Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 170526, 9 pages
doi:10.1155/2009/170526

Research Article

On Convergence of q-Series Involving ${}_{r+1}\phi_r$ Basic Hypergeometric Series

Mingjin Wang[1] and Xilai Zhao[2]

[1] *Department of Applied Mathematics, Jiangsu Polytechnic University, Changzhou, Jiangsu 213164, China*
[2] *Department of Mechanical and Electrical Engineering, Hebi College of Vocation and Technology, Hebi, Henan 458030, China*

Correspondence should be addressed to Mingjin Wang, wang197913@126.com

Received 18 December 2008; Accepted 24 March 2009

Recommended by Ondrej Dosly

We use inequality technique and the terminating case of the q-binomial formula to give some results on convergence of q-series involving ${}_{r+1}\phi_r$ basic hypergeometric series. As an application of the results, we discuss the convergence for special Thomae q-integral.

1. Introduction

q-Series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics. Convergence of a q-series is an important problem in the study of q-series. There are some results about it in [1–3]. For example, Ito used inequality technique to give a sufficient condition for convergence of a special q-series called Jackson integral. In this paper, by using inequality technique, we derive the following two theorems on convergence of q-series involving ${}_{r+1}\phi_r$ basic hypergeometric series, which can be used for convergence of special Thomae q-integral.

2. Notations and Known Results

We recall some definitions, notations, and known results which will be used in the proofs. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \tag{2.2}$$

where n is an integer or ∞.

The q-binomial theorem [4, 5] is

$$\sum_{k=0}^{\infty} \frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \ |q| < 1. \tag{2.3}$$

When $a = q^{-n}$, where n denotes a nonnegative integer

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k z^k}{(q;q)_k} = (zq^{-n};q)_n. \tag{2.4}$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by [4, 5]

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; \ q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n z^n}{(q, b_1, b_2, \ldots, b_r; q)_n}. \tag{2.5}$$

3. Main Results

The main purpose of the present paper is to establish the following two theorems on convergence of q-series involving ${}_{r+1}\phi_r$ basic hypergeometric series.

Theorem 3.1. *Suppose a_i, b_i, t are any real numbers such that $t > 0$ and $b_i < 1$ with $i = 1, 2, \ldots, r$. Let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = p < 1, \tag{3.1}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix}; \ q, tq^n\right) \tag{3.2}$$

converges absolutely.

Proof. Let $b < 1$ and

$$f(t) = \frac{1 - at}{1 - bt}, \quad 0 \le t \le 1, \tag{3.3}$$

It is easy to see that $f(t)$ is a monotone function with respect to $0 \le t \le 1$.

Consequently, one has

$$\left|\frac{1-at}{1-bt}\right| \leq \max\left\{1, \frac{|1-a|}{1-b}\right\}. \tag{3.4}$$

From (3.4), one knows

$$\left|\frac{(a_i;q)_k}{(b_i;q)_k}\right| = \left|\frac{1-a_i}{1-b_i}\right| \cdot \left|\frac{1-a_iq}{1-b_iq}\right| \cdots \left|\frac{1-a_iq^{k-1}}{1-b_iq^{k-1}}\right| \leq M_i^k, \tag{3.5}$$

where $M_i = \max\{1, |1-a_i|/(1-b_i)\}$ for $i = 1, 2, \ldots, r$.

So, one has

$$\left|\frac{(a_1, a_2, \ldots, a_r; q)_k(-1)^k}{(b_1, b_2, \ldots, b_r; q)_k}\right| = \left|\frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_r; q)_k}\right| \leq \left(\prod_{i=1}^{r} M_i\right)^k. \tag{3.6}$$

It is obvious that

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} > 0, \quad t > 0,\ k = 1, 2, \ldots, n. \tag{3.7}$$

Multiplying both sides of (3.6) by

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} \tag{3.8}$$

gives

$$\left|\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \leq \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} M_i\right)^k. \tag{3.9}$$

Hence,

$$\begin{aligned}
\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, tq^n\right)\right| &= \left|\sum_{k=0}^{n}\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \\
&\leq \sum_{k=0}^{n}\left|\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \\
&\leq \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} M_i\right)^k.
\end{aligned} \tag{3.10}$$

By using (2.4) one obtains

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r}M_i\right)^k=\left(-t\prod_{i=1}^{r}M_i;q\right)_n. \tag{3.11}$$

Substituting (3.11) into (3.10), one has

$$\left|{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r,q^{-n}\\ b_1,b_2,\ldots,b_r\end{matrix};\ q,tq^n\right)\right|\leq\left(-t\prod_{i=1}^{r}M_i;q\right)_n. \tag{3.12}$$

Multiplying both sides of (3.12) by $|c_n|$, one has

$$\left|c_n\cdot{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r,q^{-n}\\ b_1,b_2,\ldots,b_r\end{matrix};\ q,tq^n\right)\right|\leq|c_n|\left(-t\prod_{i=1}^{r}M_i;q\right)_n. \tag{3.13}$$

The ratio test shows that the series

$$\sum_{n=0}^{\infty}c_n\left(-t\prod_{i=1}^{r}M_i;q\right)_n \tag{3.14}$$

is absolutely convergent. From (3.13), it is sufficient to establish that (3.2) is absolutely convergent. □

Theorem 3.2. *Suppose a_i, b_i, t are any real numbers such that $t > 0$ and $a_i < 1$, $b_i < 1$ with $i = 1,2,\ldots,r$. Let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=p>1,\quad \text{or}\quad \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=+\infty, \tag{3.15}$$

then the q-series

$$\sum_{n=0}^{\infty}c_n\cdot{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r,q^{-n}\\ b_1,b_2,\ldots,b_r\end{matrix};\ q,-tq^n\right) \tag{3.16}$$

diverges.

Proof. Let $a < 1$, $b < 1$ and

$$f(t)=\frac{1-at}{1-bt},\qquad 0\leq t\leq 1, \tag{3.17}$$

It is easy to see that $f(t)$ is a monotone function with respect to $0 \leq t \leq 1$.

Consequently, one has

$$\frac{1-at}{1-bt} \geq \min\left\{1, \frac{1-a}{1-b}\right\}. \tag{3.18}$$

From (3.18), one knows

$$\frac{(a_i;q)_k}{(b_i;q)_k} = \frac{1-a_i}{1-b_i} \cdot \frac{1-a_i q}{1-b_i q} \cdots \frac{1-a_i q^{k-1}}{1-b_i q^{k-1}} \geq m_i^k, \tag{3.19}$$

where $m_i = \min\{1, (1-a_i)/(1-b_i)\}$ for $i = 1, 2, \ldots, r$.

So, one has

$$\frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_r; q)_k} \geq \left(\prod_{i=1}^{r} m_i\right)^k. \tag{3.20}$$

It is obvious that

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} > 0, \quad t > 0,\ k = 1, 2, \ldots, n. \tag{3.21}$$

Multiplying both sides of (3.20) by

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} \tag{3.22}$$

gives

$$\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(-tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k} \geq \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n \prod_{i=1}^{r} m_i\right)^k. \tag{3.23}$$

Hence,

$$\begin{aligned} {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\ q, -tq^n\right) &= \sum_{k=0}^{n} \frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(-tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k} \\ &\geq \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n \prod_{i=1}^{r} m_i\right)^k. \end{aligned} \tag{3.24}$$

By using (2.4) one obtains

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n \prod_{i=1}^{r} m_i\right)^k = \left(-t\prod_{i=1}^{r} m_i; q\right)_n. \tag{3.25}$$

Substituting (3.25) into (3.24), one has

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\; q, -tq^n\right) \geq \left(-t\prod_{i=1}^{r} m_i; q\right)_n . \tag{3.26} $$

Multiplying both sides of (3.26) by $|c_n|$, one has

$$ |c_n| \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\; q, -tq^n\right) \geq |c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n . \tag{3.27} $$

Since

$$ \lim_{n\to\infty} \frac{|c_{n+1}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{n+1}}{|c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|. \tag{3.28} $$

By hypothesis

$$ \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = p > 1, \quad \text{or} \quad \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = +\infty, \tag{3.29} $$

therefore, in both cases there exists a integer $N_0 > 0$ such that $\forall n > N_0$

$$ \frac{|c_{n+1}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{n+1}}{|c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n} > 1. \tag{3.30} $$

So, one can conclude that

$$ |c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n > |c_{N_0}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{N_0}, \quad \forall n > N_0. \tag{3.31} $$

Now, from (3.27) and (3.31)

$$ \begin{aligned} |c_n| \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\; q, -tq^n\right) &\geq |c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n \\ &> |c_{N_0}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{N_0} \\ &> 0. \end{aligned} \tag{3.32} $$

Thereby, (3.16) diverges. □

We want to point out that some q-integral can be written as (3.2) or (3.16). So, the results obtained here can be used to discuss the convergence of q-integrals.

4. Some Applications

In [6, 7], Thomae defined the q-integral on the interval $[0,1]$ by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n. \tag{4.1}$$

The right side of (4.1) corresponds to use a Riemann sum with partition points $t_n = q^n$, $n = 0,1,2,\ldots$. Jackson [8] extended Thomae q-integral via

$$\begin{aligned}\int_0^d f(t)d_qt &= d(1-q)\sum_{0}^{\infty} f(dq^n)q^n,\\ \int_c^d f(t)d_qt &= \int_0^d f(t)d_qt - \int_0^c f(t)d_qt.\end{aligned} \tag{4.2}$$

In this section, we use the theorems derived in this paper to discuss two examples of the convergence for Thomae q-integral. We have the following theorems.

Theorem 4.1. *Let a_i, b_i, t be any real numbers such that $t > 0$ and $b_i < 1$ with $i = 1,2,\ldots,r$. If $\alpha > -1$, then the Thomae q-integral*

$$\int_0^1 t^{\alpha} \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\ q, t\right)d_qt \tag{4.3}$$

converges absolutely.

Proof. By the definition of Thomae q-integral (4.1), one has

$$\begin{aligned}&\int_0^1 t^{\alpha} \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\ q, t\right)d_qt\\ &\quad = (1-q)\sum_{n=0}^{\infty} q^{n(1+\alpha)} {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ;\ q, q^n\right).\end{aligned} \tag{4.4}$$

Using Theorem 3.1 and noticing,

$$\lim_{n\to\infty} \frac{q^{(n+1)(1+\alpha)}}{q^{n(1+\alpha)}} = q^{1+\alpha} < 1, \tag{4.5}$$

one knows that (4.3) converges absolutely. □

Theorem 4.2. *Let a_i, b_i, t be any real numbers such that $t > 0$ and $a_i < 1$, $b_i < 1$ with $i = 1, 2, \ldots, r$. If $\alpha > 1$, then the Thomae q-integral*

$$\int_0^1 t^{-\alpha} \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} ; \ q, -t\right) d_q t \tag{4.6}$$

diverges.

Proof. By the definition of Thomae q-integral (4.1), one has

$$\begin{aligned} &\int_0^1 t^{-\alpha} \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} ; \ q, -t\right) d_q t \\ &\quad = (1-q) \sum_{n=0}^{\infty} q^{(1-\alpha)n} {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} ; \ q, -q^n\right). \end{aligned} \tag{4.7}$$

Using Theorem 3.2 and noticing,

$$\lim_{n \to \infty} \frac{q^{(1-\alpha)(n+1)}}{q^{(1-\alpha)n}} = q^{1-\alpha} > 1, \tag{4.8}$$

one knows that (4.6) diverges. □

Acknowledgment

The authors would like to express deep appreciation to the referees for the helpful suggestions. In particular, the authors thank the referees for help to improve the presentation of the paper. Mingjin Wang was supported by STF of Jiangsu Polytechnic University.

References

[1] M. Ito, "Convergence and asymptotic behavior of Jackson integrals associated with irreducible reduced root systems," *Journal of Approximation Theory*, vol. 124, no. 2, pp. 154–180, 2003.

[2] M. Wang, "An inequality for ${}_{r+1}\phi_r$ and its applications," *Journal of Mathematical Inequalities*, vol. 1, no. 3, pp. 339–345, 2007.

[3] M. Wang, "Two inequalities for ${}_r\phi_r$ and applications," *Journal of Inequalities and Applications*, vol. 2008, Article ID 471527, 6 pages, 2008.

[4] G. E. Andrews, *The Theory of Partitions*, vol. 2 of *Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, Reading, Mass, USA, 1976.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, vol. 35 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, Mass, USA, 1990.

[6] J. Thomae, "Beiträge zur Theorie der durch die Heine'sche Reihe darstellbaren Funktionen," *Journal für die reine und angewandte Mathematik*, vol. 70, pp. 258–281, 1869.

[7] J. Thomae, "Les séries Heinéennes supérieures, ou les séries de la forme $1 + \sum_{n=1}^{\infty} x^n (1-q^a)/(1-q) \cdot (1-q^{a+1})/(1-q^2) \cdots (1-q^{a+n-1})/(1-q^n) \cdot (1-q^{a'})/(1-q^{b'}) \cdot (1-q^{a'+1})/(1-q^{b'+1}) \cdots (1-q^{a'+n-1})/(1-q^{b'+n-1}) \cdots (1-q^{a^{(h)}})/(1-q^{b^{(h)}}) \cdot (1-q^{a^{(h)}+1})/(1-q^{b^{(h)}+1}) \cdots (1-q^{a^{(h)}+n-1})/(1-q^{b^{(h)}+n-1})$," *Annali di Matematica Pura ed Applicata*, vol. 4, pp. 105–138, 1870.

[8] F. H. Jackson, "On q-definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 50, pp. 101–112, 1910.