l}\right)\right) \cdot \prod_{1 \leq j \leq m} \left(\sum_{1 \leq i \leq n} x_i^{r_j} \right)^{l_j} \right]. \tag{1.4}$$

2. Main Results and the Proof

To state our results, we need some notions. Throughout this paper, let $\{\omega_i; i \geq 1\}$ be a sequence of positive real numbers, and let $W_n = \sum_{i=1}^{n} \omega_i$, $n \geq 1$, satisfy $W_n \uparrow \infty$, $\omega_n W_n^{-1} \to 0$, $n \to \infty$.

Jamison et al. [12] proved the following result. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $EX_1 = 0$. Denote $N(n) \hat{=} \sharp\{k; \omega_k^{-1} W_k \leq n\}$, that is, the number of subscripts k such that $\omega_k^{-1} W_k \leq n$. If $N(n) = O(n)$, then $\sum_{i=1}^{n} \omega_i X_i / W_i \to 0$ a.s. Chen et al. [13] extended the Jamison Theorem and obtained the following result. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $EX_1 = 0$, $E|X_1|^r < \infty$ for some $r \in [1,2)$. If $N(n) = O(n^r)$, then $\sum_{i=1}^{n} \omega_i X_i / W_i \to 0$ a.s.

The main purpose of this paper is to study the strong limit theorems for weighted sums of $\tilde{\rho}$-mixing random variables sequences and try to obtain some new results. We establish weighted partial sums and weighted product sums strong convergence theorems. Our results in this paper extend and improve the corresponding results of Chen et al. [13], Wu and Jiang [7], the classical Jamison convergence theorem, and the Marcinkiewicz strong law of large numbers for independent sequences of random variables to $\tilde{\rho}$-mixing sequences of random variables.

Theorem 2.1. *Let $\{X_i; i \geq 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables with $EX_i = 0$, and let the following conditions be satisfied:*

$$W_n^{-1}\sum_{i=1}^{n}\omega_i EX_i I_{(|X_i|\geq b_i)} \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.1}$$

$$\sum_{i=1}^{\infty} P(|X_i| \geq b_i) < \infty, \tag{2.2}$$

$$\sum_{i=1}^{\infty} b_i^{-2} \operatorname{Var} X_i I_{(|X_i|<b_i)} < \infty, \tag{2.3}$$

where $b_i = \omega_i^{-1} W_i$. Then

$$T_n \widehat{=} W_n^{-1}\sum_{i=1}^{n}\omega_i X_i \longrightarrow 0 \quad a.s.\ n \longrightarrow \infty. \tag{2.4}$$

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 hold, and also suppose*

$$\sup_{i\geq 1} E|X_i| < \infty. \tag{2.5}$$

Then for all $m \geq 1$,

$$U_n \widehat{=} W_n^{-m} \sum_{1\leq i_1<\cdots<i_m\leq n} \prod_{1\leq j\leq m} \omega_{i_j} X_{i_j} \longrightarrow 0 \quad a.s.\ n \longrightarrow \infty. \tag{2.6}$$

Corollary 2.3. *Let $\{X_n; n \geq 1\}$ be a sequence of $\widetilde{\rho}$-mixing identically distributed random variables. Let for some $1 \leq p < 2$,*

$$N(n) \widehat{=} \sharp\{k; b_k \leq n\} \leq cn^p \quad \forall n \geq 1, \text{ and some constant } c > 0, \tag{2.7}$$

$$EX_1 = 0, \qquad E|X_1|^p < \infty. \tag{2.8}$$

Then (2.6) *holds.*

Remark 2.4. Let $X_1, X_2, \ldots$ be i.i.d. random variables, and $p = 1$ in Corollary 2.3, then Corollary 2.3 is the well-known Jamison convergence theorem. Thus, our Theorem 2.2 and Corollary 2.3 generalize and improve the Jamison convergence theorem from the i.i.d. case to $\widetilde{\rho}$-mixing sequence. In addition, by Theorems 1 and 2 in Chen et al. [13] are special situation of Corollary 2.3.

Theorem 2.5. *Let $\{X_n; n \geq 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables. Let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and let the following conditions be satisfied:*

$$\sum_{n=1}^{\infty} a_n^{-2} E X_n^2 I_{(|X_n|<a_n)} < \infty, \tag{2.9}$$

$$\sum_{i=1}^{\infty} P(|X_i| \geq a_i) < \infty, \tag{2.10}$$

$$a_n^{-1} \sum_{i=1}^{n} E X_i I_{(|X_i|<a_i)} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.11}$$

Then for all $m \geq 1$,

$$a_n^{-m} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{1 \leq j \leq m} X_{i_j} \longrightarrow 0, \quad a.s.\ n \longrightarrow \infty. \tag{2.12}$$

Corollary 2.6. *Let $\{X_n; n \geq 1\}$ be a $\widetilde{\rho}$-mixing identically distributed random variable sequence, for $0 < p < 2$, $E|X_1|^p < \infty$, and for $1 \leq p < 2$, $EX_1 = 0$. Then for all $m \geq 1$,*

$$n^{-m/p} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{1 \leq j \leq m} X_{i_j} \longrightarrow 0, \quad a.s.\ n \longrightarrow \infty. \tag{2.13}$$

In particular, taking $m = 1$, the above formula is the well-known Marcinkiewicz strong law of large numbers. Thus, our Theorem 2.5 and Corollary 2.6 generalize and improve the Marcinkiewicz strong law of large numbers from the i.i.d. case to $\widetilde{\rho}$-mixing sequence. In addition, by Theorem 4 in Wu and Jiang [7] is a special case of Corollary 2.6.

Proof of Theorem 2.1. Let $X_i(b_i) = X_i I_{(|X_i|<b_i)}$. From (2.2),

$$\sum_{i=1}^{\infty} P(X_i(b_i) \neq X_i) = \sum_{i=1}^{\infty} P(|X_i| \geq b_i) < \infty. \tag{2.14}$$

By the Borel-Cantelli lemma and the Toeplitz lemma,

$$W_n^{-1} \sum_{i=1}^{n} \omega_i (X_i - X_i(b_i)) \longrightarrow 0 \quad \text{a.s. } n \longrightarrow \infty. \tag{2.15}$$

By $EX_i = 0$ and (2.1),

$$W_n^{-1} \sum_{i=1}^{n} \omega_i E X_i(b_i) = -W_n^{-1} \sum_{i=1}^{n} \omega_i E X_i I_{(|X_i| \geq b_i)} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.16}$$

By (2.3),

$$\sum_{i=1}^{\infty} b_i^{-2} \operatorname{Var} X_i(b_i) = \sum_{i=1}^{\infty} b_i^{-2} \operatorname{Var} X_i I_{(|X_i|<b_i)} < \infty. \tag{2.17}$$

Applying Lemma 1.2,

$$\sum_{i=1}^{\infty} b_i^{-1}(X_i(b_i) - EX_i(b_i)) \quad \text{a.s.} \tag{2.18}$$

converges. Hence

$$W_n^{-1}\sum_{i=1}^{n} \omega_i(X_i(b_i) - EX_i(b_i)) \longrightarrow 0 \quad \text{a.s.} \tag{2.19}$$

from the Kronecker lemma. Combining (2.15)–(2.19), (2.4) holds. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. By Lemma 1.3,

$$U_n = \sum_{(\vec{r},\vec{l})\in G(m)} A^{(m)}\left(\left(\vec{r},\vec{l}\right)\right) \cdot \prod_{1\le j\le m}\left(\sum_{i=1}^{n}\left(\omega_i X_i W_n^{-1}\right)^{r_j}\right)^{l_j}, \tag{2.20}$$

where $G(m)$ denote the set of all vectors $(\vec{r},\vec{l}) := ((r_1,r_2,\ldots,r_m),(l_1,l_2,\ldots,l_m)) \in \{0,1,2,\ldots,m\}^m \times \{0,1,2,\ldots,m\}^m$ such that $\sum_{j=1}^{m} r_j l_j = m$, and $A^{(m)}((\vec{r},\vec{l}))$ are constants which do not depend on n, $\{\omega_i; i \ge 1\}$ and $\{X_i; i \ge 1\}$. Thus, in order to prove (2.6), we only need to prove that

$$W_n^{-r}\sum_{i=1}^{n} \omega_i^r X_i^r \longrightarrow 0 \quad \text{a.s. for } 1 \le r \le m. \tag{2.21}$$

When $r = 1$, by Theorem 2.1, (2.21) holds. When $2 \le r \le m$, we get

$$W_n^{-r}\left|\sum_{i=1}^{n} \omega_i^r X_i^r\right| \le \left(W_n^{-2}\sum_{i=1}^{n}\omega_i^2 X_i^2\right)^{r/2} \quad \forall n \ge 1 \tag{2.22}$$

from the elementary inequality $(a_1 + \cdots + a_n)^p \ge a_1^p + \cdots + a_n^p$ valid for $a_i \ge 0,\ p \ge 1$ applied with $p = r/2,\ a_i = \omega_i^2 X_i^2$. Hence, in order to prove (2.21), we only need to prove that

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2 X_i^2 \longrightarrow 0 \quad \text{a.s. } n \longrightarrow \infty. \tag{2.23}$$

By (2.3), using Lemma 1.2, we get that

$$\sum_{i=1}^{\infty} b_i^{-1}(X_i(b_i) - EX_i(b_i)) \quad \text{a.s.} \tag{2.24}$$

converges. By the Kronecker lemma, $(1/W_n)\sum_{i=1}^{n}\omega_i(X_i(b_i) - EX_i(b_i)) \longrightarrow 0$ a.s. Thus,

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2(X_i(b_i) - EX_i(b_i))^2 \le \left(\frac{1}{W_n}\sum_{i=1}^{n}\omega_i(X_i(b_i) - EX_i(b_i))\right)^2 \longrightarrow 0 \quad \text{a.s.,} \tag{2.25}$$

that is,

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2 X_i^2(b_i) - 2W_n^{-2}\sum_{i=1}^{n}\omega_i^2 X_i(b_i)EX_i(b_i) + W_n^{-2}\sum_{i=1}^{n}\omega_i^2(EX_i(b_i))^2 \longrightarrow 0 \quad \text{a.s.} \tag{2.26}$$

By (2.3),

$$\begin{aligned}\sum_{i=1}^{\infty} b_i^{-4}\operatorname{Var}(X_i(b_i)EX_i(b_i)) &= \sum_{i=1}^{\infty} b_i^{-4}(EX_i(b_i))^2 \operatorname{Var} X_i(b_i)\\ &\le \sum_{i=1}^{\infty} b_i^{-2}\operatorname{Var} X_i(b_i) < \infty.\end{aligned} \tag{2.27}$$

By Lemma 1.2, we have that

$$\sum_{i=1}^{\infty} b_i^{-2}\left(X_i(b_i)EX_i(b_i) - (EX_i(b_i))^2\right) \quad \text{a.s.} \tag{2.28}$$

converges. By the Kronecker lemma,

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2\left(X_i(b_i)EX_i(b_i) - (EX_i(b_i))^2\right) \longrightarrow 0 \quad \text{a.s.} \tag{2.29}$$

By $\omega_i/W_i \to 0$, $i \to \infty$ and the Toeplitz lemma,

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2 = W_n^{-1}\sum_{i=1}^{n}\omega_i\frac{\omega_i}{W_n} \le W_n^{-1}\sum_{i=1}^{n}\omega_i\frac{\omega_i}{W_i} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.30}$$

Then combining (2.5), we obtain

$$0 \le W_n^{-2}\sum_{i=1}^{n}\omega_i^2 (EX_i(b_i))^2 \le \left(\sup_{i\ge 1} E|X_i|\right)^2 W_n^{-2}\sum_{i=1}^{n}\omega_i^2 \longrightarrow 0. \tag{2.31}$$

Substituting (2.29) and (2.31) in (2.26), we get

$$W_n^{-2}\sum_{i=1}^{n}\omega_i^2 X_i^2(b_i) \longrightarrow 0 \quad \text{a.s.} \tag{2.32}$$

Then combining (2.2) and the Borel-Cantelli lemma, (2.23) holds. This completes the proof of Theorem 2.2. □

Proof of Corollary 2.3. By Theorem 2.2, we only need to verify (2.1)–(2.3) and (2.5). From X_n having identically distribution, (2.7), (2.8) and $1 \le p < 2$, (2.5) holds automatically.

Since

$$EX_i I_{(|X_i|<b_i)} \longrightarrow EX_i = 0, \tag{2.33}$$

by the Toeplitz lemma,

$$W_n^{-1}\sum_{i=1}^{n}\omega_i EX_i I_{(|X_i|<b_i)} \longrightarrow 0, \tag{2.34}$$

That is,(2.1) holds.

By (2.7),

$$\begin{aligned}
\sum_{i=1}^{\infty} P(|X_i| > b_i) &\le \sum_{j=1}^{\infty}\sum_{j-1<b_i\le j} P\big(|X_1| > j-1\big) \\
&= \sum_{j=1}^{\infty}\big(N(j) - N(j-1)\big)\sum_{k=j}^{\infty} P((k-1 \le |X_1| \le k)) \\
&= \sum_{k=1}^{\infty} P(k-1 \le |X_1| \le k)\sum_{j=1}^{k}\big(N(j) - N(j-1)\big) \\
&\ll \sum_{k=1}^{\infty} j^p P(k-1 \le |X_1| \le k) \\
&\ll E|X_1|^p < \infty,
\end{aligned} \tag{2.35}$$

That is, (2.2) holds.

Similarly,

$$
\begin{aligned}
\sum_{i=1}^{\infty} b_i^{-2} E X_i^2 I_{(|X_i|<b_i)} &= \sum_{i=1}^{\infty} b_i^{-2} E X_1^2 I_{(|X_1|<b_i)} \\
&= \sum_{j=1}^{\infty} \sum_{j-1<b_i\le j} b_i^{-2} E X_1^2 I_{(|X_1|<b_i)} \le \sum_{j=1}^{\infty} \sum_{j-1<b_i\le j} (j-1)^{-2} E X_1^2 I_{(|X_1|<j)} \\
&= \sum_{j=1}^{\infty} (j-1)^{-2} (N(j) - N(j-1)) \sum_{k=1}^{j} E X_1^2 I_{(k-1\le|X_1|<k)} \\
&= \sum_{k=1}^{\infty} E X_1^2 I_{(k-1\le|X_1|<k)} \sum_{j=k}^{\infty} (j-1)^{-2} (N(j) - N(j-1)) \\
&\le \sum_{k=1}^{\infty} E X_1^2 I_{(k-1\le|X_1|<k)} \sum_{j=k}^{\infty} \left((j-1)^{-2} - j^{-2} \right) N(j) \\
&\ll \sum_{k=1}^{\infty} E X_1^2 I_{(k-1\le|X_1|<k)} \sum_{j=k}^{\infty} \left((j-1)^{-2} - j^{-2} \right) j^p \\
&\ll \sum_{k=1}^{\infty} E X_1^2 I_{(k-1\le|X_1|<k)} k^{p-2} \\
&< \sum_{k=1}^{\infty} E|X_1|^p I_{(k-1\le|X_1|<k)} \\
&\ll E|X_1|^p < \infty,
\end{aligned} \tag{2.36}
$$

That is, (2.3) holds. This completes proof of Corollary 2.3. □

Proof of Theorem 2.5. Similar to the proof of Theorem 2.2, by Lemma 1.3, in order to prove (2.12), we only need to prove that

$$a_n^{-r} \sum_{i=1}^{n} X_i^r \longrightarrow 0 \quad \text{a.s. for } r = 1, 2. \tag{2.37}$$

Let $X_i(a_i) = X_i I_{(|X_i|<a_i)}$, then

$$a_n^{-1} \sum_{i=1}^{n} X_i = a_n^{-1} \sum_{i=1}^{n} (X_i - X_i(a_i)) + a_n^{-1} \sum_{i=1}^{n} (X_i(a_i) - EX_i(a_i)) + a_n^{-1} \sum_{i=1}^{n} EX_i(a_i). \tag{2.38}$$

(i) When $r = 1$, by (2.10) and (2.11), in order to prove $a_n^{-1} \sum_{i=1}^{n} X_i \longrightarrow 0$ a.s., we only need to prove that

$$a_n^{-1} \sum_{i=1}^{n} (X_i(a_i) - EX_i(a_i)) \longrightarrow 0 \quad \text{a.s.} \tag{2.39}$$

By (2.9) and Lemma 1.2,

$$\sum_{n=1}^{\infty} a_n^{-1} \left(X_n(a_n) - EX_n(a_n) \right) \quad \text{a.s.} \tag{2.40}$$

converges. By the Kronecker lemma, (2.39) holds.

(ii) When $r = 2$, by (2.9) and the Kronecker lemma,

$$a_n^{-2} \sum_{i=1}^{n} X_i^2(a_i) \longrightarrow 0 \quad \text{a.s.} \tag{2.41}$$

By (2.10) and the Borel-Cantelli lemma,

$$a_n^{-2} \sum_{i=1}^{n} X_i^2 \longrightarrow 0 \quad \text{a.s.} \tag{2.42}$$

Hence, combining (2.39), (2.37) holds. This completes the proof of Theorem 2.5. □

Proof of Corollary 2.6. Let $a_n = n^{1/p}$. We can easy to verify (2.9)–(2.11). By Theorem 2.5, Corollary 2.6 holds. □

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