Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 167403, 12 pages
doi:10.1155/2009/167403

Research Article

On Some Quasimetrics and Their Applications

Imed Bachar[1] and Habib Mâagli[2]

[1] *Mathematics Department, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia*

Correspondence should be addressed to Imed Bachar, abachar@ksu.edu.sa

Received 27 September 2009; Accepted 8 December 2009

Recommended by Shusen Ding

We aim at giving a rich class of quasi-metrics from which we obtain as an application an interesting inequality for the Greens function of the fractional Laplacian in a smooth domain in $\mathbb{R}^n$.

1. Introduction

Let D be a bounded smooth domain in $\mathbb{R}^n, n \geq 1$, or $D = \mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ the half space. We denote by G_D the Green's function of the operator $u \mapsto (-\Delta)^\alpha u$ with Dirichlet or Navier boundary conditions, where α is a positive integer or $0 < \alpha < 1$.

The following inequality called 3G inequality has been proved by several authors (see [1, 2] for $\alpha = 1$ or [3, 4] for $\alpha \geq 1$ or [5] for $0 < \alpha < 1$).

There exists a constant $C > 0$ such that for each $x, y, z \in D$,

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leq C\left(\frac{\lambda(z)}{\lambda(x)}G_D(x,z) + \frac{\lambda(z)}{\lambda(y)}G_D(y,z)\right), \tag{1.1}$$

where $x \to \lambda(x)$ is a positive function which depends on the Euclidean distance between x and ∂D and the exponent α.

More precisely, to prove this inequality, the authors showed that the function $(x, y) \to \rho(x, y) = \lambda(x)\lambda(y)/G_D(x, y)$ is a quasi-metric on D (see Definition 2.1).

We emphasis that the generalized 3G inequality is crucial for various applications (see e.g., [1, Theorem 1.2], [6, Lemma 7.1]). It is also very interesting tools for analysts working on pde's. In [7, Theorem 5.1], the authors used the standard 3G inequality (see [7, Proposition 4.1]) to prove that on the unit ball B in $\mathbb{R}^n$, the inverse of polyharmonic operators that are perturbed by small lower order terms is positivity preserving. They also obtained similar results for systems of these operators. On the other hand, local maximum principles for solutions of higher-order differential inequalities in arbitrary bounded domains are obtained in [8, Theorem 2] by using estimates on Green's functions. Recently, a refined version of the standard 3G inequality for polyharmonic operator is obtained in ([3, Theorem 2.8] and [4, Theorem 2.9]). This allowed the authors to introduce and study an interesting functional Kato class, which permits them to investigate the existence of positive solutions for some polyharmonic nonlinear problems.

In the present manuscript we aim at giving a generalization of these known 3G inequalities by proving a rich class of quasimetrics (see Theorem 2.8) which in particular includes the one $\rho(x,y)=\lambda(x)\lambda(y)/G_D(x,y)$.

In order to simplify our statements, we define some convenient notations.

For $s,t\in\mathbb{R}$, we denote by

$$s\wedge t=\min(s,t),\qquad s\vee t=\max(s,t). \tag{1.2}$$

The following properties will be used several times.

For $s,t\geq 0$, we have

$$\begin{gathered}\frac{st}{s+t}\leq\min(s,t)\leq 2\frac{st}{s+t},\qquad \frac{1}{2}(s+t)\leq\max(s,t)\leq s+t,\\ \min\left(2^{p-1},1\right)(s^p+t^p)\leq(s+t)^p\leq\max\left(2^{p-1},1\right)(s^p+t^p),\quad p\in\mathbb{R}^+.\end{gathered} \tag{1.3}$$

For $\lambda,\mu>0$ and $t>0$, we have

$$\min\left(1,\frac{\mu}{\lambda}\right)\operatorname{Log}(1+\lambda t)\leq\operatorname{Log}(1+\mu t)\leq\max\left(1,\frac{\mu}{\lambda}\right)\operatorname{Log}(1+\lambda t). \tag{1.4}$$

Let f and g be two nonnegative functions on a set S. We write $f\sim g$, if there exists $c>0$ such that

$$\frac{1}{c}g(x)\leq f(x)\leq cg(x),\quad \forall x\in S. \tag{1.5}$$

Throughout this paper, we denote by c a positive generic constant whose value may vary from line to line.

2. Quasimetrics

Definition 2.1. Let E be a nonempty set. A nonnegative function $\rho(x,y)$ defined on $E\times E$ is called a quasi-metric on E if it satisfies the following properties.

(i) For all $x,y\in E$, $\rho(x,y)=\rho(y,x)$.

(ii) There exists a constant $c>0$ such that for all $x,y,z\in E$,

$$\rho(x,y)\leq c\left[\rho(x,z)+\rho(z,y)\right]. \tag{2.1}$$

Example 2.2. (1) Let d be a metric on a set E, then for each $\alpha \geq 0, d^{\alpha}$ is a quasi-metric on E.

(2) Let ρ be a quasi-metric on E and γ a non-negative symmetric function on $E \times E$ such that $\gamma \sim \rho$, then γ is a quasi-metric on E.

(3) Let ρ_1 and ρ_2 be two quasimetrics on E, then for all $a \geq 0, \rho_1 + a\rho_2$ and $\max(\rho_1, \rho_2)$ are quasimetrics on E.

Next we denote by $\mathscr{H}$ the set of nonnegative nondecreasing functions f on $[0, \infty)$ satisfying the following property

For each $a > 0$, there exists a constant $c = c(a) > 0$ such that for each $t, s \in [0, \infty)$,

$$f(a(t+s)) \leq c\big(f(t) + f(s)\big). \tag{2.2}$$

Example 2.3. The following functions belong to the set $\mathscr{H}$:

(i) $f(t) = \alpha t + \beta$, with $\alpha \geq 0$ and $\beta \geq 0$.

(ii) $f(t) = t^p$, with $p \geq 0$.

(iii) $f(t) = \mathrm{Log}(1 + \alpha t), \alpha > 0$.

Remark 2.4. (1) Let f be a function in $\mathscr{H}$. Then for each $a > 0$, there exists a constant $c = c(a) > 0$ such that for each $t \in [0, \infty)$,

$$\frac{1}{c} f(t) \leq f(at) \leq c f(t). \tag{2.3}$$

(2) Let f be a nontrivial function in $\mathscr{H}$. Then for each $t > 0, f(t) > 0$.

Proposition 2.5. *$\mathscr{H}$ is a convex cone which is stable by product and composition of functions.*

Proof. Let $f, g \in \mathscr{H}$ and $\lambda \geq 0$. First, it is clear that $f + \lambda g$, fg and the composition of functions $g \circ f$ are nonnegative nondecreasing functions on $[0, \infty)$. So we need to prove that these functions satisfy (2.2). Let $a > 0$, then from (2.2), there exists a constant $c > 0$, such that for each $t, s \in [0, \infty)$, we have

$$f(a(t+s)) \leq c\big(f(t) + f(s)\big), \qquad g(a(t+s)) \leq c\big(g(t) + g(s)\big). \tag{2.4}$$

So

$$(f + \lambda g)(a(t+s)) \leq (1+\lambda)c\big[(f+\lambda g)(t) + (f + \lambda g)(s)\big]. \tag{2.5}$$

Hence $\mathscr{H}$ is a convex cone.

On the other hand, for each $t, s \in [0, \infty)$, we have

$$\begin{aligned}(fg)(a(t+s)) &\leq c^2\big(f(t) + f(s)\big)\big(g(t) + g(s)\big)\\ &\leq c^2\big[(fg)(t) + (fg)(s) + f(t)g(s) + f(s)g(t)\big]\\ &\leq 2c^2\big[(fg)(t) + (fg)(s)\big].\end{aligned} \tag{2.6}$$

Thus, we deduce that the cone $\mathscr{H}$ is stable by product.

Finally, for each $t, s \in [0, \infty)$, we have by using again (2.2),

$$g(f(a(t+s))) \leq g(c(f(t)+f(s))) \leq c(g(f(t))+g(f(s))). \tag{2.7}$$

Hence $\mathscr{H}$ is stable by composition of functions. □

Remark 2.6. Let ρ be a quasi-metric on nonempty set E and f, g two functions in $\mathscr{H}$. Then it is clear that $f \circ \rho$ is a quasi-metric on E and so that for $(f \circ \rho) \cdot (g \circ \rho) = (f \cdot g) \circ \rho$. In particular, for each $\alpha, \beta \in (0, \infty)$, $(f \circ \rho^\alpha)(g \circ \rho^\beta)$ is a quasi-metric on E.

Indeed, the last assertion follows from the fact that the functions $t \to f(t^\alpha)$ and $t \to g(t^\beta)$ belong to the cone $\mathscr{H}$.

Before stating our main result, we need to introduce the following set $\mathcal{F}$.

Let $\mathcal{F}$ be the set of nonnegative nondecreasing functions f on $[0, \infty)$ satisfying the following two properties.

There exists a constant $c > 0$, such that for all $t \in [0, \infty)$,

$$\frac{1}{c}\frac{t}{1+t} \leq f(t) \leq ct, \tag{2.8}$$

and for all $a > 0$, there exists a constant $c = c(a) > 0$, such that for each $t \in [0, \infty)$,

$$f(at) \leq cf(t). \tag{2.9}$$

Note that functions belonging to the set $\mathcal{F}$ satisfy also properties stated in Remark 2.4.

Example 2.7. (1) For $0 \leq \alpha \leq 1$, the function $f_\alpha(t) := t/(1+t)^\alpha$ belongs to the set $\mathcal{F}$.

(2) For $\lambda > 0$ and $0 \leq \alpha \leq 1$, the function $f_{\alpha,\lambda}(t) := t^{1-\alpha}\log(1+\lambda t^\alpha)$ belongs to the set $\mathcal{F}$.

(3) The function $f(t) := \arctan(t)$ belongs to the set $\mathcal{F}$.

Indeed, the function f satisfies (2.8) with $c = 1$ and for each $a > 0$ and $t \geq 0$, we have

$$\frac{a}{\max(1, a^2)} f(t) \leq f(at) \leq \frac{a}{\min(1, a^2)} f(t). \tag{2.10}$$

Throughout this paper, (E, d) denote a metric space. Let Ω be a subset in E such that $\partial\Omega \neq \emptyset$, and let $\delta(x)$ be the distance between x and $\partial\Omega$. Our main result is the following.

Theorem 2.8. *Let $f \in \mathcal{F}$, $g \in \mathscr{H}$ and h be a nontrivial function in $\mathscr{H}$. For $(x, y) \in \Omega \times \Omega$, put*

$$\gamma(x,y) = g\Big(\max\Big(d^2(x,y), \delta(x)\delta(y)\Big)\Big)\frac{h(\delta(x)\delta(y))}{f(h(\delta(x)\delta(y))/h(d^2(x,y)))}. \tag{2.11}$$

Then γ is a quasi-metric on Ω.

For the proof, we need the following key lemma (see [9]). For completeness of this paper, we reproduce the proof of this lemma here.

Lemma 2.9 (see [9]). *Let $x, y \in \Omega$. Then one has the following properties.*

(1) *If $\delta(x)\delta(y) \le d^2(x,y)$, then $(\delta(x) \vee \delta(y)) \le ((\sqrt{5}+1)/2)d(x,y)$.*

(2) *If $d^2(x,y) \le \delta(x)\delta(y)$, then $((3-\sqrt{5})/2)\delta(x) \le \delta(y) \le ((3+\sqrt{5})/2)\delta(x)$ and $d(x,y) \le (\sqrt{5}+1/2)(\delta(x) \wedge \delta(y))$.*

Proof. (1) We may assume that $\delta(x) \vee \delta(y) = \delta(y)$. Then the inequalities

$$\delta(y) \le \delta(x) + d(x,y), \qquad \delta(x)\delta(y) \le d^2(x,y) \tag{2.12}$$

imply that

$$(\delta(y))^2 - \delta(y)d(x,y) - d^2(x,y) \le 0. \tag{2.13}$$

That is

$$\left(\delta(y) + \frac{\left(\sqrt{5}-1\right)}{2}d(x,y)\right)\left(\delta(y) - \frac{\left(\sqrt{5}+1\right)}{2}d(x,y)\right) \le 0. \tag{2.14}$$

It follows that

$$(\delta(x) \vee \delta(y)) \le \frac{\left(\sqrt{5}+1\right)}{2}d(x,y). \tag{2.15}$$

(2) For each $z \in \partial\Omega$, we have $d(y,z) \le d(x,y) + d(x,z)$ and since $d^2(x,y) \le \delta(x)\delta(y)$, we obtain

$$d(y,z) \le \sqrt{\delta(x)\delta(y)} + d(x,z) \le \sqrt{d(x,z)d(y,z)} + d(x,z). \tag{2.16}$$

That is

$$\left(\sqrt{d(y,z)} + \frac{\left(\sqrt{5}-1\right)}{2}\sqrt{d(x,z)}\right)\left(\sqrt{d(y,z)} - \frac{\left(\sqrt{5}+1\right)}{2}\sqrt{d(x,z)}\right) \le 0. \tag{2.17}$$

It follows that

$$d(y,z) \le \frac{\left(3+\sqrt{5}\right)}{2}d(x,z). \tag{2.18}$$

Thus, interchanging the role of x and y, we have

$$\left(\frac{3-\sqrt{5}}{2}\right)d(x,z) \le d(y,z) \le \left(\frac{3+\sqrt{5}}{2}\right)d(x,z). \tag{2.19}$$

Which gives that

$$\left(\frac{3-\sqrt{5}}{2}\right)\delta(x) \le \delta(y) \le \left(\frac{3+\sqrt{5}}{2}\right)\delta(x). \tag{2.20}$$

Moreover, we have

$$d^2(x,y) \le \delta(x)\delta(y) \le \left(\frac{\sqrt{5}+1}{2}\right)^2 (\delta(x)\wedge\delta(y))^2. \tag{2.21}$$

$\square$

Proof of Theorem 2.8. It is clear that γ is a non-negative symmetric function on $\Omega\times\Omega$

Put $\rho(x,y) = g(\max(d^2(x,y),\delta(x)\delta(y)))$, for x, y in Ω.

Since there exists a constant $c > 0$, such that for each $t \ge 0$,

$$\frac{1}{c}\frac{t}{1+t} \le f(t) \le ct, \tag{2.22}$$

we deduce that for each x, y in Ω,

$$\frac{1}{c}\rho(x,y)h\left(d^2(x,y)\right) \le \gamma(x,y) \le c\rho(x,y)\left[h\left(d^2(x,y)\right) + h(\delta(x)\delta(y))\right]. \tag{2.23}$$

Let z in Ω. We distinguish the following subcases.

(i) If $\delta(x)\delta(y) \le d^2(x,y)$, then by (2.23) and Remark 2.6, we have

$$\begin{aligned}\gamma(x,y) &\le cg\left(d^2(x,y)\right)h\left(d^2(x,y)\right)\\ &\le c\left[g\left(d^2(x,z)\right)h\left(d^2(x,z)\right) + g\left(d^2(y,z)\right)h\left(d^2(y,z)\right)\right]\\ &\le c(\gamma(x,z)+\gamma(y,z)).\end{aligned} \tag{2.24}$$

(ii) If $d^2(x,y) \le \delta(x)\delta(y)$, it follows by Lemma 2.9 that $\delta(x)\sim\delta(y)$.

(a) If $d^2(x,z) \le \delta(x)\delta(z)$ or $d^2(y,z) \le \delta(y)\delta(z)$, then from Lemma 2.9, we deduce that $\delta(x)\sim\delta(y)\sim\delta(z)$.

So

$$\begin{gathered}\rho(x,y)\sim\rho(x,z)\sim\rho(y,z),\\ h(\delta(x)\delta(y))\sim h(\delta(x)\delta(z))\sim h(\delta(y)\delta(z)).\end{gathered} \tag{2.25}$$

Now, since

$$h\left(d^2(x,y)\right) \le c\left[h\left(d^2(x,z)\right) + h\left(d^2(y,z)\right)\right] \le c\left(h\left(d^2(x,z)\right) \vee h\left(d^2(y,z)\right)\right), \tag{2.26}$$

we deduce from (2.25) that

$$\frac{h(\delta(x)\delta(z))}{h(d^2(x,z))} \wedge \frac{h(\delta(y)\delta(z))}{h(d^2(y,z))} \leq c\frac{h(\delta(x)\delta(y))}{h(d^2(x,y))}. \tag{2.27}$$

Which implies from (2.25) and (2.9) that

$$\begin{aligned}
&\frac{h(\delta(x)\delta(y))}{f(h(\delta(x)\delta(y))/h(d^2(x,y)))} \\
&\quad\leq c\left(\frac{h(\delta(x)\delta(z))}{f(h(\delta(x)\delta(z))/h(d^2(x,z)))} \vee \frac{h(\delta(y)\delta(z))}{f(h(\delta(y)\delta(z))/h(d^2(y,z)))}\right) \\
&\quad\leq c\left(\frac{h(\delta(x)\delta(z))}{f(h(\delta(x)\delta(z))/h(d^2(x,z)))} + \frac{h(\delta(y)\delta(z))}{f(h(\delta(y)\delta(z))/h(d^2(y,z)))}\right)
\end{aligned} \tag{2.28}$$

Hence

$$\gamma(x,y) \leq c\big(\gamma(x,z) + \gamma(y,z)\big). \tag{2.29}$$

(b) If $d^2(x,z) \geq \delta(x)\delta(z)$ and $d^2(y,z) \geq \delta(y)\delta(z)$, then by Lemma 2.9, it follows that

$$(\delta(x) \vee \delta(z)) \leq cd(x,z), \qquad (\delta(y) \vee \delta(z)) \leq cd(y,z). \tag{2.30}$$

Hence, by (2.23) and (2.2), we have

$$\begin{aligned}
\gamma(x,y) &\leq cg(\delta(x)\delta(y))h(\delta(x)\delta(y)) \\
&\leq c\Big[g\Big((\delta(x))^2\Big)h\Big((\delta(x))^2\Big) + g\Big((\delta(y))^2\Big)h\Big((\delta(y))^2\Big)\Big] \\
&\leq c\Big[g\Big(d^2(x,z)\Big)h\Big(d^2(x,z)\Big) + g\Big(d^2(y,z)\Big)h\Big(d^2(y,z)\Big)\Big] \\
&\leq c\big(\gamma(x,z) + \gamma(y,z)\big).
\end{aligned} \tag{2.31}$$

So there exists a constant $c > 0$, such that for each $x, y, z \in \Omega$, we have

$$\gamma(x,y) \leq c\big(\gamma(x,z) + \gamma(y,z)\big). \tag{2.32}$$

By taking $f(t) = t$ in Theorem 2.8, we obtain the following corollary. □

Corollary 2.10. *Let g and h be two functions in $\mathcal{H}$. Then the function*

$$\rho(x,y) := h\Big(d^2(x,y)\Big)g\Big(\max\Big(d^2(x,y), \delta(x)\delta(y)\Big)\Big) \tag{2.33}$$

is a quasi-metric on Ω.

Example 2.11. Let Ω be a subset of (E,d) such that $\partial\Omega \neq \emptyset$. We denote $\delta_\Omega(x) = \delta(x)$, the distance between x and $\partial\Omega$.

(1) For each $\alpha,\beta \geq 0$, the function

$$\rho(x,y) := (d(x,y))^{\beta}\Big(\max\big(d^2(x,y),\delta(x)\delta(y)\big)\Big)^{\alpha} \tag{2.34}$$

is a quasi-metric on Ω.

(2) For each $\lambda,\mu \geq 0$, the function

$$\rho(x,y) := \Big(\max\big(d^2(x,y),\delta(x)\delta(y)\big)\Big)^{\mu}\frac{(\delta(x)\delta(y))^{\lambda}}{\operatorname{Log}\Big(1+(\delta(x)\delta(y))^{\lambda}/(d(x,y))^{2\lambda}\Big)} \tag{2.35}$$

is a quasi-metric on Ω.

3. Applications

As applications of Theorem 2.8, we will collect many forms of the 3G inequality (1.1), which in fact depend on the shape of the domain and the choose of the operator $u \mapsto (-\Delta)^{\alpha}u$ with Dirichlet or Navier boundary conditions, where α is a positive integer or $0<\alpha<1$.

3.1. Polyharmonic Laplacian Operator with Dirichlet Boundary Conditions

In [10, page 126], Boggio gave an explicit expression for the Green function $G^{B}_{m,n}$ of $(-\Delta)^m$ on the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$, with Dirichlet boundary conditions $(\partial/\partial\nu)^j u = 0, 0 \leq j \leq m-1$:

$$G^{B}_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n}\int_1^{[x,y]/|x-y|}\frac{(v^2-1)^{m-1}}{v^{n-1}}dv, \tag{3.1}$$

where $\partial/\partial\nu$ is the outward normal derivative, m is a positive integer, $k_{m,n} = \Gamma(n/2)/(2^{2m-1}\pi^{n/2}[(m-1)!]^2)$ and $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for x,y in B.

Then we deduce that for each $a \in \mathbb{R}^n$ and $r>0$, we have

$$G^{B(a,r)}_{m,n}(x,y) = r^{2m-n}G^{B}_{m,n}\left(\frac{x-a}{r},\frac{y-a}{r}\right), \quad \text{for } x,y \in B(a,r), \tag{3.2}$$

where $B(a,r)$ denote the open ball in $\mathbb{R}^n$ with radius r, centered at a.

Using a rescaling argument, one recovers from (3.1) a similar Green function $G^{\mathbb{R}^n_+}_{m,n}$ of $(-\Delta)^m$ on the half-space $\mathbb{R}^n_+ := \{(x',x_n) \in \mathbb{R}^n : x_n > 0\}$ (see [11, page 165]):

$$G^{\mathbb{R}^n_+}_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n}\int_1^{|x-\overline{y}|/|x-y|}\frac{(v^2-1)^{m-1}}{v^{n-1}}dv, \quad \text{for } x,y \text{ in } \mathbb{R}^n_+, \tag{3.3}$$

where $\overline{y} = (y_1,\ldots,y_{n-1},-y_n)$.

Indeed, let $e = (0, \dots, 0, 1) \in \mathbb{R}^n$ and for $p \in \mathbb{N}$, we denote by $B_p = B(2^p e, 2^p)$. Then by using (3.2) and (3.1), we obtain for each x, y in $\mathbb{R}^n_+$,

$$G^{\mathbb{R}^n_+}_{m,n}(x,y) = \sup_{p\in\mathbb{N}} G^{B_p}_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n}\int_1^{|x-\overline{y}|/|x-y|}\frac{\left(v^2-1\right)^{m-1}}{v^{n-1}}dv. \tag{3.4}$$

Now to prove the 3G inequality (1.1) with $\lambda(x) = (\delta(x))^m$, for these Green's functions $G^D_{m,n}$, where $D = B$ or $D = \mathbb{R}^n_+$, we put $\rho(x,y) := (\delta(x)\delta(y))^m / G^D_{m,n}(x,y)$ and we need to show that ρ is a quasi-metric on D. To this end, we observe that from [7] or [3] for $D = B$ and from [4] for $D = \mathbb{R}^n_+$, we have the following estimates on $G^D_{m,n}$:

$$G^D_{m,n}(x,y) \sim \begin{cases} \dfrac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}\left(|x-y|^2 \vee \delta(x)\delta(y)\right)^m}, & \text{if } n > 2m, \\[2ex] \operatorname{Log}\left(1 + \dfrac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right), & \text{if } n = 2m, \\[2ex] \dfrac{(\delta(x)\delta(y))^m}{\left(|x-y|^2 \vee (\delta(x)\delta(y))\right)^{n/2}}, & \text{if } n < 2m. \end{cases} \tag{3.5}$$

From which we deduce that

$$\rho(x,y) \sim \begin{cases} |x-y|^{n-2m}\left(\max\left(|x-y|^2, \delta(x)\delta(y)\right)\right)^m, & \text{if } n > 2m, \\[2ex] \dfrac{(\delta(x)\delta(y))^m}{\log\left(1 + \left((\delta(x)\delta(y))^m / |x-y|^{2m}\right)\right)}, & \text{if } n = 2m, \\[2ex] \left(\max\left(|x-y|^2, \delta(x)\delta(y)\right)\right)^{n/2}, & \text{if } n < 2m. \end{cases} \tag{3.6}$$

So by Example 2.11, we see that ρ is a quasi-metric on D, that is, $G^D_{m,n}$ satisfies the 3G inequality (1.1) with $\lambda(x) = (\delta(x))^m$.

3.2. Polyharmonic Laplacian Operator with Navier Boundary Conditions

Let D be a bounded smooth domain in $\mathbb{R}^n$ or $D = \mathbb{R}^n_+$ the half space. We denote by $G^D_{m,n}$ the Green's function of the polyharmonic operator $u \mapsto (-\Delta)^m u$ on D, with Navier boundary conditions $(-\Delta)^k u_{|\partial D} = 0$, for $0 \le k \le m-1$, where m is a positive integer.

In [12], for D a bounded smooth domain and in [13], for $D = \mathbb{R}^n_+$, the authors have established the following estimates for the Green function $G^D_{m,n}$:

$$G^D_{m,n}(x,y) \sim \begin{cases} \dfrac{\delta(x)\delta(y)}{|x-y|^{n-2m}\left(|x-y|^2 \vee \delta(x)\delta(y)\right)}, & \text{if } n > 2m, \\ \operatorname{Log}\left(1 + \dfrac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n = 2m, \\ \dfrac{\delta(x)\delta(y)}{\left(|x-y|^2 \vee \delta(x)\delta(y)\right)^{1/2}}, & \text{if } n = 2m-1. \end{cases} \tag{3.7}$$

So we deduce that the function $\rho(x,y) := \delta(x)\delta(y)/(G^D_{m,n}(x,y))$ satisfies

$$\rho(x,y) \sim \begin{cases} |x-y|^{n-2m}\max\left(|x-y|^2, \delta(x)\delta(y)\right), & \text{if } n > 2m, \\ \dfrac{\delta(x)\delta(y)}{\operatorname{Log}\left(1 + \delta(x)\delta(y)/|x-y|^2\right)}, & \text{if } n = 2m, \\ \left(\max\left(|x-y|^2, \delta(x)\delta(y)\right)\right)^{1/2}, & \text{if } n = 2m-1. \end{cases} \tag{3.8}$$

Hence, the function ρ is a quasi-metric on D, by Example 2.11. Therefore, the Green function $G^D_{m,n}$ satisfies the 3G inequality (1.1) with $\lambda(x) = \delta(x)$.

3.3. *Fractional Laplacian with Dirichlet Boundary Conditions*

Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ $(n \geq 2)$ and G_D the Green's function of the fractional Laplacian $(-\Delta)^{\alpha/2}$, with Dirichlet boundary conditions $0 < \alpha \leq 2$. From [14], we have the following estimates on G_D:

$$G_D(x,y) \sim \frac{(\delta(x)\delta(y))^{\alpha/2}}{|x-y|^{n-\alpha}\left(\max\left(|x-y|^2, \delta(x)\delta(y)\right)\right)^{\alpha/2}}, \tag{3.9}$$

which implies that the function $\rho(x,y) := (\delta(x)\delta(y))^{\alpha/2}/G_D(x,y)$ satisfies

$$\rho(x,y) \sim |x-y|^{n-\alpha}\left(\max\left(|x-y|^2, \delta(x)\delta(y)\right)\right)^{\alpha/2}. \tag{3.10}$$

By Example 2.11, the function ρ is a quasi-metric on D, and so the Green's function G_D satisfies the 3G inequality (1.1) with $\lambda(x) = (\delta(x))^{\alpha/2}$. Note that in this case, the 3G inequality has been already proved by Chen and Song in [5].

3.4. On the Operator $-(1/A)(Au')'$ on $(0,1)$ with $u(0) = u(1) = 0$

Let A be a continuous function on $[0,1]$, which is positive and differentiable on $(0,1)$. We assume that $t \mapsto 1/(A(t))$ is integrable on $[0,1]$ and without loss of generality we may suppose that $\int_0^1 (1/A(t))dt = 1$.

We denote by $G(x,y)$ the Green's function of the operator $u \to -(1/A)(Au')'$ with $u(0) = u(1) = 0$. Clearly we have for each $x, y \in [0,1]$,

$$G(x,y) = A(y)\rho(x \wedge y)(1 - \rho(x \vee y)), \tag{3.11}$$

where $\rho(x) = \int_0^x (1/A(t))dt$.

Put $\lambda(x) = \rho(x)(1 - \rho(x))$, then we claim that the Green's function $G(x,y)$ satisfies the following 3G inequality: $\forall x, y, z \in (0,1)$,

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le \frac{\lambda(z)}{\lambda(x)}G(x,z) + \frac{\lambda(z)}{\lambda(y)}G(y,z). \tag{3.12}$$

That is

$$\frac{\Gamma(x,z)\Gamma(z,y)}{\Gamma(x,y)} \le \frac{\lambda(z)}{\lambda(x)}\Gamma(x,z) + \frac{\lambda(z)}{\lambda(y)}\Gamma(y,z), \tag{3.13}$$

where $\Gamma(x,y) := \rho(x \wedge y)(1 - \rho(x \vee y))$.

To prove (3.13), we need to show that the function

$$\gamma(x,y) := \frac{\lambda(x)\lambda(y)}{\Gamma(x,y)} = \rho(x \vee y)(1 - \rho(x \wedge y)) \tag{3.14}$$

is quasi-metric on $(0,1)$.

Indeed, since the function $\gamma(x,y)$ is symmetric in x, y, we will discuss three cases:

(i) if $z \le x \le y$, then

$$\gamma(x,y) = \rho(y)(1 - \rho(x)) \le \rho(y)(1 - \rho(z)) = \gamma(z,y) \le \gamma(x,z) + \gamma(z,y), \tag{3.15}$$

(ii) if $x \le y \le z$, then

$$\gamma(x,y) = \rho(y)(1 - \rho(x)) \le \rho(z)(1 - \rho(x)) = \gamma(x,z) \le \gamma(x,z) + \gamma(z,y), \tag{3.16}$$

(iii) if $x \le z \le y$, then by writing $\rho(z) = \alpha\rho(x) + (1-\alpha)\rho(y)$ for some $\alpha \in [0,1]$, we obtain $1 - \rho(z) = \alpha(1 - \rho(x)) + (1 - \alpha)(1 - \rho(y))$.

So we deduce that

$$\begin{aligned}\gamma(x,z)+\gamma(z,y) &= \big(\alpha\rho(x)+(1-\alpha)\rho(y)\big)\big(1-\rho(x)\big)\\ &\quad+\rho(y)\big(\alpha(1-\rho(x))+(1-\alpha)(1-\rho(y))\big)\\ &\geq (1-\alpha)\rho(y)\big(1-\rho(x)\big)+\alpha\rho(y)\big(1-\rho(x)\big)\\ &\geq \rho(y)\big(1-\rho(x)\big)=\gamma(x,y).\end{aligned} \tag{3.17}$$

This completes the proof.

Acknowledgments

The authors thank the reviewers for their valuable suggestions. The research of I. Bachar is supported by King Saud University, College of Sciences Research Center, Project no (Math/2009/51).

References

[1] N. J. Kalton and I. E. Verbitsky, "Nonlinear equations and weighted norm inequalities," *Transactions of the American Mathematical Society*, vol. 351, no. 9, pp. 3441–3497, 1999.

[2] M. Selmi, "Inequalities for Green functions in a Dini-Jordan domain in $\mathbb{R}^2$," *Potential Analysis*, vol. 13, no. 1, pp. 81–102, 2000.

[3] I. Bachar, H. Mâagli, S. Masmoudi, and M. Zribi, "Estimates for the Green function and singular solutions for polyharmonic nonlinear equation," *Abstract and Applied Analysis*, vol. 2003, no. 12, pp. 715–741, 2003.

[4] I. Bachar, H. Mâagli, and M. Zribi, "Estimates on the Green function and existence of positive solutions for some polyharmonic nonlinear equations in the half space," *Manuscripta Mathematica*, vol. 113, no. 3, pp. 269–291, 2004.

[5] Z. Q. Chen and R. Song, "General gauge and conditional gauge theorems," *The Annals of Probability*, vol. 30, no. 3, pp. 1313–1339, 2002.

[6] Y. Pinchover, "Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations," *Mathematische Annalen*, vol. 314, no. 3, pp. 555–590, 1999.

[7] H. C. Grunau and G. Sweers, "Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions," *Mathematische Annalen*, vol. 307, no. 4, pp. 589–626, 1997.

[8] H. C. Grunau and G. Sweers, "Classical solutions for some higher order semilinear elliptic equations under weak growth conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 5, pp. 799–807, 1997.

[9] H. Mâagli, "Inequalities for the Riesz potentials," *Archives of Inequalities and Applications*, vol. 1, no. 3-4, pp. 285–294, 2003.

[10] T. Boggio, "Sulle funzione di Green d'ordine m," *Rendiconti del Circolo Matemàtico di Palermo*, vol. 20, pp. 97–135, 1905.

[11] H. C. Grunau and G. Sweers, "The maximum principle and positive principal eigenfunctions for polyharmonic equations," in *Reaction Diffusion Systems*, G. Caristi and E. Mitidieri, Eds., vol. 194, pp. 163–182, Marcel Dekker, New York, NY, USA, 1998.

[12] H. C. Grunau and G. Sweers, "Sharp estimates for iterated Green functions," *Proceedings of the Royal Society of Edinburgh A*, vol. 132, no. 1, pp. 91–120, 2002.

[13] I. Bachar, "Estimates for the Green function and characterization of a certain Kato class by the Gauss semigroup in the half space," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 4, article 119, 14 pages, 2005.

[14] Z. Q. Chen and R. Song, "Estimates on Green functions and Poisson kernels for symmetric stable processes," *Mathematische Annalen*, vol. 312, no. 3, pp. 465–501, 1998.